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Two Families of Self-adjoint Indecomposable Operators in an Orthomodular Space

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Abstract

Orthomodular spaces are the counterpart of Hilbert spaces for fields other than $\mathbb{R}$ or $\mathbb{C}$. Both share numerous properties, foremost among them is the validity of the Projection theorem. Nevertheless in the study of bounded linear operators which started in [3], there appeared striking differences with the classical theory. In fact, in this paper we shall construct, on the canonical non-archimedean orthomodular space E of [5], two infinite families of self-adjoint bounded linear operators having no invariant closed subspaces other than the trivial ones. Spectrums of such operators contain exactly one point which, therefore, is not an eigenvalue. We also study relations between the subalgebras of bounded linear operators of E, which are the commutant of each of these operators, and the algebra $\mathcal{A}$ studied in [3].

1. Introduction

A vector space V provided with a hermitian form Φ is an **orthomodular space** if

$$U^{\perp\perp} = U \Longrightarrow V = U \oplus U^{\perp}$$

for all linear subspaces U of V. Until 1979 Hilbert spaces over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ were the only known examples of such spaces, but since then classes of non-classical orthomodular spaces have been constructed([5],[2]). All of these new examples are infinite dimensional vector spaces over Krull-valued complete fields where hermitian forms induce non-archimedean norms.

The orthomodular space E considered from now on was the first non-classical example, constructed in [5] (over an ordered field) and generalized –in valued fields context– in [2]. Now, an outline of its construction is presented.

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The value group of the Krull valuation of the base field K is

$$\Gamma := \bigoplus_{j\in\mathbb{N}} \Gamma_j,$$

where each Γ_j is an isomorphic copy of the additive group of integers. Γ is ordered antilexicographically, i.e., if $0 \neq (g_j)_{j\in\mathbb{N}} \in \Gamma$ and $m := \max\{j \in \mathbb{N} : g_j \neq 0\}$, then

$$(g_j)_{j\in\mathbb{N}} > 0 \iff g_m > 0 \text{ in } \Gamma_m.$$

Let $K_0 := \mathbb{R}(X_i)_{i\in\mathbb{N}}$, the field of rational functions in the variables $X_1, X_2, \ldots$ with real coefficients. For convenience of notation, we define $X_0 := 1$. The non-archimedean valuation $\nu_0 : K_0 \longrightarrow \Gamma \cup \{\infty\}$ is trivial on $\mathbb{R}$ and maps each X_i to $(0, \ldots, 0, -1, 0, \ldots) \in \Gamma$ (where the -1 is in the i-th place).

The base field K is the completion of K_0 with respect to the valuation, and ν_0 can be extended uniquely to a valuation ν on K with the same value group.

We define the $K-$vector space E by

$$E := \left\{ (\xi_i)_{i\in\mathbb{N}_0} \in K^{\mathbb{N}_0} : \sum_{i=0}^{\infty} \xi_i^2 X_i \text{ converges in the valuation topology} \right\}$$

with componentwise operations.

This vector space over K along with the anisotropic form $\Phi : E \times E \longrightarrow K$ defined by

$$\Phi\left((\xi_i)_{i\in\mathbb{N}_0}, (\eta_i)_{i\in\mathbb{N}_0}\right) = \sum_{i=0}^{\infty} \xi_i \eta_i X_i,$$

is an orthomodular space (see [5], [2]).

Then, following the notation of [3], the assignment $\|\cdot\| : E \longrightarrow \Gamma \cup \{\infty\}$, defined by $\|x\| = \nu(\Phi(x, x))$, satisfies the strong triangle inequality and induces a topology in E and the notion of Cauchy nets in E, for which E is complete.

Moreover, a subspace U of E is closed in this topology if and only if it is orthogonally closed, that is $U^{\perp\perp} = U$ ([5]).

We shall also work here with elements of $\mathcal{B}(E)$, the algebra of linear operators $B : E \longrightarrow E$ for which there exists an element $\gamma \in \Gamma$ such that, for all $x \in E$, $x \neq 0$, $\| B(x) \| - \| x \| \geq \gamma$.

In Section 2, we summarize all the geometric properties of E (its residual spaces and the definition of types in this space) and all the results concerning the algebra $\mathcal{B}(E)$ and the subalgebra $\mathcal{A}$ that will be necessary later on. In Section 3, the core of this work is developed: we define two infinite families of bounded operators on E, perturbations of the operator A studied in [3], and we prove that each element of these families is an indecomposable self-adjoint operator (Theorem 3.1 and Theorem 3.5) that has non-empty spectrum (Theorem 3.6). Both families contain a sequence of bounded operators converging to A. Finally, in Section 4, we establish that all the commutant algebras of the operators defined are mutually distinct and that the intersection of each one of these algebras and $\mathcal{A}$ is minimal (Theorem 4.4 and Theorem 4.5).

2. **Preliminaries**

The required results of [5], [3] and [4] are condensed in this section. We will use here the notation and definitions of the last section.

2.1. **E and its residual subspaces**

The standard basis of E is the set $\{e_i \in E : i \in \mathbb{N}_0\}$, where

$$e_i := (0, 0, \ldots, 0, 1, 0, \ldots),$$

where the 1 is in the $(i+1)$-th place.

$\Phi(e_i, e_j) = 0$ if $i \neq j$ and $\Phi(e_i, e_i) = X_i$. In addition, each $x \in E$ can be uniquely written as a convergent series in the $\|\cdot\|$-topology:

$$x = \sum_{i=0}^{\infty} \xi_i e_i.$$

An extremely useful technique for our work is the reduction of bounded operators to the residual spaces of E. Let us recall the definition of these spaces and some of their properties:

The convex subgroups (see [6] for a definition) of Γ are exactly the subgroups $\Delta_n = \Gamma_1 \oplus \cdots \oplus \Gamma_n \oplus \{0\} \oplus \{0\} \oplus \cdots$ $(n \in \mathbb{N}_0)$.

A valuation ring

$$R_n := \{\xi \in K : \nu(\xi) \geq \delta \text{ for some } \delta \in \Delta_n\}$$

corresponds to each Δ_n. The unique maximal ideal of R_n is $J_n := \{\xi \in K : \nu(\xi) > \delta \text{ for all } \delta \in \Delta_n\}$.

$\widehat{K}_n := R_n/J_n$ is the residual field corresponding to Δ_n (we let $\Theta_n : R_n \longrightarrow \widehat{K}_n$ be the canonical projection). It can easily be proved that $\widehat{K}_n \cong \mathbb{R}(X_1, \ldots, X_n)$.

From the strong triangle inequality of $\|\cdot\|$ it follows that

$$M_n := \{x \in E : \|x\| \geq \delta \text{ for some } \delta \in \Delta_n\}$$

is a module over R_n and

$$S_n := \{x \in E : \|x\| > \delta \text{ for all } \delta \in \Delta_n\}$$

is a submodule.

$\widehat{E}_n := M_n/S_n$ is a vector space over $\widehat{K}_n$ ($\pi_n : M_n \longrightarrow \widehat{E}_n$ is the canonical projection) by defining scalar multiplication by

$$\Theta_n(\xi)\pi_n(x) := \pi_n(\xi x). \qquad (x \in M_n, \xi \in R_n)$$

Finally, Φ induces a form $\widehat{\Phi}_n$ in $\widehat{E}_n$ defined by $\widehat{\Phi}_n(\pi_n(x), \pi_n(y)) = \Theta_n(\Phi(x, y))$.

$(\widehat{E}_n, \widehat{\Phi}_n)$ is the **residual space** of E corresponding to Δ_n.

Theorem 2.1 ([3]). *We have* $\dim(\widehat{E}_n) = n + 1$. *The vectors* $\widehat{e}_i := \pi_n(e_i)$, $i = 0, 1, \ldots, n$, *form an orthogonal basis for* $(\widehat{E}_n, \widehat{\Phi}_n)$ *and*

$$\widehat{\Phi}_n \sim \operatorname{diag}(1, X_1, X_2, \ldots, X_n).$$

A subspace $U \subseteq E$ is reduced in $\widehat{E}_n$ to a subspace

$$\pi_n(U) = \{\pi_n(x) : \ x \in U \cap M_n\}.$$

Lemma 2.2 ([3]). *Let U and V be two orthogonal subspaces of E ($U \perp V$). Then $\pi_n(U) \perp \pi_n(V)$ and $\pi_n(U \oplus V) = \pi_n(U) \oplus \pi_n(V)$.*

2.2. Types in E.

Studying linear operators on E through their "reductions" to residual spaces relies strongly on the concept of types. In this subsection, we recall this definition for our particular space ([3]) as well as some significant results using this concept.

A type $T(\gamma)$ is assigned to each $\gamma = (g_j)_{j\in\mathbb{N}} \in \Gamma$ by

$$T(\gamma) := \begin{cases} 0 & \text{if } \gamma \in 2\Gamma \\ \max\{j \in \mathbb{N} : g_j \text{ is odd}\} & \text{if } \gamma \notin 2\Gamma \end{cases}$$

A type is also assigned to every non-zero scalar and to every non-zero vector in the space: the type of $\xi \in K^*$ is defined by

$$T(\xi) := T(\nu(\xi))$$

and the type of $x \in E$, $x \neq 0$, is

$$T(x) := T(\Phi(x, x)).$$

Note that for each pair $\gamma, \gamma' \in \Gamma$, $T(\gamma) = T(\gamma + 2\gamma')$. Then, for $\xi \in K^*$, $T(\alpha^2 \xi) = T(\xi)$ for all $\alpha \in K^*$, since $\nu(\alpha^2 \xi) = 2\nu(\alpha) + \nu(\xi)$. Therefore, for all $\lambda \in K^*$ and all $0 \neq x \in E$, $T(\lambda x) = T(x)$, i.e. each line G of E has a type: $T(G)$.

The following results relate some geometric properties of E to the concept of types.

Theorem 2.3 ([5]).

i) Let $x, y \in E \setminus \{0\}$. If $x \perp y$, then $T(x) \neq T(y)$.

ii) Let U be a closed subspace of E. Then the same types occur in any two maximal orthogonal families in U.

Lemma 2.4 ([3]). *Let G be a one-dimensional subspace of E. $\pi_n(G) = \{0\}$ if and only if $T(G) > n$.*

2.3. $\mathcal{B}$(E) and the subalgebra $\mathcal{A}$.

Recall that $\mathcal{B}(E)$ is the algebra of linear operators $B : E \longrightarrow E$ for which the set $\{\|B(x)\| - \|x\| : x \in E, x \neq 0\}$ is bounded from below in Γ.

Clearly, each linear operator on E is determined by the image of the standard basis $\{e_i : i \geq 0\}$. Then it can be represented by an infinite matrix. Since $B \in \mathcal{B}(E)$ is self-adjoint if and only if

$$\Phi(B(e_i), e_j) = \Phi(e_i, B(e_j)) \qquad (i, j \geq 0)$$

we have the following lemma

Lemma 2.5. *Let $B \in \mathcal{B}(E)$ be such that $B(e_j) = \sum_{i=0}^{\infty} b_{ij}\, e_i$ for all $j \in \mathbb{N}_0$. Then B is self-adjoint if and only if $X_i\, b_{ij} = X_j\, b_{ji}$ for all pairs $i, j \in \mathbb{N}_0$.*

Thus, a bounded operator M with matrix (m_{ij}) is self-adjoint if and only if

$$X_i \, m_{ij} = X_j \, m_{ji}, \tag{2.1}$$

for all $i, j \geq 0$.

Every operator in this work aside from being self-adjoint, also has the property defined below.

Definition 2.6. A linear operator $B : E \longrightarrow E$ is **indecomposable** if it admits no closed invariant subspaces of E with the exception of $\{0\}$ and E.

We will additionally use the following results

Lemma 2.7 ([3])**.** *A map $B_0 : \{e_i : i \in \mathbb{N}_0\} \longrightarrow E$ can be extended to a bounded linear operator $B : E \longrightarrow E$ iff the set $R_0 := \{\|B_0(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\} \subset \Gamma$ is bounded from below.*

Theorem 2.8 ([3])**.** *Let $B : E \longrightarrow E$ be an injective bounded linear operator on E. If*

$$\{\|B(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\}$$

has an upper bound in Γ, then B is surjective and its algebraic inverse $B^{-1} : E \longrightarrow E$ is bounded, that is, $B^{-1} \in \mathcal{B}(E)$.

Moreover, if $\gamma_0 \in \Gamma$ is a lower bound of R_0 then γ_0 is a lower bound of the set $\{\|B(x)\| - \|x\| : x \in E, x \neq 0\}$ too.

Let $B \in \mathcal{B}(E)$ be a linear operator such that $0 \in \Gamma$ is a lower bound of the set $\{\|B(x)\| - \|x\| : x \in E, x \neq 0\}$. Then $B(M_n) \subseteq M_n$, $B(S_n) \subseteq S_n$, hence B induces a linear operator $\widehat{B}_n : \widehat{E}_n \longrightarrow \widehat{E}_n$ defined by $\widehat{B}_n(\pi_n(x)) := \pi_n(B(x))$ ($x \in M_n$, $n \in \mathbb{N}$). These are the **induced operators** that allow us to study operators on E.

In [3], the authors study the operator $A : E \longrightarrow E$ defined over the standard basis of E by

$$A(e_k) = \sum_{i=0}^{\infty} \frac{1}{X_i} \, e_i + \left(1 - \frac{1}{X_k}\right) e_k.$$

The matrix of A in that basis,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ \frac{1}{X_1} & 1 & \frac{1}{X_1} & \frac{1}{X_1} & \frac{1}{X_1} & \cdots \\ \frac{1}{X_2} & \frac{1}{X_2} & 1 & \frac{1}{X_2} & \frac{1}{X_2} & \cdots \\ \frac{1}{X_3} & \frac{1}{X_3} & \frac{1}{X_3} & 1 & \frac{1}{X_3} & \cdots \\ \frac{1}{X_4} & \frac{1}{X_4} & \frac{1}{X_4} & \frac{1}{X_4} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

satisfies (2.1), hence A is self-adjoint.

Additionally, $\|A(e_k)\| - \|e_k\| = 0$ for all $k \in \mathbb{N}_0$. Thus $\|A(x)\| - \|x\| \geq 0$ for all $x \in E$ and A induces operators $\widehat{A}_n$ on every residual space of E.

In the following results, properties of the induced operators on the spaces $\widehat{E}_n$ are lifted to properties of the operator A defined on E.

Lemma 2.9 ([3])**.** *If $n \geq 1$ then the equation*

$$\sum_{i=0}^{n} \frac{1}{1 - \rho X_i} = 1$$

in the variable ρ has no solution in $\widehat{K_n}$.

As a consequence we have

Lemma 2.10 ([3])**.** *The operator $\widehat{A}_n : \widehat{E}_n \longrightarrow \widehat{E}_n$ ($n \geq 1$) has no eigenvectors.*

Theorem 2.11 ([3])**.** *The operator A is indecomposable.*

Proof. Let $U \neq \{0\}$ be a proper closed subspace of E, invariant under A. Since E is orthomodular and U is closed, $E = U \oplus U^{\perp}$. In addition, since A is a self-adjoint operator, $U^{\perp}$ is also an invariant space under A.

Looking at the types of vectors in U and $U^{\perp}$ by Theorem 2.3(i) no type can occur in U and $U^{\perp}$ at the same time. Hence, by Theorem 2.3(ii), either U or $U^{\perp}$ contains a vector of type 0 and, without loss of generality, we can assume it is U. Hence there exists an integer $n \geq 1$ such that U contains vectors of types $0, 1, \ldots, n-1$ and $U^{\perp}$ contains a vector of type n. We examine the reduced operator $\widehat{A}_n$ on the residual space

$$\widehat{E}_n = \pi_n(E) = \pi_n(U) \oplus^{\perp} \pi_n(U^{\perp})$$

$\pi_n(U)$ and $\pi_n(U^{\perp})$ are invariants under $\widehat{A}_n$. Let G be the (1-dimensional) subspace of $U^{\perp}$ spanned by a vector of type n. Then $U^{\perp} = G \oplus^{\perp} (U^{\perp} \cap G^{\perp})$

and $\pi_n(U^\perp) = \pi_n(G) \oplus^\perp \pi_n(U^\perp \cap G^\perp)$. By the choice of n and by Theorem 2.3.(i), $U^\perp \cap G^\perp$ contains only vectors of types greater than n, therefore, by Lemma 2.4, $\pi_n(U^\perp \cap G^\perp) = \{0\}$. Hence $\pi_n(U^\perp) = \pi_n(G)$ is a one-dimensional subspace of $\widehat{E}_n$, invariant under $\widehat{A}_n$. In other words, $\widehat{A}_n$ has an eigenvector. But we know this is impossible by Lemma 2.10. $\square$

The proof of Theorem 2.11 does not use the specific definition of A. Then it can be used for any bounded self-adjoint operator, whose reduced operators have no eigenvectors.

As an immediate consequence of Theorem 2.11, we have

Corollary 2.12 ([3]). *The operator A has no eigenvectors.*

However, the spectrum of A, defined as usual by

$$\operatorname{spec}(A) := \{\lambda \in K : (A - \lambda I) \text{ has no inverse in } \mathcal{B}(E)\}$$

is not empty. In fact,

Theorem 2.13 ([3]). $\operatorname{spec}(A) = \{1\}$.

Finally, we summarize the main characteristics of the subalgebra

$$\mathcal{A} = \{C \in \mathcal{B}(E) : AC = CA\}.$$

$\mathcal{A}$ is a commutative algebra (*Corollary 5.11* of [3]) and all its elements are self-adjoint (*Corollary 5.5* of [3]). Since A is indecomposable, we have

Lemma 2.14 ([3]). *If $B, C \in \mathcal{A}$ coincide on some non-zero vector, then $B = C$.*

Therefore

Corollary 2.15 ([3]). *Every non trivial operator of $\mathcal{A}$ is injective.*

So, each element of $\mathcal{A}$ can be completely determined by its action on a single non-zero vector. In [4], the following formulas were established.

Theorem 2.16 ([4]). *Let $B \in \mathcal{A}$ be such that $B(e_0) = \sum_{k=0}^{\infty} \frac{\lambda_k}{X_k}\, e_k$. If for $m \geq 1$ $B(e_m) = \sum_{k=0}^{\infty} \beta_{km}\, e_k$, then:*

(i) if $k \neq m$, then $\beta_{km} = \frac{X_m}{X_m - X_k}\left((X_m - 1)\,\frac{\lambda_m}{X_m} - (X_k - 1)\,\frac{\lambda_k}{X_k}\right).$

(ii) if $k = m$*, then* $\beta_{mm} = \lambda_0 + \sum_{j \neq 0,m} \frac{X_m - 1}{X_m - X_j} (\lambda_j - \lambda_m)$.

3. Construction of indecomposable self-adjoint operators

3.1. The operators $\mathbf{B_{Q,s}}$.

Let $p, s \in \mathbb{N}$, such that $1 < p < s$. Consider the set $Q = \{q_1, \dots, q_p\}$ where $q_1 < \dots < q_p$ and $q_j \in \{0, 1, \dots, s-1\}$ for $j = 1, \dots, p$.

Put $u := \sum_{k=0}^{\infty} \frac{1}{X_k} e_k$ and define the map $B^0_{Q,s} : \{e_i : i \in \mathbb{N}_0\} \longrightarrow E$ by:

$$B^0_{Q,s}(e_i) = A(e_i) = u + \left(1 - \frac{1}{X_i}\right) e_i, \text{ for } i \neq q_1, \dots, q_p, s.$$

$$\begin{aligned} B^0_{Q,s}(e_{q_j}) &= A(e_{q_j}) - \frac{1}{X_s} e_s \\ &= u + \left(1 - \frac{1}{X_{q_j}}\right) e_{q_j} - \frac{1}{X_s} e_s, \text{ for } j = 1, \dots, p. \end{aligned}$$

$$\begin{aligned} B^0_{Q,s}(e_s) &= A(e_s) - \sum_{j=1}^{p} \frac{1}{X_{q_j}} e_{q_j} \\ &= u + \left(1 - \frac{1}{X_s}\right) e_s - \sum_{j=1}^{p} \frac{1}{X_{q_j}} e_{q_j}. \end{aligned}$$

It is easy to check that $\| B^0_{Q,s}(e_i) \| - \|e_i\| = 0$ for all $i \in \mathbb{N}_0$. By Lemma 2.7, $B^0_{Q,s}$ can be extended linearly to an operator in $\mathcal{B}(E)$, $B_{Q,s} : E \longrightarrow E$ satisfying

$$\|B_{Q,s}(x)\| - \|x\| \geq 0$$

for all $x \in E$. It follows that $B_{Q,s}$ induces an operator in each residual space.

The $B_{Q,s}$ matrix in the standard basis is:

$$
\begin{pmatrix}
1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots \\
\vdots & \ddots & \vdots & & \vdots & & \vdots & & \vdots & \\
\frac{1}{X_{q_1}} & \cdots & 1 & \cdots & \frac{1}{X_{q_1}} & \cdots & \frac{1}{X_{q_1}} & \cdots & 0 & \cdots \\
\vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \\
\frac{1}{X_{q_2}} & \cdots & \frac{1}{X_{q_2}} & \cdots & 1 & \cdots & \frac{1}{X_{q_2}} & \cdots & 0 & \cdots \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & \\
\frac{1}{X_{q_p}} & \cdots & \frac{1}{X_{q_p}} & \cdots & \frac{1}{X_{q_p}} & \cdots & 1 & \cdots & 0 & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & \ddots & \vdots & \\
\frac{1}{X_s} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{pmatrix}
\begin{matrix}
\\ \\ \leftarrow q_1 \\ \\ \leftarrow q_2 \\ \\ \leftarrow q_p \\ \\ \leftarrow s \\ \\
\end{matrix}
$$

$$
\begin{matrix}
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & q_1 & & q_2 & & q_p & & s
\end{matrix}
$$

This is identical to the matrix of A in the same basis except for the indicated zeros. Then clearly this matrix satisfies (2.1) too and $B_{Q,s}$ is self-adjoint.

The following is the main result of this section.

Theorem 3.1. *$B_{Q,s}$ is an indecomposable operator.*

Since $B_{Q,s}$ is a self-adjoint bounded operator, using the proof of Theorem 2.11, it is enough to prove that none of the induced operators of $B_{Q,s}$ on the residual spaces has eigenvectors.

Let $\widehat{B}_n := \widehat{(B_{Q,s})}_n$ be the operator induced by $B_{Q,s}$ on $\widehat{E}_n$. To prove that $\widehat{B}_n$ $(n \geq 1)$ has no eigenvectors, we consider two cases:

When $n < s$, $\widehat{B}_n$ is equal to the induced operator by A on $\widehat{E}_n$, hence $\widehat{B}_n = \widehat{A}_n$ has no eigenvectors (by Lemma 2.10).

The case $n \geq s$ requires a keener study. The problem of determining whether $\widehat{B}_n$ has eigenvectors is equivalent to the one of solving a finite system of equations. Thus, the goal of everything that follows will be to prove that such system has no solution.

Putting $\widehat{u} = \sum_{k=0}^{n} \frac{1}{X_k}\, \widehat{e}_k$, $\widehat{B}_n$ is defined by

$$\widehat{B}_n(\widehat{e}_i) = \widehat{u} + \left(1 - \frac{1}{X_i}\right)\widehat{e}_i, \text{ for } 0 \le i \le n,\ i \ne q_1, \dots, q_p, s.$$

$$\widehat{B}_n(\widehat{e}_{q_j}) = \widehat{u} + \left(1 - \frac{1}{X_{q_j}}\right)\widehat{e}_{q_j} - \frac{1}{X_s}\, \widehat{e}_s, \text{ for } j = 1, \dots, p.$$

$$\widehat{B}_n(\widehat{e}_s) = \widehat{u} + \left(1 - \frac{1}{X_s}\right)\widehat{e}_s - \sum_{j=1}^{p} \frac{1}{X_{q_j}}\, \widehat{e}_{q_j}.$$

So $\widehat{0} \ne \widehat{x} = \sum_{i=0}^{n} \xi_i \widehat{e}_i$ is an eigenvector of $\widehat{B}_n$ that corresponds to the eigenvalue λ if and only if

$$\sum_{i=0}^{n} \frac{1}{X_i} \sum_{\substack{j=0 \\ j \ne i}}^{n} \xi_j \widehat{e}_i - \sum_{j=1}^{p} \frac{\xi_s}{X_{q_j}} \widehat{e}_{q_j} - \frac{1}{X_s} \sum_{j=1}^{p} \xi_{q_j} \widehat{e}_s = (\lambda - 1) \sum_{i=0} \xi_i \widehat{e}_i.$$

This is equivalent to the system of $(n+1)$ equations with variables λ, ξ_0, ξ_1, …, ξ_n

$$\begin{cases} [1 + (\lambda - 1)X_i]\xi_i = \sum_{k=0}^{n} \xi_k & \text{for } 0 \le i \le n,\ i \ne q_1, \dots, q_p, s \\ [1 + (\lambda - 1)X_{q_j}]\xi_{q_j} + \xi_s = \sum_{k=0}^{n} \xi_k & \text{for } j = 1, \dots, p \\ [1 + (\lambda - 1)X_s]\xi_s + \sum_{j=1}^{p} \xi_{q_j} = \sum_{k=0}^{n} \xi_k & \end{cases} \tag{3.1}$$

having a solution in $\widehat{K_n}$.

In order to simplify the writing as well as the next calculations, we put $\eta := \sum_{k=0}^{n} \xi_k$ and $\lambda_i := 1 + (\lambda - 1)X_i$ for $i = 0, 1, \dots, n$.

By all these considerations, the next result is the only necessary fact for Theorem 3.1.

Lemma 3.2. *The system in variables* $\lambda, \xi_0, \xi_1, \ldots, \xi_n$

$$\begin{cases} (a_i) \quad \lambda_i \xi_i = \eta & \text{for } 0 \leq i \leq n,\ i \neq q_1, \ldots, q_p, s \\ (b_j) \quad \lambda_{q_j} \xi_{q_j} + \xi_s = \eta & \text{for } j = 1, \ldots, p \\ (c) \quad \lambda_s \xi_s + \sum_{j=1}^{p} \xi_{q_j} = \eta & \end{cases} \tag{3.2}$$

has no solution in $\widehat{K}_n$.

Before proving this lemma, we will establish some facts.

Lemma 3.3. *If* $\lambda_i = 0$ *for some* i, *then* $\lambda_k = 1 - \dfrac{X_k}{X_i} \neq 0$ *for all* $k \neq i$.

Lemma 3.4. *If system* (3.2) *has a solution, then*

i) $\eta \neq 0$

ii) $\eta \neq \xi_s$

iii) $\eta \neq \sum_{j=1}^{p} \xi_{q_j}$

Proof. By direct but long calculations ([1]), we prove that the system (3.2) has no solution when $\eta = 0$, $\eta = \xi_s$ or $\eta = \sum_{j=1}^{p} \xi_{q_j}$. □

Proof of Lemma 3.2. Suppose the system has a solution.

By Lemma 3.4, $\lambda_k \neq 0$ and $\xi_k \neq 0$ for all $k \in \{0, 1, \ldots, n\}$. Therefore, combining the equations (b_1), ..., (b_p) and (c) of system (3.2), it can be expressed as follows.

$$\begin{cases} \xi_i = \dfrac{\eta}{\lambda_i} & \text{for } 0 \leq i \leq n,\ i \neq q_1, \ldots, q_p, s \\ \xi_{q_j} = \dfrac{\eta}{\lambda_{q_j}} \left[1 - \dfrac{1}{\lambda_s} - \dfrac{(1-\lambda_s)\theta}{\lambda_s(\lambda_s - \theta)} \right] & \text{for } j = 1, \ldots, p \\ \xi_s = \dfrac{\eta}{\lambda_s} \left[1 + \dfrac{(1-\lambda_s)\theta}{\lambda_s - \theta} \right] & \end{cases}$$

where $\theta = \sum_{j=1}^{p} \dfrac{1}{\lambda_{q_j}}$.

Adding up all these equations and dividing the sum by η we get the next equation which has only the variable λ:

$$1 = \sum_{i=0}^{n} \frac{1}{\lambda_i} + \frac{\theta(1 - 2\lambda_s + \lambda_s\theta)}{\lambda_s(\lambda_s - \theta)}. \tag{3.3}$$

This equation must have a solution $\widetilde{\lambda} \in \widehat{K}_n$, since the system (3.2) can be solved by our initial assumption.

Putting $\widetilde{\lambda} - 1 = \dfrac{\varphi(X_n)}{\tau(X_n)}$, where $\varphi(X_n), \tau(X_n)$ have no common factors in $\widehat{K}_{n-1}[X_n]$ and substituting in (3.3), we have

$$1 = \sum_{i=0}^{n} \frac{\tau(X_n)}{\tau(X_n) + \varphi(X_n)X_i} + \frac{\theta(X_n)\ \tau(X_n)}{\tau(X_n) + \varphi(X_n)X_s} \left[\frac{\tau(X_n) + (\tau(X_n) + \varphi(X_n)X_s)(\theta(X_n) - 2)}{\tau(X_n) + \varphi(X_n)X_s - \tau(X_n)\theta(X_n)} \right] \tag{3.4}$$

with $\theta(X_n) = \displaystyle\sum_{j=1}^{p} \frac{\tau(X_n)}{\tau(X_n) + \varphi(X_n)\ X_{q_j}}$.

Let us consider the equality (3.4) in $\overline{\widehat{K}}_{n-1}(X_n)$ (where $\overline{\widehat{K}}_{n-1}$ is an algebraic closure of $\widehat{K}_{n-1}$). If $\deg \varphi(X_n) > 0$, there exists a $\xi \in \overline{\widehat{K}}_{n-1}$ such that $\varphi(\xi) = 0$ and $\tau(\xi) \neq 0$. Hence $\theta(\xi) = p$ and replacing in (3.4) we have

$$1 = \sum_{i=0}^{n} 1 + \frac{p\ \tau(\xi)}{\tau(\xi) + \varphi(\xi)X_s} \left[\frac{\tau(\xi) + \tau(\xi)(p - 2)}{\tau(\xi) - \tau(\xi)p} \right] = n + 1 - p.$$

But $n \geq s > p$, therefore $\varphi(X_n) = \varphi \in \widehat{K}_{n-1}$, $\varphi \neq 0$ and $\widetilde{\lambda} - 1 = \dfrac{\varphi}{\tau(X_n)}$.

If $\deg \tau(X_n) > 0$, we consider separately the cases $n > s$ and $n = s$. In each one, we will consider two subcases: $\tau(X_n)$ has a non-zero root in $\overline{\widehat{K}}_{n-1}$ or $\tau(X_n) = X_n^\alpha$ for some $\alpha \in \mathbb{N}$.

If $n > s$ and

i) there exists a $\zeta \in \overline{\widehat{K}}_{n-1}$, $\zeta \neq 0$ such that $\tau(\zeta) = 0$, then $\theta(\zeta) = 0$ and evaluating (3.4) in $X_n = \zeta$ we arrive to a contradiction since

$$1 = \sum_{i=0}^{n-1} \frac{\tau(\zeta)}{\tau(\zeta) + \varphi X_i} + \frac{\tau(\zeta)}{\tau(\zeta) + \varphi\zeta} + \frac{0}{\varphi^2 X_s^2} = 0.$$

ii) if $\tau(X_n) = X_n^{\alpha}$ $(\alpha \in \mathbb{N})$, then

$$\widetilde{\lambda} - 1 = \frac{\varphi}{X_n^{\alpha}} \text{ and } \theta(X_n) = \sum_{j=1}^{p} \frac{X_n^{\alpha}}{X_n^{\alpha} + \varphi X_{q_j}} .$$

Substituting in (3.4), we have

$$1 = \sum_{i=0}^{n-1} \frac{X_n^{\alpha}}{X_n^{\alpha} + \varphi X_i} + \frac{X_n^{\alpha-1}}{X_n^{\alpha-1} + \varphi} + \frac{\theta(X_n) X_n^{\alpha}}{X_n^{\alpha} + \varphi X_s} \left[\frac{X_n^{\alpha} + (X_n^{\alpha} + \varphi X_s)(\theta(X_n) - 2)}{X_n^{\alpha} + \varphi X_s - X_n^{\alpha}\ \theta(X_n)} \right].$$

Evaluating the last equality in $X_n = 0$ we conclude $\alpha = 1$ and then $1 = \dfrac{1}{1+\varphi}$. But $\varphi \neq 0$, leading again to a contradiction.

Thus, in the case $n > s$, $\tau(X_n) = \tau \in \widehat{K}_{n-1}$. Then (3.4) implies $\theta(X_n) = \theta \in \widehat{K}_{n-1}$ and $X_n \in \widehat{K}_{n-1}$ which is impossible.

Now if $n = s$ and

i) there exists a $\zeta \in \overline{\widehat{K}}_{s-1}$, $\zeta \neq 0$ such that $\tau(\zeta) = 0$, then $\theta(\zeta) = 0$ and, as in the previous case, evaluating (3.4) in $X_n = \zeta$, we get $1 = 0$.

ii) $\widetilde{\lambda} - 1 = \dfrac{\varphi}{X_s^{\alpha}}$ $(\alpha \in \mathbb{N})$, then $\theta(X_s) = \displaystyle\sum_{j=1}^{p} \frac{X_s^{\alpha}}{X_s^{\alpha} + \varphi X_{q_j}}$ and (3.4) is equivalent to

$$\begin{aligned} 1 = {} & \sum_{i=0}^{s-1} \frac{X_s^{\alpha}}{X_s^{\alpha} + \varphi X_i} + \frac{X_s^{\alpha-1}}{X_s^{\alpha-1} + \varphi} \\ & + \frac{\theta(X_s) X_s^{\alpha-1}}{X_s^{\alpha-1} + \varphi} \left[\frac{X_s^{\alpha-1} + (X_s^{\alpha-1} + \varphi)(\theta(X_s) - 2)}{X_s^{\alpha-1} + \varphi - X_s^{\alpha-1}\ \theta(X_s)} \right]. \end{aligned}$$

Again, evaluating $X_s = 0$ we conclude $\alpha = 1$ and $1 = \dfrac{1}{1+\varphi}$, another contradiction.

Then, also in this case, $\tau(X_n) = \tau \in \widehat{K}_{n-1}$ and $\theta(X_n) = \theta \in \widehat{K}_{n-1}$. A less direct algebraic work ([1]) shows that in this case we also have that (3.4) implies $X_n \in \widehat{K}_{n-1}$. Therefore, equation (3.3) has no solution in $\widehat{K}_{n-1}$ and neither does system (3.2). $\square$

We have established Theorem 3.1: the infinite family defined at the beginning of this section (of operators $B_{Q,s}$) only contains bounded self-adjoint indecomposable operators.

3.2. The operators $\mathbf{B_{pqr}}$.

Let $p, q, r \in \mathbb{N}_0$ such that $p < q < r$ and $r \geq 3$. Putting once again

$$u = \sum_{k=0}^{\infty} \frac{1}{X_k}\ e_k$$

we define $B^0_{pqr} : \{e_i : i \in \mathbb{N}_0\} \longrightarrow E$ by:

$$\begin{aligned}
B^0_{pqr}(e_i) &= A(e_i) = u + \left(1 - \frac{1}{X_i}\right) e_i, \qquad\qquad \text{for } i \neq p, q, r, r+1.\\
B^0_{pqr}(e_p) &= A(e_p) - \frac{1}{X_r}\ e_r\\
&= u + \left(1 - \frac{1}{X_p}\right) e_p - \frac{1}{X_r}\ e_r.\\
B^0_{pqr}(e_q) &= A(e_q) - \frac{1}{X_r} e_r - \frac{1}{X_{r+1}} e_{r+1}\\
&= u + \left(1 - \frac{1}{X_q}\right) e_q - \frac{e_r}{X_r} - \frac{e_{r+1}}{X_{r+1}}.\\
B^0_{pqr}(e_r) &= A(e_r) - \frac{1}{X_p}\ e_p - \frac{1}{X_q}\ e_q\\
&= u + \left(1 - \frac{1}{X_r}\right) e_r - \frac{1}{X_p}\ e_p - \frac{1}{X_q}\ e_q.\\
B^0_{pqr}(e_{r+1}) &= A(e_{r+1}) - \frac{1}{X_q}\ e_q = u + \left(1 - \frac{1}{X_{r+1}}\right) e_{r+1} - \frac{1}{X_q}\ e_q.
\end{aligned}$$

$\|B^0_{pqr}(e_i)\| - \|e_i\| = 0$ for all $i \in \mathbb{N}_0$, hence B^0_{pqr} can be linearly extended to $B_{pqr} : E \longrightarrow E \in \mathcal{B}(E)$ such that for all $x \in E$ we have $\|B_{pqr}(x)\| - \|x\| \geq 0$ (Lemma 2.7). Therefore, B_{pqr} induces operators in each residual space.

The matrix of B_{pqr} in the standard basis

$$
\begin{array}{c}
\left(\begin{array}{ccccccccc}
1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots \\
\vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots & \\
\frac{1}{X_p} & \cdots & 1 & \cdots & \frac{1}{X_p} & \cdots & 0 & \frac{1}{X_p} & \cdots \\
\vdots & & \vdots & \ddots & \vdots & & \vdots & \vdots & \\
\frac{1}{X_q} & \cdots & \frac{1}{X_q} & \cdots & 1 & \cdots & 0 & 0 & \cdots \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & \vdots & \\
\frac{1}{X_r} & \cdots & 0 & \cdots & 0 & \cdots & 1 & \frac{1}{X_r} & \cdots \\
\frac{1}{X_{r+1}} & \cdots & \frac{1}{X_{r+1}} & \cdots & 0 & \cdots & \frac{1}{X_{r+1}} & 1 & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \ddots
\end{array}\right)
\begin{array}{l}
\\ \\ \leftarrow p \\ \\ \leftarrow q \\ \\ \leftarrow r \\ \leftarrow r+1 \\ \\
\end{array}
\\
\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & \uparrow & \\
 & & p & & q & & r & r+1 &
\end{array}
\end{array}
$$

satisfies (2.1), then B_{pqr} is a self-adjoint operator.

Likewise for the family of operators $B_{Q,s}$, but through a much more difficult algebraic work ([1]), we can prove that none of the induced operators by B_{pqr} on the residual spaces has an eigenvector. Hence, proceeding analogously to Section 3.1, we get the following result.

Theorem 3.5. *B_{pqr} is a indecomposable operator.*

3.3. The spectrums of $\mathbf{B_{Q,s}}$ and $\mathbf{B_{pqr}}$.

In the previous two sections, we have proved that operators $B_{Q,s}$ as well as operators B_{pqr} are indecomposable. Hence they do not have eigenvectors and the bounded operators $B_{Q,s} - \lambda I$ and $B_{pqr} - \lambda I$ are injective for all $\lambda \in K$. Recall that, by Lemma 2.8, an injective operator $C \in \mathcal{B}(E)$ is invertible if and only if the set $\{\|C(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\}$ is bounded from above in Γ.

Given $\lambda \in K$, the sets

$$R_{Q,s} = \{\|(B_{Q,s} - \lambda\, I)(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\}$$

and

$$R_{pqr} = \{\|(B_{pqr} - \lambda\, I)(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\}$$

differ in a finite number of elements from the set

$$R_A = \{\|(A - \lambda\, I)(e_i)\| - \|e_i\| : i \in \mathbb{N}_0\}.$$

Hence, $R_{Q,s}$ and R_{pqr} are bounded from above in Γ if and only if R_A is also bounded.

By Theorem 2.13 ([3]), $\operatorname{spec}(A) = \{1\}$. Then R_A is bounded from above only when $\lambda = 1$. This proves the following statement

Theorem 3.6. *For every B that belongs to one of the families of bounded operators defined in sections 3.1 and 3.2 we have*

$$\operatorname{spec}(B) = \{1\}.$$

4. The subalgebras of $\mathcal{B}(\mathbf{E})$: $\mathcal{B}_{\mathbf{Q,s}}$ and $\mathcal{B}_{\mathbf{pqr}}$

For each of the infinite operators defined in sections 3.1 and 3.2 we will consider its commutant algebra in $\mathcal{B}(E)$. We denote the commutant algebra of the operator $B_{Q,s}$ by

$$\mathcal{B}_{Q,s} = \{C \in \mathcal{B}(E) : CB_{Q,s} = B_{Q,s}C\}, \tag{4.1}$$

and the commutant algebra of B_{pqr} by

$$\mathcal{B}_{pqr} = \{C \in \mathcal{B}(E) : CB_{pqr} = B_{pqr}C\}. \tag{4.2}$$

By Lemma 2.14, proved in [3], we have

Lemma 4.1. *If the operators C, D in $\mathcal{B}_{Q,s}$ (resp. $\mathcal{B}_{pqr}$) coincide on a non-zero vector, then $C = D$.*

This lemma has two immediate consequences. First, a non-injective operator of $\mathcal{B}_{Q,s}$ (resp. $\mathcal{B}_{pqr}$) would coincide with the zero operator in a non-zero vector. Then:

Corollary 4.2. *All non-trivial operators of $\mathcal{B}_{Q,s}$ (resp. $\mathcal{B}_{pqr}$) are injective.*

Since two operators B and C belonging to one of the families defined in sections 3.1 and 3.2 differ only in a finite number of vectors of the standard basis and coincide in the rest of this basis, by Lemma 4.1, we have that B can not belong to the commutant algebra of C. Hence:

Corollary 4.3. *All the subalgebras presented in (4.1) and (4.2) are mutually distinct.*

By using the same argument, each of the subalgebras presented in (4.1) and (4.2) is different from $\mathcal{A}$. But a stronger result can be established. Using the formulas of Theorem 2.16([4]) we will prove that the intersection of each one of these algebras and $\mathcal{A}$ is minimal.

Theorem 4.4. *Let $p, s \in \mathbb{N}$ such that $1 < p < s$, $Q = \{q_1, \ldots, q_p\}$, $q_j \in \{0, 1, \ldots, s-1\}$ and $q_1 < q_2 < \cdots < q_p$. Then*

$$\mathcal{A} \cap \mathcal{B}_{Q,s} = \{\alpha\, I : \alpha \in K\}.$$

Proof. Let $K_{Q,s} \in \mathcal{B}(E)$ be the operator defined by:

$$\begin{aligned} K_{Q,s}(e_i) &= 0 && \text{for } i \neq q_1, \ldots, q_p, s. \\ K_{Q,s}(e_{q_j}) &= -\frac{1}{X_s}\; e_s && \text{for } j = 1, \ldots, p. \\ K_{Q,s}(e_s) &= -\sum_{j=1}^{p} \frac{1}{X_{q_j}}\; e_{q_j}. \end{aligned}$$

Then $B_{Q,s} = A + K_{Q,s}$. Thus, an operator $C \in \mathcal{B}_{Q,s} \cap \mathcal{A}$ if and only if it conmutes with A and $K_{Q,s}$.

$C \in \mathcal{A}$ is self-adjoint (Section 2.3) and putting $C(e_k) = \sum_{i=0}^{\infty} c_{ik} e_i$, by Lemma 2.5, we have

$$c_{ik} X_i = c_{ki} X_k \qquad (i, k \in \mathbb{N}_0) \tag{4.3}$$

Now,

$$K_{Q,s}(C(e_k)) = -\frac{1}{X_s} \sum_{j=1}^{p} c_{q_j k} e_s - c_{sk} \sum_{j=1}^{p} \frac{1}{X_{q_j}} e_{q_j} \qquad \text{for all } k \in \mathbb{N}_0.$$

and

$$\begin{aligned} C(K_{Q,s}(e_i)) &= 0 \quad \text{for } i \neq q_1, \ldots, q_p, s. \\ C(K_{Q,s}(e_{q_j})) &= -\frac{1}{X_s} \sum_{i=0}^{\infty} c_{is} e_i \quad \text{for } j = 1, \ldots, p. \\ C(K_{Q,s}(e_s)) &= -\sum_{i=0}^{\infty} \sum_{j=1}^{p} \frac{1}{X_{q_j}} c_{i q_j} e_i. \end{aligned}$$

Therefore, $C \in \mathcal{B}_{Q,s} \cap \mathcal{A}$ iff the following four conditions are fulfilled:

$$c_{sj} = c_{js} = 0 \quad \text{for } j \neq q_1, \ldots, q_p, s \tag{4.4}$$

$$c_{sq_1} = c_{sq_2} = \cdots = c_{sq_p} \ (=: a) \ \text{ and } c_{q_j s} = \frac{X_s}{X_{q_j}} a \ \text{ for } j = 1, \ldots, p \tag{4.5}$$

$$\sum_{k=1}^{p} c_{q_k j} = 0 \quad \text{for } j \neq q_1, \ldots, q_p, s \tag{4.6}$$

$$\sum_{k=1}^{p} c_{q_k q_j} = c_{ss} \quad \text{for } j = 1, \ldots, p \tag{4.7}$$

By Theorem 2.16, if $C(e_0) = \sum_{k=0}^{\infty} \frac{\lambda_k}{X_k} e_k$ then

$$c_{kn} = \frac{X_n}{X_n - X_k}\left((X_n - 1)\frac{\lambda_n}{X_n} - (X_k - 1)\frac{\lambda_k}{X_k}\right), \qquad \text{si } k \neq n \tag{4.8}$$

$$c_{nn} = \lambda_0 + \sum_{j \neq 0, n} \frac{X_n - 1}{X_n - X_j}\,(\lambda_j - \lambda_n). \tag{4.9}$$

for $n \geq 1$. By (4.4) and (4.5)

$$C(e_s) = aX_s \sum_{j=1}^{p} \frac{1}{X_{q_j}} e_{q_j} + c_{ss} e_s. \tag{4.10}$$

With this equality and using Theorem 2.16 we will be able to determine C. If $q_1 \neq 0$, by (4.4), $0 = c_{0s} = c_{s0} = \frac{\lambda_s}{X_s}$. Then $\lambda_s = 0$. Additionally, by (4.8), (4.9) and (4.10), we have

- $\lambda_k = 0$ for $k \neq 0, q_1, \ldots, q_p$.
- $\lambda_{q_j} = \frac{X_s - X_{q_j}}{1 - X_{q_j}} a$ for $j = 1, \ldots, p$.
- $\lambda_0 = c_{ss} + a(X_s - 1) \sum_{j=1}^{p} \frac{1}{X_{q_j} - 1}$.

By (4.8), if $n = s + 1$ and $k \neq s + 1$

$$c_{k(s+1)} = \frac{X_{s+1}(1 - X_k)\lambda_k}{X_k(X_{s+1} - X_s)}.$$

And by (4.6),

$$0=\sum_{j=1}^{p} c_{q_j(s+1)} = X_{s+1}\ a\sum_{j=1}^{p}\frac{X_s-X_{q_j}}{X_{q_j}(X_{s+1}-X_{q_j})}.$$

This implies $a=0$. Hence, $\lambda_0=c_{ss}$ and $\lambda_k=0$ for $k\geq 1$. Thus, $C(e_0)=c_{ss}\ e_0=c_{ss}I(e_0)$ and, by Lemma 2.14, we have $C=c_{ss}I$.

Now, if $q_1=0$, by (4.5), $a=c_{s0}=c_{sq_1}=\dfrac{\lambda_s}{X_s}$. Hence, $\lambda_s=aX_s$.

By (4.8), (4.9) y (4.10), we get

- $\lambda_k=\dfrac{X_s-1}{X_k-1}\,aX_s$ for $k\neq q_1,\ldots,q_p,s$.
- $\lambda_{q_j}=aX_s$ for $j=2,3,\ldots,p$.
- $\lambda_{q_1}=c_{ss}+a(X_s-1)\displaystyle\sum_{k\neq q_1,\ldots,q_p,s}\frac{1}{1-X_k}$.

And by (4.8)

$$c_{k(s+1)}=\frac{X_{s+1}}{X_{s+1}-X_k}\left((X_s-1)a-(X_k-1)\frac{\lambda_k}{X_k}\right)\qquad (k\neq s+1)$$

In particular, $c_{q_l(s+1)}=\dfrac{X_{s+1}(X_s-X_{q_l})a}{X_{q_l}(X_{s+1}-X_{q_l})}$ for $l=1,2,\ldots,p$. Finally, by (4.6)

$$0=\sum_{l=1}^{p} c_{q_l(s+1)} = aX_{s+1}\underbrace{\sum_{l=1}^{p}\frac{X_s-X_{q_l}}{X_{q_l}(X_{s+1}-X_{q_l})}}_{\neq 0}.$$

Therefore, $a=0$, $C(e_0)=c_{ss}e_0$. And in this case, we have $C=c_{ss}I$ too. □

By analogous procedures (see [1]) it can be proved that

Theorem 4.5. *Let $p,q,r\in\mathbb{N}_0$ such that $p<q<r$ and $r\geq 3$. Then*

$$\mathcal{A}\cap\mathcal{B}_{pqr}=\{\alpha\ I:\alpha\in K\}.$$

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