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# P-adic Spaces of Continuous Functions II

Athanasios Katsaras

**Abstract**

Necessary and sufficient conditions are given so that the space $C(X,E)$ of all continuous functions from a zero-dimensional topological space $X$ to a non-Archimedean locally convex space $E$, equipped with the topology of uniform convergence on the compact subsets of $X$, to be polarly absolutely quasi-barrelled, polarly $\aleph_o$-barrelled, polarly $\ell^\infty$-barrelled or polarly $c_o$-barrelled. Also, tensor products of spaces of continuous functions as well as tensor products of certain $E'$-valued measures are investigated.

## Introduction

This paper is a continuation of [3]. Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X,E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space $X$ to a non-Archimedean Hausdorff locally convex space $E$. We will denote by $C_b(X,E)$ (resp. by $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of $E$. The dual space of $C_{rc}(X,E)$, under the topology $t_u$ of uniform convergence, is a space $M(X,E')$ of finitely-additive $E'$-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of $X$. Some subspaces of $M(X,E')$ turn out to be the duals of $C(X,E)$ or of $C_b(X,E)$ under certain locally convex topologies. In section 1, we give necessary and sufficient conditions for the space $C(X,E)$, equipped with the topology of uniform convergence on the compact subsets of $X$, to be polarly absolutely quasi-barrelled, polarly $\aleph_o$-barrelled, polarly $\ell^\infty$-barrelled or polarly $c_o$-barrelled. In section 2 , we study tensor products of spaces of continuous functions as well as tensor products of certain $E'$-valued measures. We refer to paper [3] for the notations used in the paper as well as some preliminaries needed for the paper.

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*Math. classification:* 46S10, 46G10.

## 1. Barrelledness in Spaces of Continuous Functions

We will denote by $C_c(X,E)$ the space $C(X,E)$ equipped with the topology of uniform convergence on compact subsets of $X$. By $M_c(X,E')$ we will denote the space of all $m \in M(X,E')$ with compact support. The dual space of $C_c(X,E)$ coincides with $M_c(X,E')$.
Recall that a zero-dimensional Hausdorff topological space $X$ is called a $\mu_o$-space (see [1]) if every bounding subset of $X$ is relatively compact. We denote by $\mu_o X$ the smallest of all $\mu_o$-subspaces of $\beta_o X$ which contain $X$. Then $X \subset \mu_o X \subset \theta_o X$ and, for each bounding subset $A$ of $X$, the set $\overline{A}^{\beta_o X}$ is contained in $\mu_o X$ (see [1]). Moreover, if $Y$ is another Hausdorff zero-dimensional space and $f : X \to Y$, then $f^{\beta_o}(\mu_o X) \subset \mu_o Y$ and so there exists a continuous extension $f^{\mu_o} : \mu_o X \to \mu_o Y$ of $f$.

Let us say that a family $\mathcal{F}$ of subsets of a a set $Z$ is finite on a subset $F$ of $Z$ if the family of all members of $\mathcal{F}$ which meet $F$ is finite.

**Definition 1.1.** A subset $D$, of a topological space $Z$, is said to be $w$-bounded if every family $\mathcal{F}$ of open subsets of $Z$, which is finite on each compact subset of $Z$, is also finite on $D$. If this happens for families of clopen sets, then $D$ is said to be $w_o$-bounded. We say that $Z$ is a $w$-space (resp. a $w_o$-space ) if every $w$-bounded (resp. $w_o$-bounded) subset is relatively compact.

**Definition 1.2.** A subset $W$, of a locally convex space $E$, is said to be absolutely bornivorous if it absorbs every subset $S$ of $E$ for which $\sup_{x\in S}|u(x)| < \infty$ for all $u \in W^o$. The space $E$ is said to be polarly absolutely quasi-barrelled if every polar absolutely bornivorous subset of $E$ is a neighborhood of zero.

**Lemma 1.3.** *Every absolutely bornivorous subset $W$, of a locally convex space $E$, absorbs bounded subsets of $E$.*

*Proof:* Let $B$ be a bounded subset of $E$ and suppose that $W$ does not absorb $B$. Let $|\lambda| > 1$. Since $B$ is not absorbed by $W$, there exists $u \in W^o$ such that $\sup_{x\in B}|u(x)| = \infty$. Choose a sequence $(x_n)$ in $B$ such that $|u(x_n)| > |\lambda|^n$ for all $n$. Since $B$ is bounded, we have that $y_n = \lambda^{-n}x_n \to 0$, and so $u(y_n) \to 0$, a contradiction.

**Definition 1.4.** A subset $A$, of a topological space $Z$, is called $aw_o$-bounded if it is $w_o$-bounded in its subspace topology. The space $Z$ is said to be an $aw_o$-space if every $aw_o$-bounded set is relatively compact.

**Theorem 1.5.** *If $D$ is an absolutely bornivorous subset of $G = C_c(X, E)$ and if $H = D^o$ is the polar of $D$ in the dual space $M_c(X, E')$ of $G$, then the set*

$$Y = S(H) = \overline{\bigcup_{m \in H} supp(m)}$$

*is $aw_o$-bounded.*

*Proof:* Assume the contrary. Then, there exists a sequence $(O_n)$ of open subsets of $X$ such that $Z_n = O_n \cap Y \neq \emptyset$, $Z_n \neq Z_k$, for $n \neq k$, and $(Z_n)$ is finite on each compact subset of $Y$. For each $n$, there exists an $m_n \in H$ with $O_n \cap supp(m_n) \neq \emptyset$. Let $W_n$ be a clopen subset of $O_n$ such that $m_n(W_n) \neq 0$. Choose $s_n \in E$ such that $m_n(W_n)s_n = 1$, and let $|\lambda| > 1$, $h_n = \lambda^n \chi_{W_n} s_n$. Consider the set $F = \{h_n : n \in \mathbf{N}\}$. For each $m \in H$, the sequence $(W_n)$ is finite on the $supp(m)$ and thus $m(W_n) = 0$ finally, which implies that $\sup_n | < m, h_n > | < \infty$ for all $m \in H$. Therefore, there exists $\alpha \neq 0$ such that $F \subset \alpha D$. But then

$$1 \geq | < \alpha^{-1} h_n, m_n > | = |\alpha^{-1} \lambda^n|,$$

for all $n$, which is impossible. This contradiction completes the proof.

**Theorem 1.6.** *Assume that $E' \neq \{0\}$. If the space $G = C_c(X, E)$ is polarly absolutely quasi-barrelled, then $E$ is polarly absolutely quasi-barrelled and $X$ an $aw_o$-space.*

*Proof:* Let $W$ be a polar absolutely bornivorous subset of $E$ and let $W^o$ be its polar in $E'$. Let $x \in X$ and, for $u \in E'$, let $u_x \in G'$, $u_x(f) = u(f(x))$. Consider the set $H = \{u_x : u \in W^o\}$, and let $D = H^o$ be its polar in $G$. Then $D$ is absolutely bornivorous. Indeed, let $M \subset G$ be such that $\sup_{f \in M} |u_x(f)| < \infty$ for all $u \in W^o$. Thus, for $u \in W^o$, we have that $\sup_{f \in M} |u(f(x))| < \infty$. Let $S = \{f(x) : f \in M\}$. Since, for $u \in W^o$, we have that $\sup_{s \in S} |u(s)| < \infty$ and since $W$ is absolutely bornivorous, there exists $\alpha \in \mathbb{K}$ such that $S \subset \alpha W$. But then $M \subset \alpha D$. So, $D$ is an absolutely bornivorous polar subset of $G$. By our hypothesis, $D$ is a neighborhood of zero in $G$. Hence, there exist a compact subset $Y$ of $X$ and $p \in cs(E)$ such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset D,$$

which implies that

$$\{s \in E : p(s) \leq 1\} \subset W^{oo} = W.$$

This proves that $E$ is polarly absolutely quasi-barrelled. To prove that $X$ is an $aw_o$-space, consider an $aw_o$-bounded subset $A$ of $X$, $x'$ a non-zero element of $E'$ and define $p(s) = |x'(s)|$. The set

$$V = \{f \in C(X,E) : \|f\|_{A,p} \leq 1\}$$

is a polar subset of $G$. Also $V$ is absolutely bornivorous. In fact, let $Z \subset G$ be such that $\sup_{f \in Z} |u(f)| < \infty$ for each $u \in V^o \subset G'$. We claim that $V$ absorbs $Z$. Assume the contrary and let $|\lambda| > 1$. There exists a sequence $(f_n)$ in $Z$, $f_n \notin \lambda^n V$. Let

$$V_n = \{x : p(f_n(x)) > |\lambda|^n\}.$$

Then $V_n \cap A \neq \emptyset$. Since $A$ is $aw_o$-bounded, there exists a compact subset $Y$ of $A$ such that $(V_n)$ is not finite on $Y$. Let $g_n = f_n|_Y$ and consider the space $F = C(Y,E)$ with the topology of uniform convergence. Let $q \in cs(F)$, $\quad q(g) = \|g\|_p$. Then $q$ is a polar seminorm on $F$ and so the normed space $F_q$ is polar. Since $(V_n)$ is not finite on $Y$, it follows that $\sup_n q(g_n) = \infty$. Let $\pi : F \to F_q$ be the canonical map and $\tilde{g}_n = \pi(g_n)$. Then $\sup_n \|\tilde{g}_n\| = \infty$. Since $F_q$ is polar, there exists $\phi \in F_q'$ such that $\sup_n |\phi(\tilde{g}_n| = \infty$. Let $u = \phi \circ \pi$. For $g \in F$, we have

$$|u(g)| = |\phi(\tilde{g})| \leq \|\phi\| \cdot \|g\|_p.$$

Let

$$\omega : C_c(X,E) \to \mathbb{K}, \quad \omega(f) = u(f|_Y).$$

Then $|\omega(f)| \leq \|\phi\| \cdot \|f\|_{Y,p}$ and so $\omega \in G'$. Let $|\gamma| > \|\phi\|$. If $v = \gamma^{-1}\omega$, then $v \in V^o$. But

$$\sup_{f \in Z} |v(f)| \geq |\gamma^{-1}| \cdot \sup_n |u(g_n| = |\gamma^{-1}| \cdot \sup_n |\phi(\tilde{g}_n)| = \infty,$$

a contradiction. This contradiction shows that $V$ absorbs $Z$ and therefore $V$ is an absolutely bornivorous barrel. Thus $V$ is a neighborhood of zero in $G$. Let $K$ be a compact subset of $X$ and $r \in cs(E)$ be such that

$$\{f \in G : \|f\|_{K,r} \leq 1\} \subset V.$$

Then $A \subset K$ and so $A$ is relatively compact. This clearly completes the proof.

**Theorem 1.7.** *Assume that $E' \neq \{0\}$. If $E$ is polarly quasi-barrelled, then $G = C_c(X,E)$ is polarly absolutely quasi-barrelled iff $X$ is an $aw_o$-space.*

*Proof:* The necessity follows from the preceding Theorem.
**Sufficiency** : Let $D$ be a polar absolutely bornivorous subset of $G$ and let $H = D^o$ be its polar in $G'$. By Theorem 9.17, the set

$$Y = S(H) = \overline{\bigcup_{m \in H} supp(m)}$$

ia $aw_o$-bounded and hence compact. Let

$$\Phi = \bigcup_{m \in H} m(K(X)).$$

Then $\Phi$ is a strongly bounded subset of $E'$. In fact, let $B$ be a bounded subset of $E$. The set

$$F = \{\chi_A s : A \in K(X), s \in B\}$$

is bounded in $G$. Since $D$ is bornivorous, there exists a non-zero $\alpha \in \mathbb{K}$ such that $F \subset \alpha D$. Thus, for $m \in H$, $s \in B$, $A \in K(X)$, we have that $\alpha^{-1}\chi_A s \in D$ and so $|m(A)s| \leq |\alpha|$. Therefore

$$\sup_{\phi \in \Phi, s \in B} |\phi(s)| \leq |\alpha|,$$

which proves that $\Phi$ is strongly bounded in $E'$. But then $\Phi$ is equicontinuous. Hence, there exists $p \in cs(E)$ such that

$$\Phi \subset \{s \in E : p(s) \leq 1\}^o.$$

Now

$$W = \{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o = D.$$

Indeed, let $\|f\|_{Y,p} \leq 1$ and let $V = \{x : p(f(x)) \leq 1\}$. For each clopen subset $V_1$ of $V^c$, we have that $m(V_1) = 0$ for all $m \in H$. For $A$ a clopen subset of $V$ and $x \in A$, we have $p(f(x)) \leq 1$ and so $|m(A)f(x)| \leq 1$, which implies that

$$\left|\int f\, dm\right| = \left|\int_V f\, dm\right| \leq 1.$$

Thus $W \subset D$ and the result follows.

**Corollary 1.8.** *$C_c(X)$ is polarly absolutely quasi-barrelled iff $X$ is an $aw_o$-space.*

**Corollary 1.9.** *Assume that $E' \neq \{0\}$. If $E$ is a bornological space and $X$ an $aw_o$-space, then $C_c(X, E)$ is polarly absolutely quasi-barrelled. In particular this happens when $E$ is metrizable.*

**Definition 1.10.** A locally convex space $E$ is said to be :

(1) polarly $\aleph_o$-barrelled if every $w^\star$-bounded countable union of equicontinuous subsets of $E'$ is equicontinuous.

(2) polarly $\ell^\infty$-barrelled if every $w^\star$-bounded sequence in $E'$ is equicontinuous.

(3) polarly co-barrelled if every $w^\star$-null sequence in $E'$ is equicontinuous.

**Theorem 1.11.** *Assume that $E' \neq \{0\}$ and let $G = C_c(X, E)$. Consider the following conditions.*

(1) *$G$ is polarly $\aleph_o$-barrelled.*

(2) *$G$ is polarly $\ell^\infty$-barrelled .*

(3) *$G$ is polarly co-barrelled.*

(4) *If a $\sigma$-compact subset $A$ of $X$ is bounding, then $A$ is relatively compact.*

*Then: (a.* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.
*(b). If $E$ is a Fréchet space, then the four properties (1), (2), (3), (4) are equivalent.*

*Proof:* Clearly $(1) \Rightarrow (2) \Rightarrow (3)$.
$(3) \Rightarrow (4)$. Let $(Y_n)$ be a sequence of compact subsets of $X$, such that $A = \bigcup Y_n$ is bounding, and choose a non-zero element $u$ of $E'$. Let $p$ be defined on $E$ by $p(s) = |u(s)|$. Then $\|u\|_p = 1$. By [5, p. 273] there exists $\mu_n \in M_\tau(X)$ with $N_{\mu_n}(x) = 1$ if $x \in Y_n$ and $N_{\mu_n}(x) = 0$ if $x \notin Y_n$. Let

$$m_n \in M(X, E'), \quad m_n(A) = \mu_n(A)u$$

for all $A \in K(X)$. Let $0 < |\lambda| < 1$. For each $f \in C(X, E)$, we have

$$\left| \int f \ dm_n \right| \leq \|f\|_{Y_n,p} \cdot \|m_n\|_p \leq \|f\|_{A,p}.$$

It follows that the sequence $H = (\lambda^n m_n)$ is $w^\star$-null and hence by (3) equicontinuous. Let $Y$ be a compact subset of $X$ and $q \in cs(E)$ be such that

$$\{f \in G : \|f\|_{Y,q} \leq 1\} \subset H^o.$$

But then $A \subset Y$ and so $A$ is relatively compact. Finally, suppose that $E$ is a Fréchet space and let (4) hold. Let $(H_n)$ be a sequence of equicontinuous subsets of the dual space $M_c(X, E')$ of $G$ such that $H = \bigcup H_n$ is $w^\star$-bounded. For each $n$, the set

$$Y_n = S(H_n) = \overline{\bigcup_{m \in H_n} supp(m)}$$

is compact. Also, the set

$$A = S(H) = \overline{\bigcup Y_n}$$

is bounding by [2, Prop. 6.6]. By our hypothesis, $A$ is compact. Since $E$ is a Fréchet space, the space $F = (C_{rc}(X, E), \tau_u)$ is a Fréchet space whose dual can be identified with $M(X, E')$. As $H$ is $\sigma(F', F)$-bounded, it follows that $H$ is $\tau_u$-equicontinuous. Thus, there exists $p \in cs(E)$ such that

$$\{f \in C_{rc}(X, E) : \|f\|_p \leq 1\} \subset H^o.$$

If $|\lambda| > 1$, then $\|m\|_p \leq |\lambda|$ for all $m \in H$. Now

$$\{f \in G : \|f\|_{A,p} \leq |\lambda^{-1}|\} \subset H^o.$$

This clearly completes the proof.

## 2. **Tensor Products**

Throughout this section, $X, Y$ will be zero-dimensional Hausdorff topological spaces and $E, F$ Hausdorff locally convex spaces. Let $B_{ou}(X)$ denote the collection of all $\phi \in \mathbb{K}^X$ for which $|\phi|$ is bounded, upper-semicontinuous and vanishes at infinity. For $\phi \in B_{ou}(X)$ and $p \in cs(E)$, let $p_\phi$ be the seminorm on $C_b(X, E)$ defined by

$$p_\phi(f) = \sup_{x \in X} p(\phi(x) f(x)).$$

As it is shown in [4], the topology $\beta_o$ is generated by the family of seminorms

$$\{p_\phi : \phi \in B_{ou}(X),\, p \in cs(E)\}.$$

For $\phi_1, \phi_2 \in B_{ou}(X)$, it is proved in [4] that there exists $\phi \in B_{ou}(X)$ such that $|\phi| = \max\{|\phi_1|, |\phi_2|\}$. If $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, then the function

$$\phi = \phi_1 \times \phi_2 : X \times Y \to \mathbb{K},\; \phi(x, y) = \phi_1(x)\phi_2(y),$$

is in $B_{ou}(X \times Y)$ and, for each locally convex space $G$, the topology $\beta_o$ on $C_b(X \times Y, G)$ is generated by the seminorms

$$p_{\phi_1 \times \phi_2}, \quad \phi_1 \in B_{ou}(X), \quad \phi_2 \in B_{ou}(Y), \quad p \in cs(G).$$

Let $E \otimes F$ be the tensor product of $E, F$ equipped with the projective topology. For $f \in C_b(X, E)$, $g \in C_b(Y, F)$, define

$$f \odot g : X \times Y \to E \otimes F, \quad f \odot g(x, y) = f(x) \otimes g(y).$$

The bilinear map

$$\psi : E \times F \to E \otimes F, \quad \psi(a, b) = a \otimes b,$$

is continuous. Also the map $(x, y) \mapsto (f(x), g(x))$, from $X \times Y$ to $E \times F$, is continuous. Hence the composition $f \odot g$ is continuous. Since

$$p \otimes q(f \odot g(x, y)) = p(f(x)) \cdot q(g(y)) \leq \|f\|_p \cdot \|g\|_q,$$

$f \odot g$ is also bounded.

**Theorem 2.1.** *The space $G$ spanned by the functions*

$$(\chi_A s) \odot (\chi_B t),\ A \in K(X),\ B \in K(Y),\ s \in E,\ t \in F,$$

*is $\beta_o$-dense in $C_b(X \times Y, E \otimes F)$.*

*Proof:* Let $p \in cs(E)$, $q \in cs(F)$, $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, $\phi = \phi_1 \times \phi_2$. Consider the set

$$W = \{f \in C_b(X \times Y, E \otimes F) : (p \otimes q)_\phi(f) \leq 1\}$$

and let $f \in C_b(X \times Y, E \otimes F)$. We will finish the proof by showing that there exists $h \in G$ such that $f - h \in W$. To this end, we consider the set

$$D = \{(x, y) : |\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x, y)) \geq 1/2\}.$$

Then $D$ is a compact subset of $X \times Y$. Let $D_1$, $D_2$ be the projections of $D$ on $X, Y$, respectively. Then $D \subset D_1 \times D_2$. Choose $d > \|\phi_1\|, \|\phi_2\|$ and let $x \in D_1$. There exists a $y$ such that $(x, y) \in D$ and so $\phi_1(x) \neq 0$. The set

$$Z_x = \{z \in X : |\phi_1(z)| < 2|\phi_1(x)|\}$$

is open and contains $x$. Using the compactness of $D_2$, we can find a clopen neighborhood $W_x$ of $x$ contained in $Z_x$ such that $p \otimes q(f(z, y) - f(x, y)) <$

$1/d^2$ for all $z \in W_x$ and all $y \in D_2$. In view of the compactness of $D_1$, there are $x_1, x_2, \cdots, x_m \in D_1$ such that $D_1 \subset \bigcup_{k=1}^m W_{x_k}$. Let

$$A_1 = W_{x_1}, \quad A_{k+1} = W_{x_{k+1}} \setminus \bigcup_{j=1}^{k} W_{x_j}, \quad k = 1, 2, \ldots, m-1.$$

Keeping those of the $A_i$ which are not empty, we may assume that $A_k \neq \emptyset$ for all $1 \leq k \leq m$. For $k = 1, \ldots, m$, there are pairwise disjoint clopen subsets $B_{k,1}, \ldots, B_{k,n_k}$ of $Y$ covering $D_2$ and $y_{kj} \in B_{k,j}$ such that

$$p \otimes q(f(x_k, y) - f(x_k, y_{kj})) < 1/d^2$$

if $y \in B_{k,j}$. Let

$$h = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \chi_{A_k} \times \chi_{B_{k,j}} \cdot f(x_k, y_{kj}).$$

Then $h \in G$. We will prove that

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1$$

for all $x \in X, y \in Y$. To see this, we consider the three possible cases.
Case I. $x \notin \bigcup_{k=1}^m A_k$. Then $h(x,y) = 0$. Also $(x,y) \notin D$ and thus

$$|\phi_1(x)\phi_2(y) \cdot p \otimes q(f(x,y)) \leq 1/2.$$

Case II. $x \in A_k$, $y \in D_2$. There exists $j$ such that $y \in B_{k,j}$. Now

$$p \otimes q(f(x,y) - f(x_k, y)) < 1/d^2 \quad and \quad p \otimes q(f(x_k, y) - f(x_k, y_{kj})) \leq 1/d^2.$$

Since $h(x,y) = f(x_k, y_{kj})$, we have

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1.$$

Case III. $x \in A_k, y \notin D_2$. Then $(x,y) \notin D$ and so $|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y)) < 1/2$. If $h(x,y) \neq 0$, then $y \in B_{k,j}$, for some $j$, and so $h(x,y) = f(x_k, y_{kj})$ and $p \otimes q(f(x_k,y) - f(x_k, y_{kj})) < 1/d^2$. Since $x \in W_{x_k}$, we have $|\phi_1(x)| < 2|\phi_1(x_k)|$. Thus

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x_k,y)) \leq 2|\phi_1(x_k)\phi_2(y)| \cdot p \otimes q(f(x_k,y)) \leq 1$$

since $(x_k, y) \notin D$. It follows that

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1.$$

Thus $f - h \in W$, which completes the proof.

**Lemma 2.2.** *Let $p \in cs(E)$, $q \in cs(F)$ and $u \in E \otimes F$. Then :*

(1) *If* $u = \sum_{i=1}^n x_i \otimes y_i = \sum_{j=1}^m a_j \otimes b_j$, *then for all* $x' \in E'$, *we have*

$$\sum_{i=1}^n x'(x_i)y_i = \sum_{j=1}^m x'(a_j)b_j.$$

(2) *If* $p$ *is polar, then, for any* $u = \sum_{i=1}^n x_i \otimes y_i$, *we have*

$$p \otimes q(u) = \sup\{q(\sum_{i=1}^n x'(x_i)y_i) : x' \in E', |x'| \leq p\}.$$

*Proof:* (1). Let $h \in F^\star$ and consider the bilinear map

$$\omega : E \times F \to \mathbb{K}, \quad \omega(x,y) = x'(x)h(y).$$

Let $\hat{\omega} : E \otimes F \to \mathbb{K}$ be the corresponding linear map. Then

$$\sum_{i=1}^n x'(x_i)h(y_i) = \hat{\omega}\left(\sum_{i=1}^n x_i \otimes y_i\right) = \hat{\omega}\left(\sum_{j=1}^m a_j \otimes b_j\right) = \sum_{j=1}^m x'(a_j)h(b_j).$$

Since this holds for all $h \in F^\star$, (1) follows.
(2). Let $d = \sup_{|x'| \leq p} q(\sum_{i=1}^n x'(x_i)y_i)$. For any representation

$$u = \sum_{j=1}^m a_j \otimes b_j$$

of $u$ and any $x' \in E'$, with $|x'| \leq p$, we have

$$q\left(\sum_{j=1}^m x'(a_j)b_j\right) \leq \sup_j |x'(a_j)|q(b_j) \leq \sup_j p(a_j)q(b_j)$$

and so $d \leq sup_j p(a_j)q(b_j)$, which proves that $d \leq p \otimes q(u)$. On the other hand, let $u = \sum_{i=1}^n x_i \otimes y_i$ and let $G$ be the space spanned by the the set $\{y_1, \ldots, y_n\}$. Given $0 < t < 1$, there exists a basis $\{w_1, \ldots, w_m\}$ of $G$ which is $t$-orthogonal with respect to the seminorm $q$. We may write $u$ in the form $u = \sum_{k=1}^m z_k \otimes w_k$. For $x' \in E', |x'| \leq p$, we have

$$q\left(\sum_{k=1}^m x'(z_k)w_k\right) \geq t \cdot \max_{1 \leq k \leq m} |x'(z_k)|q(w_k),$$

and so

$$\sup_{|x'|\le p} q\left(\sum_{k=1}^{m} x'(z_k)w_k\right) \ge t\cdot \sup_{|x'|\le p}\max_k |x'(z_k)|q(w_k)$$
$$= t\cdot \max_k\left[\sup_{|x'|\le p}|x'(z_k)|\right] q(w_k)$$
$$= t\cdot \max_k p(z_k)q(w_k) \ge t\cdot p\otimes q(u).$$

Since $0 < t < 1$ was arbitrary, we get that $d \ge p\otimes q(u)$ and so $d = p\otimes q(u)$.

**Lemma 2.3.** *If $p \in cs(E)$ is polar and $\phi \in B_{ou}(X)$, then $p_\phi$ is a polar continuous seminorm on $(C_b(X,E),\beta_o)$.*

*Proof* Let $p_\phi(f) > \theta > 0$. There exists $x \in X$ such that $|\phi(x)|p(f(x)) > \theta$ and so $p(f(x)) > \alpha = \theta/|\phi(x)|$. Since $p$ is polar, there exists $x' \in E', |x'| \le p$, such that $|x'(f(x))| > \alpha$. Let

$$v : C_b(X,E) \to \mathbb{K}, \quad v(g) = \phi(x)x'(g(x)).$$

Then $v$ is linear and $|v| \le p_\phi$. Moreover, $|v(f)| > \theta$, which proves that $p_\phi$ is polar.

**Theorem 2.4.** *If $E$ is polar, then there exists a linear homeomorphism*

$$\omega : (C_b(X,E),\,\beta_o)\otimes(C_b(Y,F),\,\beta_o) \to (C_b(X\times Y, E\otimes F),\,\beta_o)$$

*onto a $\beta_o$-dense subspace of $C_b(X\times Y,\, E\otimes F)$. Moreover $\omega(f\otimes g) = f\odot g$ for all $f \in C_b(X,E)$, $g \in C_b(Y,F)$.*

*Proof:* Let

$$G = (C_b(X,E),\,\beta_o)\otimes(C_b(Y,F),\,\beta_o).$$

The bilinear map

$$T : (C_b(X,E),\,\beta_o)\times(C_b(Y,F),\,\beta_o) \to (C_b(X\times Y, E\otimes F),\,\beta_o),$$

$T(f,g) = f\odot g$, is continuous. Indeed, let $p \in cs(E)$ be polar, $q \in cs(F)$, $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, $\phi = \phi_1\times\phi_2$. Then

$$(p\otimes q)_\phi(f\odot g) = \sup_{x,y}|\phi_1(x)\phi_2(y)|p\otimes q((f(x)\otimes g(y))$$
$$= \sup_{x,y}|\phi(x,y)|p(f(x))q(g(y)) = p_{\phi_1}(f)q_{\phi_2}(g),$$

and hence $T$ is continuous. Let

$$\omega : G \to (C_b(X \times Y,\, E \otimes F),\, \beta_o)$$

be the corresponding continuous linear map.
**Claim.** For each $u \in G$, we have

$$(p \otimes q)_\phi(\omega(u)) = p_{\phi_1} \otimes q_{\phi_2}(u).$$

Indeed, if $u = \sum_{k=1}^n f_k \otimes g_k$, then

$$\begin{aligned} |\phi_1(x)\phi_2(y)| \cdot p \otimes q(\omega(u)(x,y)) &= |\phi_1(x)\phi_2(y)| \cdot p \otimes q\left(\sum_{k=1}^n f_k(x) \otimes g_k(y)\right) \\ &\le |\phi_1(x)\phi_2(y)| \cdot \max_k p(f_k(x))q(g_k(y)) \\ &\le \max_k p_{\phi_1}(f_k)q_{\phi_2}(g_k). \end{aligned}$$

Thus

$$(p \otimes q)_\phi(\omega(u)) \le \max_k p_{\phi_1}(f_k)q_{\phi_2}(g_k),$$

which proves that

$$(p \otimes q)_\phi(\omega(u)) \le p_{\phi_1} \otimes q_{\phi_2}(u).$$

On the other hand, given $0 < t < 1$, there exists a representation $u = \sum_{k=1}^n f_k \otimes g_k$ of $u$ such that the set $\{g_1, \ldots, g_n\}$ is $t$-orthogonal with respect to the seminorm $q_{\phi_2}$. Now

$$\begin{aligned}(p\otimes q)_\phi(\omega(u)) &= \sup_{x,y}|\phi_1(x)\phi_2(y)|p\otimes q\left(\sum_{k=1}^n f_k(x)g_k(y)\right)\\ &= \sup_{x,y}\left[|\phi_1(x)\phi_2(y)|\cdot\sup\{q\left(\sum_{k=1}^n x'(f_k(x))g_k(y)\right) : |x'|\le p\}\right]\\ &= \sup_x\left[|\phi_1(x)|\cdot\sup_{|x'|\le p}\{\sup_y|\phi_2(y)|\cdot q(\sum_{k=1}^n x'(f_k(x))g_k(y))\}\right]\\ &= \sup_x\left[|\phi_1(x)|\cdot\sup_{|x'|\le p} q_{\phi_2}\left(\sum_{k=1}^n x'(f_k(x))g_k\right)\right]\\ &\ge t\cdot\sup_x\left[|\phi_1(x)|\cdot\sup_{|x'|\le p}\max_k|x'(f_k(x))|\cdot q_{\phi_2}(g_k)\right]\\ &= t\cdot\sup_x\left[|\phi_1(x)|\cdot\left(\max_k p(f_k(x))q_{\phi_2}(g_k)\right)\right]\\ &= t\cdot\max_k p_{\phi_1}(f_k)q_{\phi_2}(g_k)\ge t\cdot p_{\phi_1}\otimes q_{\phi_2}(u).\end{aligned}$$

Since $0<t<1$ was arbitrary, we get that $(p\otimes q)_\phi(\omega(u))\ge p_{\phi_1}\otimes q_{\phi_2}(u)$ and the claim follows.
It is now clear that $\omega$ is one-to-one and, for $M=\omega(G)$, the map $\omega : G\to(M,\beta_o)$ is a homeomophism. Since, for $A\in K(X)$, $B\in K(Y)$, $a\in E$, $b\in F$, we have that $(\chi_A a)\odot(\chi_B b)\in M$, it follows that $M$ is $\beta_o$-dense in $(C_b(X\times Y, E\otimes F),\beta_o)$ in view of Theorem 2.1. This completes the proof.

For $x'\in E'$ and $y'\in F'$, we denote by $x'\otimes y'$ the unique element of $(E\otimes F)'$ defined by

$$x'\otimes y'(s_1\otimes s_2)=x'(s_1)y'(s_2).$$

**Theorem 2.5.** *Assume that $E$ is polar and let $m_1\in M_t(X,E')$, $m_2\in M_t(Y,F')$. Then there exists a unique $\bar m\in M_t(X\times Y,(E\otimes F)')$ such that*

$$\bar m(A\times B)=m_1(A)\otimes m_2(B)$$

*for $A\in K(X)$, $B\in K(Y)$. Moreover, for $g\in C_b(X,E)$, $f\in C_b(Y,F)$, $h=g\odot f$, we have*

$$\int h\,d\bar m=\left(\int g\,dm_1\right)\cdot\left(\int f\,dm_2\right).$$

*Proof:* Since $m_1$ is $\beta_o$-continuous on $C_b(X,E)$, there exist $\phi_1 \in B_{ou}(X)$ and a polar continuous seminorm $p$ on $E$ such that $|\int g\,dm_1| \le p_{\phi_1}(g)$ for all $g \in C_b(X,E)$. Similarly, there exist $\phi_2 \in B_{ou}(Y)$ and $q \in cs(F)$ such that $|\int f\,dm_2| \le q_{\phi_2}(f)$ for all $f \in C_b(Y,F)$. Consider the bilinear map

$$T : (C_b(X,E),\beta_o) \times (C_b(Y,F),\beta_o) \to \mathbb{K},$$

$$T(g,f) = \left(\int g\,dm_1\right) \cdot \left(\int f\,dm_2\right).$$

Then $T$ is continuous since $|T(g,f)| \le p_{\phi_1}(g) \cdot q_{\phi_2}(f)$. Hence the corresponding linear map

$$\psi : G = (C_b(X,E),\beta_o) \otimes (C_b(Y,F),\beta_o) \to \mathbb{K}$$

is continuous. Let $\omega$ be as in the preceding Theorem and $M = \omega(G)$. The linear map

$$v : (M,\beta_o) \to \mathbb{K}, \quad v = \psi \circ \omega^{-1},$$

is continuous. Since $M$ is $\beta_o$-dense in $C_b(X \times Y, E \otimes F)$, there exists a unique $\beta_o$-continuous linear extension $\tilde{v}$ of $v$ to all of $C_b(X \times Y, E \otimes F)$. Let

$$\bar{m} \in M_t(X \times Y, (E \otimes F)')$$

be such that $\tilde{v}(h) = \int h\,d\bar{m}$ for all $h \in C_b(X \times Y, E \otimes F)$. Taking

$$h = (\chi_A s_1) \odot (\chi_B s_2) = \chi_{A \times B} s_1 \otimes s_2,$$

where $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, we get that

$$\begin{aligned}
\bar{m}(A \times B)(s_1 \otimes s_2) &= \int h\,d\bar{m} = \psi((\chi_A s_1) \otimes (\chi_B s_2)) \\
&= (m_1(A)s_1) \otimes (m_2(B)s_2)) \\
&= [m_1(A) \otimes m_2(B)](s_1 \otimes s_2).
\end{aligned}$$

Thus $\bar{m}(A \times B) = m_1(A) \otimes m_2(B)$. If $g \in C_b(X,E)$, $f \in C_b(Y,F)$ and $h = g \odot f$, then

$$\int h\,d\bar{m} = \tilde{v}(h) = \psi(g \otimes f) = \left(\int g\,dm_1\right) \cdot \left(\int f\,dm_2\right).$$

Finally, let $\mu \in M_t(X \times Y, (E \otimes F)')$ be such that $\mu(A \times B) = m_1(A) \otimes m_2(B)$ for all $A \in K(X)$, $B \in K(Y)$. The map

$$v_1 : C_b(X \times Y, E \otimes F) \to \mathbb{K}, \quad v_1(h) = \int h\,d\mu,$$

is $\beta_o$-continuous. Taking

$$h = (\chi_A s_1) \odot (\chi_B s_2) = \chi_{A\times B} s_1 \otimes s_2),$$

where $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, we have that $v_1(h) = \tilde{v}(h)$. In view of Theorem 2.1, we see that $v_1 = \tilde{v}$ on a $\beta_o$-dense subspace of $C_b(X \times Y, E \otimes F)$ and hence $v_1 = \tilde{v}$, which implies that $\bar{m} = \mu$. This completes the proof.

**Definition 2.6.** If $m_1, m_2, \bar{m}$ are as in the preceding Theorem, we will call $\bar{m}$ the tensor product of $m_1, m_2$ and denote it by $m_1 \otimes m_2$.

**Theorem 2.7.** *Assume that $E$ is polar and let $m_1 \in M_{t,p}(X, E')$, $m_2 \in M_{t,q}(Y, F')$. Suppose that $p$ is polar. Then*

(1) $\bar{m} = m_1 \otimes m_2 \in M_{t,p\otimes q}(X \times Y, (E \otimes F)')$ *and*

$$\|\bar{m}\|_{p\otimes q} = \|m_1\|_p \|m_2\|_q.$$

(2) *If* $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$ *are such that* $|\int g\,dm_1| \le p_{\phi_1}(g)$, *for all* $g \in C_b(X, E)$, *and* $|\int f\,dm_2| \le p_{\phi_2}(f)$, *for all* $f \in C_b(Y, F)$, *then for* $\phi = \phi_1 \times \phi_2$, *we have*

$$\left|\int h\,d\bar{m}\right| \le (p \otimes q)_\phi(h), \quad \textit{for all} \quad h \in C_b(X \times Y, E \otimes F).$$

*Proof:* Let $\phi_1$ and $\phi_2$ be as in the Theorem. For $g \in C_b(X, E)$, $f \in C_b(Y, F)$ and $h = g \odot f$, we have

$$\left|\int h\,d\bar{m}\right| = \left|\left(\int g\,dm_1\right) \cdot \left(\int f\,dm_2\right)\right| \le p_{\phi_1}(g) q_{\phi_2}(f).$$

It is easy to see that $\|\phi h\|_{p\otimes q} = \|\phi_1 g\|_p \cdot \|\phi_2 f\|_q$. Thus

$$\left|\int h\,d\bar{m}\right| \le \|\phi h\|_{p\otimes q}.$$

Since both maps $h \mapsto (p \otimes q)_\phi(h)$ and $h \mapsto \int h\,\bar{m}$ are $\beta_o$-contiuous and $M$ is $\beta_o$-dense, it follows that

$$\left|\int h\,d\bar{m}\right| \le \|\phi h\|_{p\otimes q}.$$

for all $h \in C_b(X \times Y, E \otimes F)$. Hence $\bar{m} \in M_{t,p\otimes q}(X \times Y, (E \otimes F)')$. For $g \in C_b(X,E)$, $f \in C_b(Y,F)$, $h = g \odot f$, we have

$$\left|\int h\, d\bar{m}\right| = \left|\left(\int g\, dm_1\right) \cdot \left(\int f\, dm_2\right)\right| \leq \|m_1\|_p \cdot \|g\|_p \cdot \|m_2\|_q \cdot \|f\|_q$$
$$= [\|m_1\|_p \cdot \|m_2\|_q] \cdot [\|h\|_{p\otimes q}] .$$

Thus $\|\bar{m}\|_{p\otimes q} \leq \|m_1\|_p \cdot \|m_2\|_q = d$. If $d > 0$ and $0 < \epsilon_1 < \|m_1\|_p$, $0 < \epsilon_2 < \|m_2\|_q$, then there are $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, such that

$$\frac{|m_1(A)s_1|}{p(s_1)} > \|m_1\|_p - \epsilon_1, \quad \frac{|m_2(B)s_2|}{q(s_2)} > \|m_2\|_q - \epsilon_2.$$

Now

$$\|\bar{m}\|_{p\otimes q} \geq \frac{|\bar{m}(A \times B)s_1 \otimes s_2|}{p \otimes q(s_1 \otimes s_2)} > (\|m_1\|_p - \epsilon_1) \cdot (\|m_2\|_q - \epsilon_2).$$

Taking $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, we get $\|\bar{m}\|_{p\otimes q} \geq \|m_1\|_p \cdot \|m_2\|_q$, which completes the proof.

**Lemma 2.8.** *Let $m \in M_p(X, E')$, $V \in K(X)$ and*

$$\alpha = \sup\{|m(A)s| : A \in K(X), A \subset V, p(s) \leq 1\}.$$

*Then*

(1) *for any $\lambda \in \mathbb{K}$, with $|\lambda| > 1$, we have $\alpha \leq m_p(V) \leq |\lambda|\alpha$.*

(2) *If the valuation of $\mathbb{K}$ is dense or if it is discrete and $p(E) \subset |\mathbb{K}|$, then*
*$m_p(V) = \alpha$.*

*Proof:* (1). If $p(s) \leq 1$ and $A \in K(X)$, $A \subset V$, then $|m(A)s)| \leq m_p(V) \cdot p(s) \leq m_p(V)$ and so $\alpha \leq m_p(V)$.On the other hand, if $p(s) > 0$, then there exists $\gamma \in \mathbb{K}$ with $|\gamma| \leq p(s) \leq |\gamma\lambda|$. Now, for $A \subset V$, we have

$$\alpha \geq |m(A)(\gamma^{-1}\lambda^{-1}s)| \geq |\lambda^{-1}| \cdot \frac{|m(A)s|}{p(s)}.$$

It follows that $\alpha|\lambda| \geq m_p(V)$.
(2). It is clear from (1) that $\alpha = m_p(V)$ if the valuation is dense. Suppose that the valuation is discrete and $p(E) \subset |\mathbb{K}|$. If $p(s) > 0$, then there exists $\gamma \in \mathbb{K}$, with $p(s) = |\gamma|$. For $A \subset V$, we have $\frac{|m(A)s|}{p(s)} = |m(A)(\gamma^{-1}s)| \leq \alpha$ and so $m_p(V) \leq \alpha$, which completes the proof.

**Theorem 2.9.** *Assume that $E$ is polar and let $p \in cs(E)$ be polar, $q \in cs(F)$. If $m_1 \in M_{t,p}(X,E')$, $m_2 \in M_{t,q}(Y,F')$ and $\bar{m} = m_1 \otimes m_2$, then, for $|\lambda| > 1$, we have*

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq N_{\bar{m},p\otimes q}(x,y) \leq |\lambda| N_{m_1,p}(x) \cdot N_{m_2,q}(y).$$

*If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then*

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) = N_{\bar{m},p\otimes q}(x,y)$$

*Proof:* Let $Z$ be a clopen neighborhood of $(x,y)$. There are $A \in K(X)$, $B \in K(Y)$ such that $(x,y) \in A \times B \subset Z$. For $s_1 \in E$, $s_2 \in F$, $s = s_1 \otimes s_2$, with $p(s_1) \leq 1$, $q(s_2) \leq 1$, we have

$$\sup_{A_1 \subset A, B_1 \subset B} \frac{|m_1(A_1)s_1| \cdot |m_2(B_1)s_2|}{p \otimes q(s)} \leq |\bar{m}|_{p\otimes q}(Z)$$

and so

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq |m_1|_p(A) \cdot |m_2|_q(B) \leq |\bar{m}|_{p\otimes q}(Z).$$

Hence

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq N_{\bar{m},p\otimes q}(x,y).$$

On the other hand, let $N_{m_1,p}(x) \cdot N_{m_2,q}(y) < \theta$. There are clopen sets $V_1$, $V_2$,
$x \in V_1$, $y \in V_2$, $|m_1|_p(V_1) \cdot |m_2|_q(V_2) < \theta$. Let

$$d = \sup\{|\bar{m}(D)u| : D \subset V_1 \times V_2,\ p \otimes q(u) \leq 1\}.$$

Let $u \in E \otimes F$ with $p \otimes q(u) \leq 1$. Given $0 < t < 1$, there exists a representation $u = \sum_{j=1}^{N} s_j \otimes a_j$ of $u$ such that the set $\{a_1, \ldots, a_N\}$ is t-orthogonal with respect to the seminorm $q$. Now

$$\begin{aligned} 1 \geq p \otimes q(u) &= \sup_{|x'| \leq p} q\left(\sum_{j=1}^{N} x'(s_j)a_j\right) \\ &\geq t \cdot \sup_{|x'| \leq p} \max_j |x'(s_j)| q(a_j) \\ &= t \cdot \max_j p(s_j) q(a_j). \end{aligned}$$

Let $0 < \epsilon < \theta$. There exists a compact subset $G$ of $X \times Y$ such that $|\bar{m}|_{p\otimes q}(W) < \epsilon$ if the clopen set $W$ is disjoint from $G$. Let $D$ be a clopen subset of $V_1 \times V_2$. For each $z = (a,b) \in G \cap D$, there are clopen neighborhoods $W_z$, $M_z$ of $a, b$, respectively, with $(a,b) \in W_z \times M_z \subset D$.

In view of the compactness of $G \cap D$, there are $z_i = (x_i, y_i) \in G \cap D$, $i = 1, \ldots, n$, such that

$$G \cap D \subset D_1 = \bigcup_{i=1}^{n} W_{z_i} \times M_{z_i} \subset D.$$

There are pairwise disjoint clopen rectangles $A_j \times B_j$, $j = 1, \ldots, k$, such that

$$D_1 = \bigcup_{j=1}^{k} A_j \times B_j.$$

Now

$$\bar{m}(D)s_i \otimes a_i = \bar{m}(D \setminus D_1)s_i \otimes a_i + \sum_{j=1}^{k} \bar{m}(A_j \times B_j)s_i \otimes a_i.$$

Since $D \setminus D_1$ is disjoint from $G$, it follows that

$$|\bar{m}(D \setminus D_1)s_i \otimes a_i| \leq |\bar{m}|_{p\otimes q}(D \setminus D_1) \cdot p \otimes q(s_i \otimes a_i) \leq \epsilon/t < \theta/t.$$

Also,

$$\begin{aligned} |\bar{m}(A_j \times B_j)s_i \otimes a_i| &= |m_1(A_j)s_i| \cdot |m_2(B_j)a_i| \\ &\leq |m_1|_p(V_1)p(s_i) \cdot |m_2|_q(V_2)q(a_i) \\ &\leq \frac{|m_1|_p(V_1) \cdot |m_2|_q(V_2)}{t} < \theta/t. \end{aligned}$$

Thus $|\bar{m}(D)s_i \otimes a_i| < \theta/t$ and hence

$$|\bar{m}(D)u| \leq \max_i |\bar{m}(D)s_i \otimes a_i| < \theta/t.$$

This proves that $d \leq \theta/t$ and so $|\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq |\lambda| \cdot \theta/t$, which shows that $N_{\bar{m},p\otimes q}(x, y) \leq |\lambda|\theta/t$. Therefore

$$N_{\bar{m},p\otimes q}(x, y) \leq \frac{|\lambda|}{t} \cdot N_{m_1,p}(x)N_{m_2,q}(y).$$

Since $0 < t < 1$ was arbitrary, we get that

$$N_{\bar{m},p\otimes q}(x, y) \leq |\lambda| \cdot N_{m_1,p}(x)N_{m_2,q}(y).$$

If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then

$$d = |\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq \theta/t$$

and hence $N_{\bar{m},p\otimes q}(x,y) \leq \theta/t$. Since $0 < t < 1$ was arbitrary, we have that $N_{\bar{m},p\otimes q}(x,y) \leq \theta$, which shows that

$$N_{\bar{m},p\otimes q}(x) \leq N_{m_1,p}(x) \cdot N_{m_2,q}(y),$$

and the result follows.

**Note 1.** *Assume that $\mathbb{K}$ is discrete. If $p$ is polar and $q(F) \subset |\mathbb{K}|$ , then*

$$p \otimes q(E \otimes F) \subset |\mathbb{K}|.$$

This follows from the fact that, for $u = \sum_{i=1}^{n} x_i \otimes y_i$, we have

$$p \otimes q(u) = \sup_{|x'|\leq p} q\left(\sum_{i=1}^{n} x'(x_i) y_i\right).$$

We have the following easily established

**Theorem 2.10.** *Let $m_1$, $m_2$, $\bar{m}$ be as in Theorem 2.9. If $V_1 \in K(X)$, $V_2 \in K(Y)$ and $|\lambda| > 1$, then*

$$|m_1|_p(V_1) \cdot |m_2|_q(V_2) \leq |\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq |\lambda| \cdot |m_1|_p(V_1) \cdot |m_2|_q(V_2).$$

*If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then*

$$|m_1|_p(V_1) \cdot |m_2|_q(V_2) = |\bar{m}|_{p\otimes q}(V_1 \times V_2).$$

**Theorem 2.11.** *Let $m_1$, $m_2$, $\bar{m}$ be as in Theorem 2.9. Then*

$$supp(\bar{m}) = supp(m_1) \times supp(m_2).$$

*Proof:* Let $A_1 = \{x \in X : N_{m_1,p}(x) \neq 0\}$, $A_2 = \{y \in Y : N_{m_2,q}(y) \neq 0\}$, and $A = \{(x,y) : N_{\bar{m},p\otimes q}(x,y) \neq 0\}$. Then $A = A_1 \times A_2$. The result now follows from [3, Thm. 2.1].

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