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Abstract

Let $G_{\mathbb{R}}$ be a real form of a complex semisimple Lie group G. Recall that Rossmann defined a Weyl group action on Lagrangian cycles supported on the conormal bundle of the flag variety of G. We compute the signed average of the Weyl group action on the characteristic cycle of the standard sheaf associated to an open $G_{\mathbb{R}}$-orbit on the flag variety. This result is applied to find the value of the constant term in Harish-Chandra's limit formula for the delta function at zero.

1. Introduction

Let $G_{\mathbb{R}}$ be a semisimple Lie group, $\mathfrak{g}_{\mathbb{R}}$ the Lie algebra of $G_{\mathbb{R}}$, $\mathfrak{g}$ the complexification of $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{h}_{\mathbb{R}}$ a Cartan subalgebra, $\mathfrak{h}$ the complexification of $\mathfrak{h}_{\mathbb{R}}$, and X the flag variety of $\mathfrak{g}$. A classical formula of Harish-Chandra [4] for the delta function at zero states that

$$\lim_{\lambda\to 0}\prod_{\alpha>0}\partial(\alpha)m_\lambda = cm_{\{0\}}.$$

Here $\lambda\in i\mathfrak{h}_{\mathbb{R}}^*$ is regular, m_λ resp. $m_{\{0\}}$ is the canonical measure on the coadjoint orbit $G_{\mathbb{R}}\cdot\lambda$ resp. $\{0\}$, and $\partial(\alpha)$ is the differential operator on $\mathfrak{h}^*$ defined by a positive root α. Furthermore, the constant $c\neq 0$ if and only if $\mathfrak{h}_{\mathbb{R}}$ is a fundamental Cartan subalgebra.

In [8] Rossmann suggested a geometric approach to Harish-Chandra's formula, which was based on his results relating invariant eigendistributions on $\mathfrak{g}_{\mathbb{R}}$ and homology classes of the conormal variety of $G_{\mathbb{R}}$-action on X, and on the properties of the coherent continuation representation of the Weyl group. Rossmann's argument does not give the exact value of the

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constant c in case $\mathfrak{h}_\mathbb{R}$ is fundamental. One of the main motivations for the present paper was to compute the non-zero c. It turns out that considerably more theory is needed to obtain this information. In fact, our results rely heavily on the work of Schmid and Vilonen [11], [12]. In more details, the integral formula for the character associated to a $G_\mathbb{R}$-equivariant sheaf on X [12], Theorem 3.8, and a local expression of the character [12], Theorem 5.27 are two of the main ingredients in our analysis. The argument we use to prove Harish-Chandra's formula is quite standard: instead of measures on the coadjoint orbits one studies the asymptotic behaviour of their Fourier transforms. By the general philosophy that goes back to the work of Harish-Chandra, these Fourier transforms represent the characters of representations. In explicit terms, the Fourier transform of the canonical measure m_λ, under the appropriate positivity assumption on the parameter λ, represents the character of an induced representation. It is interesting to point out that Harish-Chandra's formula does not follow from the information about the leading term in the asymptotic expansion of the character of this induced representation. Rather, one has to consider the signed average of the character of the induced representation over the Weyl group. It turns out that this virtual character is up to a constant term, which we compute explicitly, equal to the character of a finite dimensional representation. These facts are actually established as an identity between homology cycles supported in the conormal variety of $G_\mathbb{R}$-action on X in Thorem 3.2, and the translation to the language of invariant eigendistributions is explained in Proposition 3.3 below. Harish-Chandra's formula is then deduced from the identity of homology cycles in Theorem 3.2, using the results in [12] and [8]. When $G_\mathbb{R}$ has a compact Cartan subgroup the character identity from Proposition 3.3 appears already without proof in [1]. For this reason Proposition 3.3 can be considered as a generalization of a known result. Further applications of Schmid and Vilonen theory to the characters of Vogan-Zuckerman modules, and related limit formulas will be taken up in future publications.

2. Preliminaries

Suppose $G_\mathbb{R}$ is a real, connected, linear, semisimple Lie group. We embed $G_\mathbb{R}$ into a complexification G and denote by

$$\tau : G \longrightarrow G$$

the involution on G having $G_{\mathbb{R}}$ as the identity component of the set of fixed points. Next we choose a Cartan involution

$$\theta : G_{\mathbb{R}} \longrightarrow G_{\mathbb{R}},$$

and extend it to G. Denote by $K_{\mathbb{R}}$ resp. K the set of fixed points of θ on $G_{\mathbb{R}}$ resp. G. Observe that $\theta\tau$ is a Cartan involution on G. We denote by $U_{\mathbb{R}}$ the set of fixed points. Write $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{u}_{\mathbb{R}}$ for the Lie algebras of G, K, $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $U_{\mathbb{R}}$ respectively. Denote the involutions on $\mathfrak{g}$ induced by θ, τ by the same letters. In addition, let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}\,, \;\; \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the eigenspace decompositions defined by θ. Let $(\ ,\ )$ be the Killing form on $\mathfrak{g}$. We will use it whenever convenient to identify $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$.

Next we introduce the notation related to the geometry of the flag variety. Write X for the flag variety of Borel subalgebras of $\mathfrak{g}$. Let $n = \dim_{\mathbb{C}} X$. Given $x \in X$ we denote by $\mathfrak{b}_x$ the Borel subalgebra which fixes x , and by $B_x \subset G$ the Borel subgroup which stabilizes x via the adjoint action. Consider G-homogenous bundles $\mathcal{B}$ and $[\mathcal{B}, \mathcal{B}]$ over X with fiber $\mathfrak{b}_x$ resp. $[\mathfrak{b}_x, \mathfrak{b}_x]$ at $x \in X$. Observe that B_x acts trivially on $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$, hence the G-bundle $\mathcal{B}/[\mathcal{B}, \mathcal{B}]$ is trivial. We denote by $\mathfrak{h}$ its fiber, and call it the universal Cartan subalgebra. Note that $\mathfrak{h} \simeq \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$ canonically, for any $x \in X$. Let $\mathfrak{c} \subset \mathfrak{g}$ be a Cartan subalgebra. Denote by $\Delta(\mathfrak{g}, \mathfrak{c})$ the root system. Then $\mathfrak{c}$ has $|W|$ fixed points on X ($|\ \ |$ stands for the cardinality), and we choose one of them: $x \in X$. Then $\mathfrak{c} \subset \mathfrak{b}_x$, and we have a canonical isomorphism $\tau_x : \mathfrak{c} \to \mathfrak{h}$. We denote by $\tau_x^* : \mathfrak{h}^* \to \mathfrak{c}^*$ the dual isomorphism. Then

$$\Delta = \tau_x^{*-1}(\Delta(\mathfrak{g}, \mathfrak{c}))$$

is independent on the choice of the pair $(\mathfrak{c}, x)$, and is called the universal root system. Set $\Delta_x^+ = \Delta(\mathfrak{g}/\mathfrak{b}_x, \mathfrak{c})$. A positive root system in Δ is defined by the condition

$$\Delta^+ = \tau_x^{*-1}(\Delta_x^+).$$

Given $\lambda \in \Delta$, and a pair $(\mathfrak{c}, x)$ as above, we write $\lambda_x = \tau_x^*(\lambda)$. The universal Weyl group W is defined as the Weyl group of the root system Δ. Denote by $\rho \in \mathfrak{h}^*$ half the sum of the positive roots, and by $\mathfrak{h}'^*$ the set of regular elements. Note that $\mathfrak{h}$ resp. $\mathfrak{h}^*$ comes equipped with W-invariant symmetric bilinear form $(\cdot, \cdot)$ whose specialization at $x \in X$ coincides with the Killing form. In particular, if $\lambda \in \mathfrak{h}^*$ we write h_λ for the element in $\mathfrak{h}$ such that $\lambda(h) = (h, h_\lambda)$, $h \in \mathfrak{h}$.

Let us recall the definition of the moment map and of the twisted moment map. Denote by T^*X the cotangent bundle of the variety X. Given $x \in X$, denote by $\mathfrak{b}_x^\perp \subset \mathfrak{g}^*$ the space of linear forms vanishing on $\mathfrak{b}_x$. We use the identification

$$T^*X \cong \left\{(x, \xi) : x \in X,\ \xi \in \mathfrak{b}_x^\perp\right\},$$

to consider T^*X as a submanifold of $X \times \mathfrak{g}^*$ The moment map is defined by

$$\mu : T^*X \longrightarrow \mathfrak{g}^*, \qquad \mu(x, \xi) = \xi.$$

Denote by $\mathcal{N}^*$ the cone of nilpotent elements in $\mathfrak{g}^*$. Note that $\mu(T^*X) = \mathcal{N}^*$. The definition of the twisted moment map is due to Rossmann [9], §2. Note that any $x \in X$ is fixed by a unique maximal torus $C_\mathbb{R} \subset U_\mathbb{R}$. We can use the decomposition $\mathfrak{g} = \mathfrak{c} + [\mathfrak{c}, \mathfrak{g}]$ to view $\mathfrak{c}^*$ as a subspace of $\mathfrak{g}^*$. Now we define the twisted moment map by

$$\mu_\lambda : T^*X \longrightarrow \mathfrak{g}^*, \ \ \mu_\lambda(x, \xi) = \lambda_x + \mu(x, \xi), \quad \xi \in \mathfrak{b}_x^\perp.$$

If λ is regular, one can show that μ_λ is a $U_\mathbb{R}$-equivariant, real algebraic isomorphism of T^*X with complex coadjoint orbit $\mathrm{Ad}^*(G)\lambda_x$. Note that $\mathrm{Ad}^*(G)\lambda_x$ is independent on $x \in X$. We shall write $G \cdot \lambda = \mathrm{Ad}^*(G)\lambda_x$.

By the result of Matsuki [7] $G_\mathbb{R}$ acts on X with finitely many orbits. Moreover, $G_\mathbb{R}$-action on X is real algebraic, hence the orbits define a semi-algebraic Whitney stratification of X. We shall denote by $T^*_{G_\mathbb{R}}X$ union of the conormal bundles of the $G_\mathbb{R}$-orbits on X. Via the characteristic cycle map the K-group of $G_\mathbb{R}$-equivariant sheaves on X can be related to the top-dimensional homology group of the conormal variety of $G_\mathbb{R}$-action. In order to explain this, we need some additional notation. If Y is a locally compact space, we denote by $H_i(Y, \mathbb{C})$, $i \in \mathbb{Z}$, the Borel-Moore homology groups with complex coefficients. Suppose that Y is a real algebraic manifold. The characteristic cycle $CC(\mathcal{F})$ of a constructible sheaf $\mathcal{F}$ was defined by Kashiwara [5], Ch.IX, and [11]. Recall that $CC(\mathcal{F})$ is defined as a Lagrangian cycle in the real cotangent bundle T^*Y. In fact, let $\mathcal{S}$ be a semi-algebraic Whitney stratification on Y, and $\mathcal{F}$ a sheaf constructible for $\mathcal{S}$. Denote by $T^*_\mathcal{S}Y$ union of the conormal bundles to the strata. Then

$$CC(\mathcal{F}) \in H_m(T^*_\mathcal{S}Y, \mathbb{C}), \qquad m = \dim_\mathbb{R} Y.$$

We remark that $CC(\mathcal{F})$ is actually an integral homology cycle. However, in view of the applications we have in mind, it will be more convenient to consider complex coefficients.

Returning to the setting of the flag variety, we denote by $Sh_{G_\mathbb{R}}(X)$ the category of $G_\mathbb{R}$-equivariant constructible sheaves on X [2], Ch.0. Note that objects from $Sh_{G_\mathbb{R}}(X)$ are constructible for the orbit stratification on X. Let $K(Sh_{G_\mathbb{R}}(X))$ be the Grothendieck group of the abelian category $Sh_{G_\mathbb{R}}(X)$. Since CC is additive on short exact sequences we obtain a homomorphism

$$CC : K(Sh_{G_\mathbb{R}}(X)) \longrightarrow H_{2n}(T^*_{G_\mathbb{R}}X, \mathbb{C}). \tag{2.1}$$

Let $\mathcal{V}$ be a coadjoint G-orbit in $\mathfrak{g}^*$ or a coadjoint $G_\mathbb{R}$-orbit in $i\mathfrak{g}^*_\mathbb{R}$. To treat both cases simultaneously write $V = G$ or $V = G_\mathbb{R}$, and denote by $\mathfrak{v}$ the Lie algebra of V. The space

$$\mathfrak{v} \cdot \xi = \{\mathrm{ad}^*(x)(\xi) : x \in \mathfrak{v}\}$$

identifies with tangent space $T_\xi\mathcal{V}$ of $\mathcal{V}$ at ξ, and we define a V-equivariant 2-form $\sigma_\mathcal{V}$ on $\mathcal{V}$ by the formula

$$\sigma_{\mathcal{V},\xi}(x \cdot \xi, y \cdot \xi) = \xi[x, y]\,, \;\; x, y \in \mathfrak{v}.$$

In case $V = G_\mathbb{R}$, the form $-i\sigma_\mathcal{V}$ is real valued, and we use the form $(-i\sigma_\mathcal{V})^k$, $2k = \dim_\mathbb{R} \mathcal{V}$ to orient $\mathcal{V}$. Then we define the measure $m_\mathcal{V}$ by the formula

$$m_\mathcal{V} = \frac{1}{(2\pi i)^k k!}\sigma^k_\mathcal{V}, \tag{2.2}$$

and call it the Liouville measure. When $\mathcal{V} = G \cdot \lambda$, $\lambda \in \mathfrak{h}^*$, we shall write $\sigma_\mathcal{V} = \sigma_\lambda$. Let $\lambda \in \mathfrak{h}^*$. Then a $U_\mathbb{R}$-equivariant 2-form τ_λ on X is defined at x by

$$\tau_\lambda(a_x, b_x) = \lambda_x([a, b]).$$

Here a_x and b_x denote the tangent vectors which a, $b \in \mathfrak{u}_\mathbb{R}$ induce by differentiation of the $U_\mathbb{R}$-action.

Denote by $\pi_X : T^*X \longrightarrow X$ the natural projection, and by σ the canonical symplectic form on T^*X. For $\lambda \in \mathfrak{h}'^*$ the following formula holds [12], Proposition 3.3:

$$\mu^*_\lambda(\sigma_\lambda) = -\sigma + \pi^*_X(\tau_\lambda). \tag{2.3}$$

Next we recall, following [12], § 3, the definition of invariant distributions on the Lie algebra as integrals of certain differential forms over the semi-algebraic cycles in T^*X. The Fourier transform of a test function $\phi \in$

$C_c^\infty(\mathfrak{g}_\mathbb{R})$ will be defined by

$$\hat{\phi}(\xi) = \int_{\mathfrak{g}_\mathbb{R}} e^{\xi(x)}\phi(x)dx\,, \;\; \xi \in \mathfrak{g}^*,$$

without the usual i in the exponential. Here dx denotes a suitably normalized Lebesgue measure on $\mathfrak{g}_\mathbb{R}$. Let Γ be a semi-algebraic chain in T^*X. We say that Γ is $\mathbb{R}$-bounded if

$$\operatorname{Re}\mu(\operatorname{supp}(\Gamma)) \subset \mathfrak{g}^*$$

is bounded. Here Re is defined with respect to $\mathfrak{g}_\mathbb{R}^*$. If Γ is a semi-algebraic, $\mathbb{R}$-bounded, $2n$-chain in T^*X one can prove that for a test function $\phi \in C_c^\infty(\mathfrak{g}_\mathbb{R})$ and $\lambda \in \mathfrak{h}^*$ the integral

$$\Theta(\Gamma,\lambda)(\phi) = \frac{1}{(2\pi i)^n n!}\int_\Gamma \mu_\lambda^*(\hat{\phi})(-\sigma + \pi_X^*\tau_\lambda)^n \tag{2.4}$$

converges and depends holomorphically on λ. In particular, this is true for a cycle $\Gamma \in H_{2n}(T^*_{G_\mathbb{R}}X,\mathbb{C})$. In this case $\Theta(\Gamma,\lambda)$ is a $G_\mathbb{R}$-invariant distribution on $\mathfrak{g}_\mathbb{R}$.

Denote by $\mathfrak{g}'_\mathbb{R}$ the set of regular semisimple elements in $\mathfrak{g}_\mathbb{R}$, and given a Cartan subalgebra $\mathfrak{c}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ let $\mathfrak{c}'_\mathbb{R} = \mathfrak{c}_\mathbb{R} \cap \mathfrak{g}'_\mathbb{R}$. By the work of Harish-Chandra $\Theta(CC(\mathcal{F}),\lambda)|\mathfrak{g}'_\mathbb{R}$ is a real analytic function. It is computed in [12] using the fixed point formalism of Goresky and MacPherson. Denote by $X^{C_\mathbb{R}}$ the set of fixed points of $C_\mathbb{R}$ on X. Let $\zeta \in \mathfrak{c}'_\mathbb{R}$ and $x \in X^{C_\mathbb{R}}$. We select a subset $\Delta'_x \subset \Delta^+_x$ closed under addition and having the property:

$$\operatorname{Re}(\alpha_x(\zeta)) \neq 0 \implies (\alpha_x \in \Delta'_x \Leftrightarrow \operatorname{Re}(\alpha_x(\zeta)) < 0).$$

We set further

$$\mathfrak{n}_x^+ = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{\alpha_x}, \;\; \mathfrak{n}'_x(\zeta) = \sum_{\alpha_x\in\Delta'_x} \mathfrak{g}_{\alpha_x},$$

$$N_x^+ = \exp(\mathfrak{n}_x^+)x, \;\; N'_x(\zeta) = \exp(\mathfrak{n}'_x(\zeta))x.$$

Let $\mathcal{F}$ be a $G_\mathbb{R}$-equivariant sheaf on X. Write $\mathbb{D}\mathcal{F}$ for the Verdier dual of $\mathcal{F}$ [5], Ch.III, and $\mathbb{D}\mathcal{F}(x)$ for the restriction of $\mathbb{D}\mathcal{F}$ to the open set N_x^+ of X. Let E be the connected component of $\mathfrak{c}'_\mathbb{R}$ containing ζ. Finally we define the integers

$$d_{E,x} = d_{\zeta,x} = \chi(R\Gamma_{N'_x(\zeta)}(\mathbb{D}\mathcal{F}(x))_x), \tag{2.5}$$

where $R\Gamma_{N'_x(\zeta)}(\cdot)$ stands for the local cohomology, and $\chi((\cdot)_x)$ for the Euler characteristic of the stalk. Let $\mathcal{F}$ be an object from $Sh_{G_\mathbb{R}}(X)$. Then the restriction of $\Theta(CC(\mathcal{F}),\lambda)$ to $\mathfrak{c}'_{\mathbb{R}}$ can be computed as follows [12], Theorem 5.27:

$$\Theta(CC(\mathcal{F}),\lambda)(\zeta) = \sum_{x\in X^{C_{\mathbb{R}}}} \frac{d_{E,x}e^{\lambda_x}(\zeta)}{\prod_{\alpha\in\Delta^+}\alpha_x(\zeta)}, \quad \zeta\in E\subset \mathfrak{c}'_{\mathbb{R}}.$$

We should point out that [12], Theorem 5.27 is deduced under the assumption that $\mathcal{F}$ is a $(-\lambda-\rho)$-monodromic sheaf. To prove (2.5) for a $G_{\mathbb{R}}$-equivariant sheaf, and arbitrary $\lambda\in\mathfrak{h}^*$, one would have to argue similarly as in [12], §8-10. Alternatively, (2.5) follows from the main result in [6].

Now we fix a θ-stable fundamental Cartan subalgebra $\mathfrak{c}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$. Let

$$\mathfrak{c}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}+\mathfrak{a}_{\mathbb{R}},\ \ \mathfrak{t}_{\mathbb{R}} = \mathfrak{c}_{\mathbb{R}}\cap\mathfrak{k}_{\mathbb{R}},\ \mathfrak{a}_{\mathbb{R}} = \mathfrak{c}_{\mathbb{R}}\cap\mathfrak{p}_{\mathbb{R}} \tag{2.6}$$

be the Cartan decomposition, and $\mathfrak{c}$ the complexification of $\mathfrak{c}_{\mathbb{R}}$. Next we recall the definition of the real Weyl group. Write $Z_{G_{\mathbb{R}}}(A)$ (resp. $N_{G_{\mathbb{R}}}(A)$) for the centralizer (resp. normalizer) of $A\subset\mathfrak{g}$. Let $C_{\mathbb{R}} = Z_{G_{\mathbb{R}}}(\mathfrak{c}_{\mathbb{R}})$ be the Cartan subgroup defined by $\mathfrak{c}_{\mathbb{R}}$. Set

$$W(G_{\mathbb{R}},C_{\mathbb{R}}) = N_{G_{\mathbb{R}}}(\mathfrak{c}_{\mathbb{R}})/C_{\mathbb{R}}.$$

On the other hand, we denote by $W(\mathfrak{g},\mathfrak{c})$ the Weyl group of the root system $\Delta(\mathfrak{g},\mathfrak{c})$. Recall that $W(\mathfrak{g},\mathfrak{c})$ is generated by the reflections s_α, $\alpha\in\Delta(\mathfrak{g},\mathfrak{c})$. We consider $W(\mathfrak{g},\mathfrak{c})$ also as a group of linear endomorphisms of $\mathfrak{c}$ and $\mathfrak{c}^*$. It is then not difficult to deduce

$$W(G_{\mathbb{R}},C_{\mathbb{R}})\subset W(\mathfrak{g},\mathfrak{c}).$$

Note that the involution θ acts naturally on $\Delta(\mathfrak{g},\mathfrak{c})$. Denote by

$$\Delta_I(\mathfrak{g},\mathfrak{c})\subset\Delta(\mathfrak{g},\mathfrak{c})$$

the subset of roots vanishing on $\mathfrak{a}$. We choose positive root systems $\Delta_0^+\subset\Delta(\mathfrak{g},\mathfrak{c})$, and $\Delta_1^+\subset\Delta(\mathfrak{g},\mathfrak{c})$ such that

$$\theta\Delta_0^+ = \Delta_0^+ \ \text{ resp. } \ -\theta(\Delta_1^+\setminus\Delta_I(\mathfrak{g},\mathfrak{c})) = \Delta_1^+\setminus\Delta_I(\mathfrak{g},\mathfrak{c}).$$

Denote by $x_0\in X$, and $x_1\in X$ the points defined by the pairs $(\mathfrak{c},-\Delta_0^+)$ and $(\mathfrak{c},-\Delta_1^+)$ respectively. Define the orbits $S_0 = G_{\mathbb{R}}\cdot x_0$, $S_1 = G_{\mathbb{R}}\cdot x_1$, and denote by

$$j_0 : S_0\hookrightarrow X,\ j_1 : S_1\hookrightarrow X$$

the inclusion maps. The definition of $\mathfrak{b}_{x_0}$ implies $\mathfrak{b}_{x_0}\cap\tau\mathfrak{b}_{x_0} = \mathfrak{c}$, hence

$$\dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}/\mathfrak{b}_{x_0}\cap\mathfrak{g}_{\mathbb{R}}) = \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{b}_{x_0}).$$

We conclude that the orbit S_0 is open in X. To the orbits S_0 and S_1 we associate standard sheaves

$$\mathcal{F}_0 = j_{0*}\mathbb{C}_{S_0}, \quad \mathcal{F}_1 = j_{1*}\mathbb{C}_{S_1}. \tag{2.7}$$

Here, $\mathbb{C}_{S_i}$ denotes the trivial local system on S_i, $i = 0, 1$. This notation for standard sheaves will be used in the rest of the paper.

Proposition 2.1. *Let $\mathfrak{c}_{\mathbb{R}}$ be a fundamental Cartan subalgebra, and $\zeta \in \mathfrak{c}'_{\mathbb{R}}$. Then*

$$\Theta(CC(\mathcal{F}_0), \lambda)(\zeta) = \frac{1}{\prod_{\alpha\in\Delta^+} \alpha_{x_0}(\zeta)} \sum_{w\in W(G_{\mathbb{R}},C_{\mathbb{R}})} (-1)^{l(w)} e^{w\lambda_{x_0}(\zeta)}.$$

Proof. Let $x \in X^{C_{\mathbb{R}}}$. Since there are no real roots in the root system $\Delta(\mathfrak{g}, \mathfrak{c})$ we can apply the same argument as in the proof of [12], Equation 7.37 to compute $d_{\zeta,x}$. We obtain

$$d_{\zeta,x} = 0 \quad \text{if} \quad x \notin S_0, \qquad d_{\zeta,x} = 1 \quad \text{if} \quad x \in S_0.$$

Write $x = g \cdot x_0$, where $g \in G_{\mathbb{R}}$. Then

$$\mathrm{Ad}(g^{-1})\mathfrak{c} \subset \mathfrak{b}_{x_0} \cap \tau\mathfrak{b}_{x_0} = \mathfrak{c}.$$

We conclude $\mathrm{Ad}(g)\mathfrak{c} = \mathfrak{c}$, hence $gC_{\mathbb{R}} \in W(G_{\mathbb{R}}, C_{\mathbb{R}})$, as desired. □

3. Weyl group modules

We begin the section be recalling some facts about Weyl group representations. When $U \subset \mathfrak{g}^*$ satisfies certain natural assumptions [9], Section 4.4 Rossmann defines a W-module structure on homology groups

$$H_*(\mu^{-1}(U), \mathbb{C}).$$

In particular, these assumptions are fulfilled in the following cases:

$$U = i\mathcal{N}^*_{\mathbb{R}},\ U = \overline{\mathcal{O}},\ U = \mathcal{O}, U = \{\nu\}\,.$$

Here we set $i\mathcal{N}^*_{\mathbb{R}} = i\mathfrak{g}^*_{\mathbb{R}} \cap \mathcal{N}^*$, $\mathcal{O} \subset i\mathcal{N}^*_{\mathbb{R}}$ is a $G_{\mathbb{R}}$-orbit, and $\nu \in \mathcal{N}^*$. Note that in the first case we have

$$\mu^{-1}(i\mathcal{N}^*_{\mathbb{R}}) = T^*_{G_{\mathbb{R}}} X\,.$$

Rossmann shows [9], Section 4.4, that inclusions of the orbit closures are compatible with W-module structure on homology groups. Moreover, if we set

$$\mathcal{N}^*_{k,\mathbb{R}} = \{G_{\mathbb{R}} \cdot \xi :\ \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot \xi \le k,\ \xi \in i\mathcal{N}^*_{\mathbb{R}}\}\,,$$

then we obtain an exhaustive filtration

$$\cdots \subset H_{2n}(\mu^{-1}(\mathcal{N}^*_{k,\mathbb{R}}),\mathbb{C}) \subset H_{2n}(\mu^{-1}(\mathcal{N}^*_{k+1,\mathbb{R}}),\mathbb{C}) \subset \cdots \subset H_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C})$$

by W-submodules. Rossmann computes the corresponding graded module and shows that:

$$H_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C}) \cong \sum_{\mathcal{O}\in i\mathcal{N}^*_{\mathbb{R}}/G_{\mathbb{R}}} H_{2n}(\mu^{-1}(\mathcal{O}),\mathbb{C}). \tag{3.1}$$

In the case $U = \{\nu\}$ denote by $C_G(\nu)$ the group of connected components of the centralizer of ν in G. Let $d = \dim_{\mathbb{C}} \mu^{-1}(\nu)$. Then $C_G(\nu)$ acts on $H_{2d}(\mu^{-1}(\nu),\mathbb{C})$ by permuting the irreducible components, and this action commutes with W-action. Hence

$$H_{2d}(\mu^{-1}(\nu),\mathbb{C})^{C_G(\nu)} \subset H_{2d}(\mu^{-1}(\nu),\mathbb{C})$$

is a W-submodule. In fact, W-module $H_{2d}(\mu^{-1}(\nu),\mathbb{C})^{C_G(\nu)}$ is irreducible [9], Theorem 4.5. This is the Springer representation associated to the orbit $G\cdot\nu$, and we denote the corresponding character by χ_ν. If V is a W-module and χ an irreducible character of W we denote by $[V:\chi]$ the multiplicity of χ in V. If $\mathcal{O}\subset i\mathcal{N}^*_{\mathbb{R}}$ is a $G_{\mathbb{R}}$-orbit Rossmann shows that [9], Section 4.4,

$$[H_{2n}(\mu^{-1}(\mathcal{O}),\mathbb{C}):\chi_\nu] = 1$$

if $\mathcal{O}\subset G\cdot\nu$, and

$$[H_{2n}(\mu^{-1}(\mathcal{O}),\mathbb{C}):\chi_\nu] = 0$$

otherwise. The preceding discussion implies

$$[H_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C})):\chi_0] = 1. \tag{3.2}$$

Next we review briefly the definition of intertwining operators. Given a real algebraic manifold Y we denote by $D(Y)$ the bounded derived category of sheaves (of complex vector spaces) on Y constructible for semialgebraic stratifications. For $w\in W$ denote by $Y_w\subset X\times X$ the variety of pairs of Borel subalgebras in the relative position w, and by

$$p_1,\, p_2 : Y_w \longrightarrow X$$

projections onto the first and second factor in $X\times X$. Then we define the intertwining functor attached to w by the formula:

$$I_w = Rp_{1*}p_2^*[l(w)] : D(X) \longrightarrow D(X),$$

One can show that I_w is an equivalence of categories. Moreover, the equivalences I_w induce an action of the Weyl group W on the K-group $K(D(X))$.

Since the maps p_1 and p_2 are $G_{\mathbb{R}}$-equivariant, the subspace $K(Sh_{G_{\mathbb{R}}}(X))$ is invariant for this W-action. It follows then from [11], Theorem 9.1 that the characteristic cycle map (2.1) is a homomorphism of W-modules.

Let V be a W-module, and π an irreducible representation of W. We shall denote by P_π the projection to the isotypical component of type π. Explicitly:

$$P_\pi : V \longrightarrow V, \;\; P_\pi(v) = \frac{\deg \pi}{|W|} \sum_{w \in W} \chi_\pi(w^{-1}) wv \,.$$

Next we explain the notation appearing in the following lemma. Denote by $T^*_X X \subset T^*X$ the zero section. We use the isomorphism $T^*_X X \simeq X$ and the complex structure on X to put the orientation on $T^*_X X$. Note that this is compatible with the orientation of the real conormal bundle on $T^*_X X$ [11],§2. We denote by

$$[T^*_X X] \in H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C}))$$

the corresponding fundamental cycle. Write sgn for the one-dimensional representation $w \mapsto (-1)^{l(w)}$, where $l(w)$ denotes the length of $w \in W$.

Lemma 3.1. *We have $P_{sgn}(H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C})) = \mathbb{C} \cdot [T^*_X X]$.*

Proof. Observe that $\mu^{-1}(0) = T^*_X X$, hence

$$H_{2n}(T^*_X X, \mathbb{C}) \simeq \mathbb{C} \cdot [T^*_X X]$$

is a W-submodule of $H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C})$. In view of (3.2) it will suffice to show $\chi_0 = \chi_{\mathrm{sgn}}$. One can check by a straightforward computation that $I_w(\mathbb{C}_X) = \mathbb{C}_X[l(w)]$. Since $CC(\mathbb{C}_X) = [T^*_X X]$, we obtain

$$w \cdot [T^*_X X] = CC(\mathbb{C}_X[l(w)]) = (-1)^{l(w)} [T^*_X X].$$

This implies the claim. □

Before stating the next result we recall one additional property of Rossmann's Weyl group action on $H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C})$. If $\Gamma \in H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C})$, $\lambda \in \mathfrak{h}^*$, and $w \in W$, then we have [8], Lemma 3.1

$$\Theta(w\Gamma, \lambda) = \Theta(\Gamma, w^{-1}\lambda) \qquad (3.3)$$

Theorem 3.2. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be standard sheaves defined in (2.7). The following identities hold in $H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C})$:*

$$\sum_{w \in W} (-1)^{l(w)} w \cdot CC(\mathcal{F}_0) = \sum_{w \in W} (-1)^{l(w)} w \cdot CC(\mathcal{F}_1) = |W(G_{\mathbb{R}}, C_{\mathbb{R}})| [T^*_X X].$$

Proof. It follows from [12], §9 that we can find $w_0 \in W$ such that

$$I_{w_0}\mathcal{F}_1 = \mathcal{F}_0[l(w_0)].$$

Now we use the fact that CC is a homomorphism of W-modules to obtain the first identity. To prove the second identity we fix $\zeta \in \mathfrak{c}'_{\mathbb{R}}$, and set $D = \prod_{\alpha\in\Delta^+} \alpha_{x_0}(\zeta)$. We use the same argument as in Proposition 2.1 to show

$$\Theta([T^*_X X], \lambda)(\zeta) = \Theta(CC(\mathbb{C}_X), \lambda)(\zeta) = \frac{1}{D} \sum_{w\in W} (-1)^{l(w)} e^{w\lambda_{x_0}(\zeta)}.$$

By Lemma 3.1 there exists $k \in \mathbb{C}$ such that

$$\sum_{w\in W} (-1)^{l(w)} w \cdot CC(\mathcal{F}_0) = k[T^*_X X].$$

This implies

$$\sum_{w\in W} (-1)^{l(w)} \Theta(w \cdot CC(\mathcal{F}_0), \lambda) = k\Theta([T^*_X X], \lambda).$$

We apply (3.3) and Proposition 2.1 to conclude

$$\begin{aligned}
\sum_{w\in W} (-1)^{l(w)} &\Theta(w \cdot CC(\mathcal{F}_0), \lambda)(\zeta) \\
&= \sum_{w\in W} (-1)^{l(w)} \Theta(CC(\mathcal{F}_0), w^{-1}\lambda)(\zeta) \\
&= \frac{1}{D} \sum_{w\in W} (-1)^{l(w)} \sum_{v\in W(G_{\mathbb{R}}, C_{\mathbb{R}})} (-1)^{l(v)} e^{vw^{-1}\lambda_{x_0}(\zeta)} \\
&= \frac{|W(G_{\mathbb{R}}, C_{\mathbb{R}})|}{D} \sum_{w\in W} (-1)^{l(w)} e^{w\lambda_{x_0}(\zeta)}.
\end{aligned}$$

It follows that $k = |W(G_{\mathbb{R}}, C_{\mathbb{R}})|$, and the proof is complete. □

Now we can explain how Theorem 3.2 can be used to obtain a formula for the coherent continuation of an induced representation. We shall restrict our consideration to the case of the trivial infinitesimal character. To simplify notation we write $\rho_1 = \rho_{x_1}$. Recall that x_1 defines a real parabolic subgroup $P_{\mathbb{R}} \subset G_{\mathbb{R}}$ [12], Equations 9.7(a)-(b), with Levi decomposition

$$P_{\mathbb{R}} = M_{\mathbb{R}} \cdot A_{\mathbb{R}} \cdot N_{\mathbb{R}},$$

where $M_{\mathbb{R}} \cdot A_{\mathbb{R}}$ is the centralizer of $\mathfrak{a}_{\mathbb{R}}$ in $G_{\mathbb{R}}$, and $N_{\mathbb{R}} \subset G_{\mathbb{R}}$ certain unipotent group. Note that in the present case the component group of

$P_{\mathbb{R}}$, denoted by F in loc.cit., is trivial. Denote by $\mathfrak{m}$ the complexified Lie algebra of $M_{\mathbb{R}}$. Write $\rho_{1,\mathfrak{t}}$ and $\rho_{1,\mathfrak{a}}$ for the restriction of ρ_1 to $\mathfrak{t}$ and $\mathfrak{a}$ respectively. Our choice of x_1 implies $\rho_{1,\mathfrak{t}} = \rho(\Delta^+(\mathfrak{m}))$, where $\Delta^+(\mathfrak{m}) = \Delta^+_{x_1} \cap \Delta(\mathfrak{m})$. Let $\pi_{\rho_{1,\mathfrak{t}}}$ be the discrete series representation of $M_{\mathbb{R}}$ defined by the Harish-Chandra parameter $\rho_{1,\mathfrak{t}}$. Denote by $I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}$ the normalized parabolic induction [12], Equation 9.13. In particular, we have an induced representation $I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}})$, and we denote the lift of the corresponding character to $\mathfrak{g}_{\mathbb{R}}$ via the exponential map by $\Theta_{\mathfrak{g}_{\mathbb{R}}}(I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}}))$. Then by [12], Proposition 9.17 we have

$$\Theta_{\mathfrak{g}_{\mathbb{R}}}(I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}})) = (-1)^q \Theta(CC(\mathcal{F}_1), \rho_1), \quad q = \frac{1}{2} \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m} \cap \mathfrak{k}.$$

Recall that the coherent continuation representation of the Weyl group is defined on the space of invariant eigendistributions on $\mathfrak{g}_{\mathbb{R}}$ with fixed infinitesimal character [8], §3. In the present situation, for $w \in W$, we have

$$w \cdot \Theta_{\mathfrak{g}_{\mathbb{R}}}(I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}})) = (-1)^q \Theta(CC(\mathcal{F}_1), w^{-1}\rho_1).$$

Denote by π_{triv} the trivial representation of $G_{\mathbb{R}}$, and by $\Theta_{\mathfrak{g}_{\mathbb{R}}}(\pi_{triv})$ the corresponding character on $\mathfrak{g}_{\mathbb{R}}$. The local expression for $\Theta([T^*_X X], \rho_1)$ (compare the proof of Proposition 2.1) implies that

$$\Theta_{\mathfrak{g}_{\mathbb{R}}}(\pi_{triv}) = \Theta([T^*_X X], \rho_1).$$

The next proposition, un unpublished result of Hecht and Schmid, summarizes the preceding discussion.

Proposition 3.3. *The induced module* $I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}})$ *associated to a fundamental Cartan subgroup* $C_{\mathbb{R}}$*, and the trivial character of* $C_{\mathbb{R}}$ *satisfies the following identity*

$$\sum_{w \in W} (-1)^{l(w)} w \cdot \Theta_{\mathfrak{g}_{\mathbb{R}}}(I^{G_{\mathbb{R}}}_{P_{\mathbb{R}}}(\pi_{\rho_{1,\mathfrak{t}}} \otimes e^{\rho_{1,\mathfrak{a}}})) = (-1)^q |W(G_{\mathbb{R}}, C_{\mathbb{R}})| \Theta_{\mathfrak{g}_{\mathbb{R}}}(\pi_{triv}).$$

4. **Limit formula**

Recall from the previous section, the choice was made of a fundamental Cartan subalgebra $\mathfrak{c} = \mathfrak{t} + \mathfrak{a}$, $x_0 \in X$, $x_1 \in X$, and $\mathfrak{m} + \mathfrak{a}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Note that the root system $\Delta(\mathfrak{m}, \mathfrak{t})$ naturally identifies with $\Delta_I(\mathfrak{g}, \mathfrak{c})$. The following theorem is proved in [12], §7-9. We point out that we work with dominant parameter, which results in a different sign in front of the orbital integral than in loc.cit.

Theorem 4.1. *Let $\lambda \in \mathfrak{h}'^*$ be such that $\lambda_{x_1} \in i\mathfrak{c}^*_{\mathbb{R}}$ and*

$$(\lambda_{x_1}, \alpha_{x_1}) > 0 \quad \textit{if} \quad \alpha_{x_1} \in \Delta(\mathfrak{m}, \mathfrak{t}) \cap \Delta^+_{x_1}.$$

Let $\phi \in C^\infty_c(\mathfrak{g}_{\mathbb{R}})$. Then the following formula holds

$$\frac{1}{(2\pi i)^n n!} \int_{CC(\mathcal{F}_1)} \mu^*_\lambda(\hat{\phi}\sigma^n_\lambda) = (-1)^q \int_{G_{\mathbb{R}} \cdot \lambda_{x_1}} \hat{\phi}\sigma^n_\lambda \,.$$

Our goal is to study the asymptotic behaviour of the distribution $\Theta(CC(\mathcal{F}_1), \lambda)$ when $\lambda \in \mathfrak{h}'^*$ approaches zero. Some additional results are needed for this analysis. Denote by $\Theta_{\mathcal{O}}$ the Fourier transform of the Liouville measure $m_{\mathcal{O}}$. In more details

$$\Theta_{\mathcal{O}}(\phi) = \frac{1}{(2\pi i)^k k!} \int_{\mathcal{O}} \hat{\phi}\sigma^k_{\mathcal{O}} \,, \quad 2k = \dim_{\mathbb{R}} \mathcal{O}, \, \phi \in C^\infty_c(\mathfrak{g}_{\mathbb{R}}).$$

Our computation will be based on the following simple formula:

$$\Theta([T^*_X X], \lambda) = (\int_X \tau^n_\lambda)\Theta_{\{0\}} + o(\lambda^n). \tag{4.1}$$

This formula is a special case of [8], Theorem 4.1. We remark that it can be also established by a straightforward calculation. The term $o(\lambda^n)$ can be described as follows. For any $\phi \in C^\infty_c(\mathfrak{g}_{\mathbb{R}})$, $o(\lambda^n)(\phi)$ is a holomorphic function of λ and

$$\lim_{t \to 0} \frac{o((t\lambda)^n)(\phi)}{t^n} = 0.$$

Denote by $\mathbb{C}[\mathfrak{h}]$ resp. $\mathbb{C}[\mathfrak{h}^*]$ the algebra of polynomial functions on $\mathfrak{h}$ resp. $\mathfrak{h}^*$. Write $S(\mathfrak{h})$ resp. $S(\mathfrak{h}^*)$ for the symmetric algebra of $\mathfrak{h}$ resp. $\mathfrak{h}^*$. Recall that we have canonical isomorphisms

$$\mathbb{C}[\mathfrak{h}] \cong S(\mathfrak{h}^*) \text{ and } \mathbb{C}[\mathfrak{h}^*] \cong S(\mathfrak{h}).$$

On the other hand the map

$$v \mapsto \partial(v), \quad \partial(v)f(\lambda) = \lim_{t \to 0}(f(\lambda + tv) - f(\lambda))/t, \quad \lambda, \, v \in \mathfrak{h}^*, \; f \in C^\infty(\mathfrak{h}^*)$$

extends to an isomorphism of $S(\mathfrak{h}^*)$ and the algebra $D(\mathfrak{h}^*)$ of differential operators on $\mathfrak{h}^*$ with constant coefficients. Thus we obtain an isomorphism of algebras

$$\mathbb{C}[\mathfrak{h}] \cong D(\mathfrak{h}^*), \;\; p \mapsto \partial(p), \, p \in \mathbb{C}[\mathfrak{h}].$$

Set $\Theta(\Gamma, \lambda_{x_1}) = \Theta(\Gamma, \lambda)$, and observe that the specialization $\tau^*_{x_1}$ defines an isomorphism $S(\mathfrak{h}^*) \simeq S(\mathfrak{c}^*)$, to be denoted $p \mapsto p_{x_1}$. With this notation we have

$$\partial(p_{x_1})\Theta(\Gamma, \lambda_{x_1}) = \partial(p)\Theta(\Gamma, \lambda).$$

The following lemma is stated in [8], Lemma 5.2. A more detailed discussion of this result can be found in [3].

Lemma 4.2. *Let* $\Gamma \in H_{2n}(T^*_{\mathbb{R}}X, \mathbb{C})$*,* $\lambda \in \mathfrak{h}^*$*,* $p \in \mathbb{C}[\mathfrak{h}]$ *and* $w \in W$*.*

(1) $\lim_{\lambda\to 0} \partial(p)\Theta(\Gamma, \lambda)$ *exists as a distribution on* $\mathfrak{g}_{\mathbb{R}}$*.*

(2) $\lim_{\lambda\to 0} \partial(w^{-1}p)\Theta(\Gamma, \lambda) = \lim_{\lambda\to 0} \partial(p)\Theta(w\Gamma, \lambda)$.

Define polynomials $\pi^+ \in \mathbb{C}[\mathfrak{h}]$ and $\omega^+ \in \mathbb{C}[\mathfrak{h}^*]$ by the formulas:

$$\pi^+ = \prod_{\alpha\in\Delta^+} \alpha, \quad \omega^+ = \prod_{\alpha\in\Delta^+} h_\alpha.$$

The following lemma can be deduced from the Weyl character formula [13], Section 4.14.

Lemma 4.3. $\partial(\pi^+)\omega^+ = |W| \prod_{\alpha\in\Delta^+}(\rho, \alpha)$*.*

Important role in our computation will play the formula for the integral of τ^n_λ over X. We refer to [14], §4 for a proof of the next lemma.

Lemma 4.4. *Let* $\lambda \in \mathfrak{h}^*$*. Then*

$$\frac{1}{(2\pi i)^n n!} \int_X \tau^n_\lambda = \prod_{\alpha\in\Delta^+} \frac{(\lambda, \alpha)}{(\rho, \alpha)}.$$

Now we can state and prove the main result of the paper.

Theorem 4.5. *Suppose* $G_{\mathbb{R}}$ *is a connected, linear, semisimple Lie group. Let* $\mathfrak{c}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ *be a fundamental Cartan subalgebra, and* $\lambda_1 \in i\mathfrak{c}'^*_{\mathbb{R}}$*. Let* m_{λ_1} *be the Liouville measure on* $G_{\mathbb{R}} \cdot \lambda_1$ *defined in (2.2). The following limit formula for the orbital measures holds*

$$\lim_{\lambda_1\to 0(i\mathfrak{c}'^*_{\mathbb{R}})} \partial(\pi^+)m_\lambda = (-1)^q |W(G_{\mathbb{R}}, C_{\mathbb{R}})| m_{\{0\}}.$$

Proof. We choose a positive system $\Delta'^+ \subset \Delta(\mathfrak{g}, \mathfrak{c})$ such that

$$\mathrm{Re}\,(\lambda_1, \beta) \geq 0 \quad \text{if} \quad \beta \in \Delta'^+.$$

By [10], Equation 6.16, we can find $\Delta_1^+ \subset \Delta(\mathfrak{g}, \mathfrak{c})$ satisfying the following conditions:

$$-\theta(\Delta_1^+ \setminus \Delta(\mathfrak{m}, \mathfrak{t})) = \Delta_1^+ \setminus \Delta(\mathfrak{m}, \mathfrak{t}), \quad \text{and} \quad \Delta_1^+ \cap \Delta(\mathfrak{m}, \mathfrak{t}) = \Delta'^+ \cap \Delta(\mathfrak{m}, \mathfrak{t}).$$

Let $x_1 \in X$ be the point determined by the pair $(\mathfrak{c}, -\Delta_1^+)$. Recall that $\mathcal{F}_1$ is the standard sheaf associated to the orbit $S_1 = G_{\mathbb{R}} \cdot x_1$. Set $\lambda = \tau_{x_1}^{*-1} \lambda_1$, and define

$$C = \left\{ \xi \in i\mathfrak{c}'^*_{\mathbb{R}} : (\xi, \alpha_{x_1}) > 0, \ \alpha_{x_1} \in \Delta_1^+ \cap \Delta(\mathfrak{m}, \mathfrak{t}) \right\}.$$

Now we apply $\lim_{\lambda \to 0} \partial(\pi^+)\Theta(\cdot, \lambda)$ to the identity from Theorem 3.2, and use Lemma 4.2, (3.3), and (4.1) to obtain

$$|W| \lim_{\lambda \to 0} \partial(\pi^+)\Theta(CC(\mathcal{F}_1), \lambda) = |W(G_{\mathbb{R}}, C_{\mathbb{R}})| \partial(\pi^+)(\int_X \tau_\lambda^n)\Theta_{\{0\}}.$$

Next we specialize $\lambda_{x_1} \in C$, and use Theorem 4.1, Lemma 4.3, Lemma 4.4. to conclude

$$\lim_{\lambda_1 \to 0(C)} \frac{1}{(2\pi i)^n n!} \partial(\pi_{x_1}^+) \int_{G_{\mathbb{R}} \cdot \lambda_1} \hat{\phi} \sigma_\lambda^n = (-1)^q |W(G_{\mathbb{R}}, C_{\mathbb{R}})| \Theta_{\{0\}}(\phi),$$

where $\phi \in C^\infty(\mathfrak{g}_{\mathbb{R}})$. The formula from the theorem follows now by taking the inverse Fourier transform. □

References

[1] D. Barbasch and D. Vogan, *Weyl group representations and nilpotent orbits*, Representations of Reductive Groups, Progr. Math. 40 (P. Trombi, ed.), Birkhäuser, Boston, 1982, pp. 21–32.

[2] J. Bernstein and V. Lunts, *Equivariant sheaves and functors, Lecture Notes in Mathematics 1578*, Springer-Verlag, Berlin, 1994.

[3] M. Božičević, *Limit formulas for groups with one conjugacy class of Cartan subgroups*, Ann. Inst. Fourier **58** (2008), 1213–1232.

[4] Harish-Chandra, *Fourier transform on a semisimple Lie algebra II*, Amer. J. Math. **79** (1957), 733–760.

[5] M. Kashiwara and P. Schapira, *Sheaves on Manifolds, Grundlehren Math. Wiss. 292*, Springer-Verlag, Berlin, 1990.

[6] M. Libine, *A localization argument for characters of reductive Lie groups*, J. Funct. Anal. **203** (2003), 197–236.

[7] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.

[8] W. Rossmann, *Nilpotent orbital integrals in a real semisimple Lie algebra and representations of the weyl groups*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progr. Math. 92 (A. Connes, M. Duflo, A. Joseph, and R. Rentschler, eds.), Birkhäuser, Boston, 1990, pp. 263–287.

[9] ———, *Picard-lefschetz theory for the coadjoint quotient of a semisimple Lie algebra*, Invent. Math. **121** (1995), 531–578.

[10] W. Schmid, *Construction and classification of irreducible Harish-Chandra modules*, Harmonic analysis on reductive groups, Progr. Math. 101 (W. Barker and P. Sally, eds.), Birkhäuser, Boston, 1991, pp. 235–275.

[11] W. Schmid and K. Vilonen, *Characteristic cycles of constructible sheaves*, Invent. Math. **124** (1996), 451–502.

[12] ———, *Two geometric character formulas for reductive Lie groups*, J. Amer. Math. Soc. **11** (1998), 799–867.

[13] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations, Graduate Texts in Math. 102*, Springer-Verlag, New York, 1984.

[14] M. Vergne, *Polynômes de Joseph et représentation de Springer*, Ann. Sci. École Norm. Sup. (4) **23** (1990), 543–562.

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