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Abstract

Properties of the so called θ_o-complete topological spaces are investigated. Also, necessary and sufficient conditions are given so that the space $C(X,E)$ of all continuous functions, from a zero-dimensional topological space X to a non-Archimedean locally convex space E, equipped with the topology of uniform convergence on the compact subsets of X to be polarly barrelled or polarly quasi-barrelled.

Introduction

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X,E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. We will denote by $C_b(X,E)$ (resp. by $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. The dual space of $C_{rc}(X,E)$, under the topology t_u of uniform convergence, is a space $M(X,E')$ of finitely-additive E'-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of X. Some subspaces of $M(X,E')$ turn out to be the duals of $C(X,E)$ or of $C_b(X,E)$ under certain locally convex topologies.
In section 2 of this paper, we give some results about the space $M(X,E')$. The notion of a θ_0-complete topological space was given in [2]. In section 3 we study some of the properties of θ_o-complete spaces. Among other results, we prove that a Hausdorff zero-dimensional space is θ_o-complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces. In section 4, we give necessary and sufficient conditions for the space $C(X,E)$, equipped with the topology of uniform convergence on the compact subsets of X, to be polarly barrelled or polarly quasi-barrelled,

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1. Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [9]). Unless it is stated explicitly otherwise, X will be a Hausdorff zero-dimensional topological space , E a Hausdorff locally convex space and $cs(E)$ the set of all continuous seminorms on E. The space of all $\mathbb{K}$-valued linear maps on E is denoted by $E^\star$, while E' denotes the topological dual of E. A seminorm p, on a vector space G over $\mathbb{K}$, is called polar if $p = \sup\{|f| : f \in G^\star, |f| \leq p\}$. A locally convex space G is called polar if its topology is generated by a family of polar seminorms. A subset A of G is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \leq 1$. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [3]) and by $\upsilon_o X$ the $\mathbf{N}$-repletion of X, where $\mathbf{N}$ is the set of natural numbers. We will let $C(X,E)$ denote the space of all continuous E-valued functions on X and $C_b(X,E)$ (resp. $C_{rc}(X,E)$) the space of all $f \in C(X,E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. In case $E = \mathbb{K}$, we will simply write $C(X), C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A. Also, for $X \subset Y \subset \beta_o X$, we denote by $\bar{B}^Y$ the closure of B in Y. If $f \in E^X, p$ a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X,E)$ (see [4]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for every $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if $\notin Y$.

Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X \setminus X$. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X,E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X,E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H, p \in cs(E)$.

The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology β. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$, which are disjoint from X, we get the topology β_1. Recall that a $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$, for some $g \in C(\beta_o X)$. We get the topologies β_u and β'_u replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i\in I}$ of X such that Q is disjoint from each $\overline{A_i}^{\beta_o X}$. Now β_u is the inductive limit of the topologies $\beta_Q,\ \ Q \in \Omega_u$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω_u, is denoted by $\beta_{u,p}$, while β'_u is the projective limit of the topologies $\beta_{u,p},\ \ p \in cs(E)$. For the definition of the topology β_e on $C_b(X)$ we refer to [7].

Let now $K(X)$ be the algebra of all clopen subsets of X. We denote by $M(X, E')$ the space of all finitely-additive E'-additive measures m on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of E'. For each such m, there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0,\quad A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X, E')$. If $m \in M_p(X, E')$, then for $x \in X$ we define

$$N_{m,p}(x) = \inf\{m_p(V) : x \in V \in K(X)\}.$$

In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded $\mathbb{K}$-valued measures on $K(X)$. An element m of $M(X)$ is called τ-additive if $m(V_\delta) \to 0$ for each decreasing net (V_δ) of clopen subsets of X with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all τ-additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all σ-additive m, i.e. those m with $m(V_n) \to 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X, E')$ and $s \in E$, we denote by ms the element of $M(X)$ defined by $(ms)(V) = m(V)s$.

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X, E')$. For a non-empty clopen subset A of X, let $\mathcal{D}_\mathcal{A}$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_k \in A_k$. We make $\mathcal{D}_\mathcal{A}$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For an $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_\mathcal{A}$ and $m \in M(X, E')$, we define

$$\omega_\alpha(f,m)=\sum_{k=1}^{n} m(A_k)f(x_k).$$

If the limit $\lim \omega_\alpha(f,m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f\,dm$. We define the integral over the empty set to be 0. For $A=X$, we write simply $\int f\,dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every clopen subset A of X and $\int_A f\,dm=\int \chi_A f\,dm$. If τ_u is the topology of uniform convergence, then every $m\in M(X,E')$ defines a τ_u-continuous linear functional ϕ_m on $C_{rc}(X,E)$, $\phi_m(f)=\int f\,dm$. Also every $\phi\in(C_{rc}(X,E),\tau_u)'$ is given in this way by some $m\in M(X,E')$.

For $p\in cs(E)$, we denote by $M_{t,p}(X,E')$ the space of all $m\in M_p(X,E')$ for which m_p is tight, i.e. for each $\epsilon>0$, there exists a compact subset Y of X such that $m_p(A)<\epsilon$ if the clopen set A is disjoint from Y. Let

$$M_t(X,E')=\bigcup_{p\in cs(E} M_{t,p}(X,E').$$

Every $m\in M_{t,p}(X,E')$ defines a β_0-continuous linear functional u_m on $C_b(X,E)$,

$u_m(f)=\int f\,dm$. The map $m\mapsto u_m$, from $M_t(X,E')$ to $(C_b(X,E),\beta_o)'$, is an algebraic isomorphism. For $m\in M_\tau(X)$ and $f\in\mathbb{K}^X$, we will denote by $(VR)\int f\,dm$ the integral of f , with respect to m, as it is defined in [9]. We will call $(VR)\int f\,dm$ the (VR)-integral of f.

For all unexplained terms on locally convex spaces, we refer to [8] and [9].

2. Some results on $M(X,E')$

Theorem 2.1. *Let $m\in M(X,E')$ be such that $ms\in M_\tau(X)$, for all $s\in E$, and let $p\in cs(E)$ with $\|m\|_p<\infty$. Then :*

(1) *$m_p(V)=\sup_{x\in V} N_{m,p}(x)$ for every $V\in K(X)$.*

(2) *The set*

$$supp(m)=\bigcap\{V\in K(X): m_p(V^c)=0\}$$

is the smallest of all closed support sets for m.

(3) $supp(m) = \overline{\{x : N_{m,p}(x) \neq 0\}}$.

(4) *If* V *is a clopen set contained in the union of a family* $(V_i)_{i\in I}$ *of clopen sets, then*

$$m_p(V) \leq \sup\{m_p(V_i) : i \in I\}.$$

Proof: (1). If $x \in V$, then $N_{m,p}(x) \leq m_p(V)$ and so

$$m_p(V) \geq \alpha = \sup_{x\in V} N_{m,p}(x).$$

On the other hand, let $m_p(V) > d$. There exists a clopen set W, contained in V, and $s \in E$ with $|m(W)s|/p(s) > d$. Let $\mu = ms \in M_\tau(X)$. Then

$$|\mu|(W) = \sup_{x\in W} N_\mu(x).$$

Let $x \in W$ be such that $N_\mu(x) > d \cdot p(s)$. Now $N_{m,p}(x) \geq d$. In fact, assume the contrary and let Z be a clopen neighborhood of x contained in W and such that $m_p(Z) < d$. Now

$$N_\mu(x) \leq |\mu|(Z) = \sup\{|m(Y)s| : Z \supset Y \in K(X)\} \leq p(s)\cdot m_p(Z) \leq d\cdot p(s).$$

This contradiction proves (1).
(2).

$$X \setminus supp(m) = \bigcup\{W \in K(X) \colon m_p(W) = 0\}.$$

Let $V \in K(X)$ be disjoint from $supp(m)$. For each $x \in V$, there exists $W \in K(X)$, with $x \in W$ and $m_p(W) = 0$ and so $N_{m,p}(x) = 0$. It follows that

$$m_p(V) = \sup_{x\in V} N_{m,p}(x) = 0,$$

which proves that $supp(m)$ is a support set for m. On the other hand, let Y be a closed support set for m. There exists a decreasing net (V_δ) of clopen sets with $Y = \bigcap V_\delta$. Let $W \in K(X)$ be disjoint from Y. For each clopen set V contained in W and each $s \in E$, we have $V \cap V_\delta \downarrow \emptyset$ and so $\lim_\delta (ms)(V \cap V_\delta) = 0$. Since V_δ^c is disjoint from Y, we have $m(V_\delta^c) = 0$ and so $m(V) = m(V_\delta \cap V)$, which implies that $m(V)s = 0$, for all $s \in E$, i.e. $m(V) = 0$, and hence $m_p(W) = 0$. Therefore $supp(m) \subset W^c$. Taking V_δ^c in place of W, we get that $supp(m) \subset \bigcap V_\delta = Y$, which proves (2).
(3) Let $G = \overline{x : N_{m,p}(x) \neq 0\}}$. If $V \in K(X)$ is disjoint from G, then

$$m_p(V) = \sup_{x\in V} N_{m,p}(x) = 0,$$

and so $supp(m) \subset V^c$, which implies that $supp(m) \subset G$. On the other hand, let $x \notin supp(m)$. There exists a clopen neighborhood W of x disjoint from $supp(m)$. Since $supp(m)$ is a support set for m, we have that $m_p(W) = 0$ and thus $N_{m,p} = 0$ on W, which proves that $x \notin G$. Thus G is contained in $supp(m)$ and (3) follows.
(4). Let $m_p(V) > \alpha > 0$. There exists a clopen set A contained in V and $s \in E$ such that $|m(A)s|/p(s) > \alpha$. If $\mu = ms \in M_\tau(X)$, then $|\mu|(V) \geq |m(A)s| > \alpha \cdot p(s)$. In view of [9, p. 250] there exists an i such that $m_p(V_i) \geq |\mu|(V_i)/p(s) > \alpha$, which clearly completes the proof.

Theorem 2.2. *Let $m \in M(X, E')$ be such that $ms \in M_\sigma(X)$ for alll $s \in E$ (this in particular holds if $m \in M_\sigma(X, E')$). Let $p \in cs(E)$ be such that $m_p(X) < \infty$. If a clopen set V is contained in the union of a sequence (V_n) of clopen sets, then $m_p(V) \leq \sup_n m_p(V_n)$.*

Proof : We show first that, for $\mu \in M_\sigma(X)$, then there exists an n with $|\mu|(V) \leq |\mu|(V_n)$. In fact, this is clearly true if $|\mu|(V) = 0$. Assume that $|\mu|(V) > 0$ and let $W_n = \bigcup_1^N V_k$. Since $W_n^c \cap V \downarrow \emptyset$, there exists n such that $|\mu|(V \cap W_n^c) < |\mu|(V)$. Since $V \subset (V \bigcap W_n^c) \bigcup W_n$, it follows that

$$|\mu|(V) \leq |\mu|(W_n) = \max_{1 \leq k \leq n} |\mu|(V_k),$$

and the claim follows for μ. Suppose now that $m_p(V) > r > 0$. There exists a clopen subset W of V and $s \in E$ such that $|m(W)s| > r \cdot p(s)$. Let $\mu = ms$. Then $\mu \in M_\sigma(X)$ and $|\mu|(V) \geq |m(W)s| > r \cdot p(s)$. By the first part of the proof, there exists an n such that $|\mu|(V_n) > r \cdot p(s)$. Hence, there exists a clopen subset D of V_n such that $|\mu(D)| > r \cdot p(s)$. Now $|m|_p(V_n) \geq |m(D)s|/p(s) > r$, which completes the proof.

For $X \subset Y \subset \beta_o X$, and $m \in M(X)$, we denote by m^Y the element of $M(Y)$ defined by $m^Y(V) = m(V \cap X)$. We denote by m^{υ_o} and m^{β_o} the m^Y for $Y = \upsilon_o X$ and $Y = \beta_o X$, respectively.
We have the following easily established

Theorem 2.3. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. Then :*

(1) $supp(m^{\beta_o}) = \overline{supp(m)}^{\beta_o X}$.

(2) $supp(m) = supp(m^{\beta_o}) \cap X$.

(3) *If m has compact support, then supp*$(m) =$ *supp*(m^{β_o}).

Theorem 2.4. *Let* $m \in M_p(X, E')$ *and* $\mu = m^{\beta_o}$. *The following are equivalent:*

(1) *supp*$(\mu) \subset \upsilon_o X$.

(2) *If* $V_n \downarrow \emptyset$, *then there exists an* n_o *such that* $m(V_n) = 0$ *for every* $n \geq n_o$.

(3) *If* $V_n \downarrow \emptyset$, *then there exists an* n *such that* $m(V) = 0$ *for every clopen set* V *contained in* V_n.

(4) *For every* $Z \in \Omega_1$ *there exists a clopen subset* A *on* $\beta_o X$ *disjoint from* Z *and such that supp*$(\mu) \subset A$.

(5) *If* $V_n \downarrow \emptyset$, *then there exists an* n *such that* $m_p(V_n) = 0$.

Proof: (1) $\Rightarrow$ (2). If $V_n \downarrow \emptyset$, then the set $\bigcap \overline{V_n}^{\beta_o X}$ is disjoint from $\upsilon_o X$ and so *supp*$(\mu) \subset \bigcup_n \overline{V_n^c}^{\beta_o X}$. In view of the compactness of *supp*(μ), there exists an n_o with *supp*$(\mu) \subset \overline{V_{n_o}^c}^{\beta_o X}$. If now $n \geq n_o$, then $m(V_n) = 0$.
(2) $\Rightarrow$ (3). Let $V_n \downarrow \emptyset$ and suppose that, for each n, there exists a clopen subset A of V_n such that $m(A) \neq 0$.
Claim. For each n, there exists $k > n$ and a clopen set B with $V_k \subset B \subset V_n$ and $m(B) \neq 0$. Indeed there exists a clopen subset A of V_n such that $m(A) \neq 0$. For each k, let $B_k = V_k \cap A$, $D_k = V_k \setminus B_k$. Then $D_k \downarrow \emptyset$. By our hypothesis, there exists $k > n$ such that $m(D_k) = 0$. Let $B = A \cup D_k$. Then $V_k \subset B \subset V_n$. Since A and D_k are disjoint, we have that $m(B) = m(A) \neq 0$ and the claim follows. By induction, we choose $n_1 = 1 < n_2 < \ldots$ and clopen sets B_k such that $V_{n_{k+1}} \subset B_k \subset V_{n_k}$ and $m(B_k) \neq 0$. Since $B_k \downarrow \emptyset$ and $m(B_k) \neq 0$ for every k, we arrived at a contradiction.
(3) $\Rightarrow$ (4). Let $Z \in \Omega_1$. There exists a decreasing sequence (V_n) of clopen sets with $Z = \bigcap \overline{V_n}^{\beta_o X}$. By our hypothesis, there exists an n suich that $m(V) = 0$ for each clopen subset V of V_n. Now it suffices to take $A = \overline{V_n^c}^{\beta_o X}$.
(4) $\Rightarrow$ (1). Let $z \in \beta_o X \setminus \upsilon_o X$. There exists a decreasing sequence (V_n) of clopen sets with $z \in Z = \bigcap \overline{V_n}^{\beta_o X}$. Clearly $Z \in \Omega_1$. Thus, there exists a

clopen subset A of $\beta_o X$ disjoint from Z and containing $supp(\mu)$. Hence z is not in $supp(\mu)$.
(3) $\Rightarrow$ (5). It is trivial.
(5) $\Rightarrow$ (1). Let $z \in \beta_o X \setminus \upsilon_o X$. There exists a decreasing sequence (V_n) of clopen sets with $z \in Z = \bigcap \overline{V_n}^{\beta_o X}$. Let n be such that $m_p(V_n) = 0$. If $G = \overline{V_n}^{\beta_o X}$, then $\mu_p(G) = 0$ and so $supp(\mu) \subset \beta_o X \setminus G$, which implies that $z \notin supp(\mu)$. This completes the proof.

Theorem 2.5. *For an $m \in M_p(X, E')$, the following are equivalent :*

(1) *m has a compact support, i.e. $m \in M_c(X, E')$.*

(2) *$supp(m^{\beta_o}) \subset X$.*

(3) *If $V_\delta \downarrow \emptyset$, then there exists a δ_o such that $m(V_\delta) = 0$ for all $\delta \geq \delta_o$.,*

(4) *If $V_\delta \downarrow \emptyset$, then there exists a δ such $m(V) = 0$ for each clopen subset V of V_δ.*

(5) *If $H \in \Omega$, then there exists a clopen subset A of $\beta_o X$, disjoint from H and containing $supp(m^{\beta_o})$.*

(6) *If $V_\delta \downarrow \emptyset$, then there exists a δ such that $m_p(V_\delta) = 0$.*

Proof : In view of Theorem 2.3, (1) implies (2).
(2) $\Rightarrow$ (3). Let $V_\delta \downarrow \emptyset$. By the compactness of $supp(m_o^\beta)$, there exists δ_o such that $supp(m^{\beta_o}) \subset V_{\delta_o}^c$ and so $m(V_\delta) = 0$ for $\delta \geq \delta_o$.
(3) $\Rightarrow$ (4). Let $V_\delta \downarrow \emptyset$ and suppose that, for each δ, there exists a clopen subset V of V_δ with $m(V) \neq 0$.
Claim: For each δ there exist $\gamma \geq \delta$ and a clopen set A such that $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. In fact, there exists a clopen subset G of V_δ with $m(G) \neq 0$. For each γ, let $Z_\gamma = V_\gamma \cap G, W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists $\gamma \geq \delta$ with $m(V_\gamma) = 0$. Let $A = G \cup W_\gamma$. Since the sets G and W_γ are disjoint, we have that $m(A) = m(G) \neq 0$. Since $V_\gamma \subset A \subset V_\delta$, the claim follows.
Let now $\mathcal{F}$ be the family of all clopen subsets A of X with the following property: There are γ, δ, with $\gamma \geq \delta$, $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. Since $\mathcal{F} \downarrow \emptyset$, we got a contradiction.
(4) $\Rightarrow$ (5). If $H \in \Omega$, then there exists a decreasing net (V_δ) of clopen

subsets of X with $\bigcap \overline{V_\delta}^{\beta_o X} = H$. Since $V_\delta \downarrow \emptyset$, there exists δ such that $m(V) = 0$ for each clopen subset V of V_δ. Now it suffices to take $A = \overline{V_\delta^c}^{\beta_o X}$.

(5) $\Rightarrow$ (1). Let $z \in \beta_o X \setminus X$. By (5), there exists a clopen subset A of $\beta_o X$ containing $supp(m^{\beta_o})$ and not containing z.

(4) $\Rightarrow$ (6). It is trivial.

(6) $\Rightarrow$ (2). Let $z \in \beta_o X \setminus X$. There exists a decreasing net (V_δ) of clopen sets with $\{z\} = \bigcap \overline{V_\delta}^{\beta_o X}$. Let δ be such that $m_p(V_\delta) = 0$. If $\mu = m^{\beta_o}$, then $\mu_p(\overline{V_\delta}^{\beta_o X}) = m_p(V_\delta) = 0$ and so $supp(\mu)$ is disjoint from the closure of V_δ in $\beta_o X$, which implies that $z \notin supp(\mu)$.

This completes the proof.

3. θ_o-Complete Spaces

Recall that $\theta_o X$ is the set of all $z \in \beta_o X$ with the following property: For each clopen partition (V_i) of X there exists i such that $z \in \overline{V_i}^{\beta_o X}$ (see [2]). By [2, Lemma 4.1] we have $X \subset \theta_o X \subset \upsilon_o X$. For each clopen partition $\alpha = (V_i)_{i \in I}$ of X, let

$$W_\alpha = \bigcup_{i \in I} V_i \times V_i.$$

Then the family of all W_α, α a clopen partition of X, is a base for a uniformity $\mathcal{U}_c = \mathcal{U}_c^X$, compatible with the topology of X, and $(\theta_o X, \mathcal{U}_c^{\theta_o X})$ coincides with the completion of $(X, \mathcal{U}_c)$. We will say that X is θ_o-complete iff $X = \theta_o X$. As it is shown in [2], if Y is a θ_o-complete and $f : X \to Y$ is a continuous function, then f has a continuous extension $f^{\theta_o} : \theta_o X \to Y$. A subset A of X is called bounding if every $f \in C(X)$ is bounded on A. Note that several authors use the term bounded set instead of bounding. But in this paper we will use the term bounding to distinguish from the notion of a bounded set in a topological vector space. A set $A \subset X$ is bounding iff $\overline{A}^{\upsilon_o X}$ is compact. In this case (as it is shown in [2, Theorem 4.6]) we have that $\overline{A}^{\theta_o X} = \overline{A}^{\upsilon_o X} = \overline{A}^{\beta_o X}$. Clearly a continuous image of a bounding set is bounding. Let us say that a family $\mathcal{F}$ of subsets of X is finite on a subset A of X if the family $\{f \in \mathcal{F} : F \cap A \neq \emptyset\}$ is finite. We have the following easily established

Lemma 3.1. *For a subset A of X, the following are equivalent :*

(1) *A is bounding.*

(2) *Every continuous real-valued function on X is bounded on A.*

(3) *Every locally finite family of open subsets of X is finite on A.*

(4) *Every locally finite family of clopen subsets of X is finite on A.*

By [1, Theorem 4.6] every ultraparacompact space (and hence every ultrametrizable space) is θ_o-complete.

Theorem 3.2. *Every complete Hausdorff locally convex space E is θ_o-complete.*

Proof: Let $\mathcal{U}$ be the usual uniformity on E, i.e. the uniformity having as a base the family of all sets of the form

$$W_{p,\epsilon} = \{(x, y) : p(x - y) \leq \epsilon\},\ p \in cs(E),\ \epsilon > 0.$$

Given $W_{p,\epsilon}$, we consider the clopen partition $\alpha = (V_i)_{i\in I}$ of E generated by the equivalence relation $x \sim y$ iff $p(x - y) \leq \epsilon$. Then $W_{p,\epsilon} = W_\alpha$ and so $\mathcal{U}$ is coarser that $\mathcal{U}_c$. Since $(E,\mathcal{U})$ is complete and $\mathcal{U}_c$ is compatible with the topology of E, it follows that $(E,\mathcal{U}_c)$ is complete and the result follows.

Corollary 3.3. *A subset B, of a complete Hausdorff locally convex space E, is bounding iff it is totally bounded.*

Proof: If B is bounding, then $\overline{B} = \overline{B}^{\theta_o E}$ is compact and hence totally bounded, which implies that B is totally bounded. Conversely, if B is totally bounded, then $\overline{B}$ is totally bounded. Thus $\overline{B}$ is compact and hence B is bounding.

Theorem 3.4. *If G is a locally convex space (not necessarily Hausdorff), then every bounding subset A of G is totally bounded.*

Proof: Assume first that G is Hausdorff. Let $\hat{G}$ be the completion of G. The closure B of A in $\hat{G}$ is bounding and hence B is totally bounded, which implies that A is totally bounded. If G is not Hausdorff, we consider the quotient space $F = G/\overline{\{0\}}$ and let $u : G \to F$ be the quotient map. Since u is continuous, the set $u(A)$ is bounding, and hence totally bounded, in F. Let now V be a convex neighborhood of zero in G. Then, $u(V)$

is a neighborhood of zero in F. Let S be finite subset of A such that $u(A) \subset u(S) + u(V)$. But then

$$A \subset S + V + \overline{\{0\}} \subset S + V + V = S + V,$$

which proves that A is totally bounded.

Theorem 3.5. *We have:*

(1) *Closed subspaces of θ_o-complete spaces are θ_o-complete.*

(2) *If $X = \prod X_i$, with $X_i \neq \emptyset$ for all i, then X is θ_o-complete iff each X_i is θ_o-complete .*

(3) *If $(Y_i)_{i\in I}$ is a family of θ_o-complete subspaces of X, then $Y = \bigcap Y_i$ is θ_o-complete.*

(4) *$\theta_o X$ is the smallest of all θ_o-complete subspaces of $\beta_o X$ which contain X.*

Proof: (1). Let Z be a closed subspace of a θ_o-complete space X and let (x_δ) be a $\mathcal{U}_c^Z$-Cauchy net in Z. Then (x_δ) is $\mathcal{U}_c^X$-Cauchy and hence $x_\delta \to x \in X$. Moreover, $x \in Z$ since Z is closed.
(2). Each X_i is homeomorphic to a closed subspace of X. Thus X_i is θ_o-complete if X is θ_o-complete. Conversely, suppose that each X_i is θ_o-complete. If (x^δ) is a $\mathcal{U}_c^X$-Cauchy net, then (x_i^δ) is a $\mathcal{U}_c^{X_i}$-Cauchy net in X_i and hence $x_i^\delta \to x_i \in X_i$. If $x = (x_i)$, then $x^\delta \to x$, which proves that $(X,\mathcal{U}_c)$ is complete.
(3). Let $X = \prod Y_i$ and consider the map $f : Y \to X$, $\quad f(x)_i = x$ for all i. Then $f : Y \to f(Y) = D$ is a homeomorphism. Also D is a closed subspace of X. Since X is θ_o-complete, it follows that D is θ_o-complete and hence Y is θ_o-complete.
(4). Since $\theta_o X$ is θ_o-complete (by [2, Theorem 4.9]) and $X \subset \theta_o X \subset \beta_o X$, the result follows from (3).

Theorem 3.6. *For a point $z \in \beta_o X$, the following are equivalent :*

(1) *$z \in \theta_o X$.*

(2) *If Y is a Hausdorff ultraparacompact space and $f : X \to Y$ continuous, then $f^{\beta_o}(z) \in Y$, where $f^{\beta_o} : \beta_o X \to \beta_o Y$ is the continuous extension of f.*

(3) *For every ultrametric space Y and every $f : X \to Y$ continuous, we have that $f^{\beta_o}(z) \in Y$.*

Proof: (1) $\Rightarrow$ (2). Since $\theta_o Y = Y$, the result follows from [2, Theorem 4.4].
(2) $\Rightarrow$ (3). It is trivial.
(3) $\Rightarrow$ (1). Assume that $z \notin \theta_o X$. Then, there exists a clopen partition (A_i) of X such that $z \notin \bigcup_i \overline{A_i}^{\beta_o X}$. Let $f_i = \chi_{A_i}$ and define

$$d : X \times X \to \mathbf{R}, \quad d(x,y) = \sup_i |f_i(x) - f_i(y)|.$$

Then d is a continuous ultrapseudometric on X. Let $Y = X_d$ be the corresponding ultrametric space and let $\pi : X \to Y_d$ be the quotient map, $x \mapsto \tilde{x}_d = \tilde{x}$. Since π is continuous, there exists (by (3)) an $x \in X$ such that $\pi^{\beta_o}(z) = \tilde{x}_d$. Let (x_δ) be a net in X converging to z. Then $\tilde{x_\delta} = \pi^{\beta_o}(x_\delta) \to \pi^{\beta_o}(z) = \tilde{x}$, and so $d(x_\delta, x) \to 0$. If $x \in A_i$, then $|f_i(x_\delta) - 1| \to 0$, and so there exists δ_o such that $x_\delta \in A_i$ when $\delta \geq \delta_o$. But then $z \in \overline{A_i}^{\beta_o X}$, a contradiction. This completes the proof.

Theorem 3.7. *Let X be a dense subspace of a Hausdorff zero-dimensional space Y. The following are equivalent :*

(1) *$Y \subset \theta_o X$ (more precisely, Y is homeomorphic to a subspace of $\theta_o X$).*

(2) *Each continuous function, from X to any ultrametric space Z, has a continuous extension to all of Y.*

Proof: (1) implies (2) by the preceding Theorem.
(2) $\Rightarrow$ (1). We will prove first that, for each clopen subset V of X, we have that $\overline{V}^Y \cap \overline{V^c}^Y = \emptyset$, and so $\overline{V}^Y$ is clopen in Y. Indeed, define

$$d : X \times X \to \mathbf{R}, \quad d(x,y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\},$$

where $f_1 = \chi_V, f_2 = \chi_{V^c}$. Then d is a continuous ultrapseudometric on X. Let $\pi : X \to X_d$ be the quotient map. By our hypothesis, there exists a continuous extension $h : Y \to X_d$ of π. Suppose that $z \in \overline{V}^Y \cap \overline{V^c}^Y$. There are nets $(x_\delta), (y_\gamma)$, in V, V^c respectively, such that $x_\delta \to z$, and $y_\gamma \to z$. Let $\tilde{d}$ be the ultrametric of X_d and let δ_o, γ_o be such that

$$\tilde{d}(\pi(x_\delta), h(z)) < 1 \quad \textit{and} \quad \tilde{d}(\pi(x_\gamma), h(z)) < 1$$

when $\delta \geq \delta_o, \gamma \geq \gamma_o$. Now

$$d(x_{\delta_o}, y_{\gamma_o}) = \tilde{d}(\pi(x_{\delta_o}), \pi(y_{\delta_o})) < 1,$$

a contradiction. Thus $\overline{V}^Y$ is clopen in Y. If $A = \overline{V}^Y, B = \overline{V^c}^Y$, then

$$\overline{A}^{\beta_o Y} \bigcap \overline{B}^{\beta_o Y} = \overline{V}^{\beta_o Y} \bigcap \overline{V^c}^{\beta_o Y} = \emptyset.$$

This, being true for each clopen subset V of X, implies that $\beta_o X = \beta_o Y$ and so $X \subset Y \subset \beta_o Y = \beta_o X$. Now our hypothesis (2) and the preceding Theorem imply that $Y \subset \theta_o X$, and the result follows.

Theorem 3.8. *For each continuous ultrapseudometric d on X, there exists a continuous ultrapseudometric d^{θ_o} on $\theta_o X$ which is an extension of d. Moreover, d^{θ_o} is the unique continuous extension of d.*

Proof: Consider the ultrametric space X_d and let $\tilde{d}$ be its ultrametric. Let h be the coninuous extension of the quotient map $\pi : X \to X_d$ to all of $\theta_o X$. Define

$$d^{\theta_o} : \theta_o X \times \theta_o X \to \mathbf{R},\ d^{\theta_o}(y, z) = \tilde{d}(h(y), h(z)).$$

It is easy to see that d^{θ_o} is a continuous ultrapseudometric which is an extension of d. Finally, let ϱ be any continuous ultrapseudometric on $\theta_o X$, which is an extension of d, and let $y, z \in \theta_o X$. There are nets $(y_\delta)_{\delta \in \Delta}$, $(z_\gamma)_{\gamma \in \Gamma})$ in X which convergence to y, z, respectively. Let $\Phi = \Delta \times \Gamma$ and consider on Φ the order $(\delta_1, \gamma_1) \geq (\delta_2, \gamma_2)$ iff $\delta_1 \geq \delta_2$ and $\gamma_1 \geq \gamma_2$. For $\phi = (\delta, \gamma) \in \Phi$, we let $a_\phi = y_\delta$, $b_\phi = z_\gamma$. Then $a_\phi \to y$, $b_\phi \to z$. Thus

$$\varrho(y, z) = \lim \varrho(a_\phi, b_\phi) = \lim \tilde{d}(h(a_\phi), h(b_\phi)) \tag{3.1}$$

$$= \lim d^{\theta_o}(a_\phi, b_\phi) = d^{\theta_o}(y, z) \tag{3.2}$$

and hence $\varrho = d^{\theta_o}$, which completes the proof.

Theorem 3.9. *Let (H_n) be a sequence of equicontinuous subsets of $C(X)$. If $z \in \theta_o X$, then there exists $x \in X$ such that $f^{\theta_o}(z) = f(x)$ for all $f \in \bigcup H_n = H$.*

Proof: Define

$$d : X^2 \to \mathbf{R}, \quad d(x, y) = \max_n \min\{1/n, \sup_{f \in H_n} |f(x) - f(y)|\}.$$

Then d is a continuous ultrapseudometric on X. Take $Y = X_d$ and let $\pi : X \to Y$ be the quotient map. Then $\pi^{\beta_o}(z) = u \in Y$. Choose $x \in$

X with $\pi(x) = u$, and let (x_δ) be a net in X converging to z in $\beta_o X$. Now $f(x_\delta) \to f^{\beta_o}(z)$ for all $f \in H$. Since $\pi(x_\delta) \to \pi(x)$, we have that $d(x_\delta, x) \to 0$, and so $|f(x_\delta) - f(x)| \to 0$ for all $f \in H$. Thus, for $f \in H$, we have $f(x) = \lim f(x_\delta) = f^{\beta_o}(z)$, and the result follows.

Theorem 3.10. *If $H \subset C(X)$ is equicontinuous, then the family*

$$H^{\theta_o} = \{f^{\theta_o} : f \in H\}$$

is equicontinuous on $\theta_o X$. Moreover, if H is pointwise bounded, then the same holds for H^{θ_o}

Proof: Define

$$d : X^2 \to \mathbf{R}, \quad d(x, y) = \min\{1, \sup_{f \in H} |f(x) - f(y)|\}.$$

Let $\pi^{\theta_o} : \theta_o X \to X_d$ be the continuous extension of the quotient map $\pi : X \to X_d$. Let $z \in \theta_o X$ and $\epsilon > 0$. There exists $x \in X$ such that $\pi^{\theta_o}(z) = \pi(x)$. Let (x_δ) be a net in X converging to z. Then $\pi(x_\delta) \to \pi^{\theta_o}(z) = \pi(x)$ and so $d(x_\delta, x) \to 0$. Thus, for $f \in H$, we have $f^{\theta_o}(z) = \lim f(x_\delta) = f(x)$. The set $W = \{y \in X : d(x, y) \le \epsilon\}$ is d-clopen (hence clopen) in X and so $\overline{W}^{\theta_o X} = V$ is clopen in $\theta_o X$. Since $x_\delta \in W$ eventually, it follows that $z \in V$. Now, for $f \in H$ and $a \in V$, we have that $|f^{\theta_o}(a) - f^{\theta_o}(z)| \le \epsilon$. In fact, there exists a net (y_γ) in W converging to a. Thus

$$|f^{\theta_o}(a) - f^{\theta_o}(z)| = |f(x) - f^{\theta_o}(a)| = \lim_\gamma |f(x) - f(y_\gamma)| \le \epsilon.$$

This proves that H^{θ_o} is equicontinuous on $\theta_o X$. The last assertion follows from the preceding Theorem.

Theorem 3.11. *$\mathcal{U}_c = \mathcal{U}_c^X$ is the uniformity $\mathcal{U}$ generated by the family of all continuous ultrapseudometrics on X.*

Proof: Let (A_i) be a clopen partition of X and let $W = \bigcup A_i \times A_i$. Define

$$d(x, y) = \sup_i |f_i(x) - f_i(y)|,$$

where $f_i = \chi_{A_i}$. Then d is a continuous ultrapseudometric on X. Since

$$W = \{(x, y) : d(x, y) < 1/2\},$$

it follows that $\mathcal{U}_c$ is coarser than $\mathcal{U}$. Conversely, let d be a continuous ultrapseudometric on X, $\epsilon > 0$ and $D = \{(x, y) : d(x, y) \le \epsilon\}$. If α is the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x,y) \leq \epsilon$, then $D = W_\alpha$ and the result follows.

Theorem 3.12. *Let $(Y_i, f_i)_{i\in I}$ be the family of all pairs (Y, f), where Y is an ultrametric space and $f : X \to Y$ a continuous map. Then*

$$\theta_o X = \bigcap_{i\in I} (f_i^{\beta_o})^{-1}(Y_i).$$

Proof: It follows from Theorem 3.6.

Theorem 3.13. *A Hausdorff zero-dimensional space X is θ_o-complete iff it is homeomorphic to a closed subspace of a product of ultrametic spaces.*

Proof: Every ultrametric space is θ_o-complete. Thus the sufficiency follows from Theorem 3.5. Conversely, assume that X is θ_o-complete and let $(Y_i, f_i)_{i\in I}$ be as in the preceding Theorem. Then $X = \bigcap_i Z_i$, $Z_i = (f_i^{\beta_o})^{-1}(Y_i)$. Let $Y = \prod Y_i$ with its product topology. The map $u : X \to Y$, $u(x)_i = f_i(x)$, is one-to-one. Indeed, let $x \neq y$ and choose a clopen neighborhood V of x not containing y. Let $f = \chi_V$ and

$$d : X \times Y \to \mathbf{R}, \quad d(a,b) = |f(a) - f(b)|.$$

The quotient map $\pi : X \to X_d$ is continuous and $\pi(x) \neq \pi(y)$, which implies that $u(x) \neq u(y)$. Clearly u is continuous. Also $u^{-1} : u(X) \to X$ is continuous. Indeed, let V be a clopen subset of X containing x_o and consider the pseudometric $d(x,y) = |\chi_V(x) - \chi_V(y)|$. Let $\pi : X \to X_d$ be the quotient map. There exists a $i \in I$ such that $Y_i = X_d$ and $f_i = \pi$. Then

$$f_i(V) = \pi(V) = \{\pi(x) : \tilde{d}(\pi(x) - \pi(x_o)) < 1\}.$$

The set $\pi(V)$ is open in $Y_i = X_d$. Let $\pi_i : Y \to Y_i$ be the *ith*-projection map and $G = \pi_i^{-1}(\pi(V))$. If $x \in X$ is such that $u(x) \in G$, then $f_i(x) = u(x)_i \in \pi(V)$ and so $d(x, x_o) < 1$, which implies that $x \in V$ since $x_o \in V$. This proves that $u : X \to u(X)$ is a homeomorphism. Finally, $u(X)$ is a closed subspace of Y. In fact, let (x_δ) be a net in X with $u(x_\delta) \to y \in Y$. Then $f_i(x_\delta) \to y_i$ for all i. Going to a subnet if necessary, we may assume that $x_\delta \to z \in \beta_o X$. Now $f_i(x_\delta) \to f_i^{\beta_o}(z)$ in $\beta_o Y_i$. But then $f_i^{\beta_o}(z) = y_i \in Y_i$, for all i, and hence $z \in \theta_o X = X$, by the preceding Theorem. Thus $y_i = f_i(z)$, for all i, and hence $y = u(z)$. This proves that X is homeomorphic to a closed subspace of Y and the result follows.

Corollary 3.14. *Every Hausdorff ultraparacompact space is homeomorphic to a closed subspace of a product of ultrametric spaces.*

Theorem 3.15. *For a subset A of X, the following are equivalent :*

(1) *A is bounding.*

(2) *A is $\mathcal{U}_c$-totally bounded.*

(3) *For each continuous ultrapseudometric d on X, A is d-totally bounded.*

Proof: In view of Theorem 3.11, (2) is equivalent to (3). Also, by [2, Theorem 4.6], (1) implies (2).
(2) $\Rightarrow$ (1). Let $f \in C(X)$,

$$A_1 = \{x : |f(x)| \leq 1\}, \quad A_{n+1} = \{x : n < |f(x)| \leq n+1\}$$

for $n \geq 1$. Then (A_n) is a clopen partition of X. Let $W = \bigcup_n A_n \times A_n$. By our hypothesis, there are $x_1, \ldots, x_N$ in A such that $A \subset \bigcup_1^N W[x_k]$. For each $1 \leq k \leq N$, there exists n_k such that $x_k \in A_{n_k}$. Then $A \subset \bigcup_1^N A_{n_k}$ and so

$$\|f\|_A \leq \max_{1 \leq k \leq N} n_k,$$

which proves that A is bounding.

4. Polarly Barrelled Spaces of Continuous Functions

Definition 4.1. A Hausdorff locally convex space E is called :

(1) polarly barrelled if every bounded subset of $E'_\sigma = (E', \sigma(E', E))$ is equiconinuous.

(2) polarly quasi-barrelled if every strongly bounded subset of E' is equicontinuous.

We will denote by $C_c(X, E)$ the space $C(X, E)$ equipped with the topology of uniform convergence on compact subsets of X. By $M_c(X, E')$ we will denote the space of all $m \in M(X, E')$ with compact support. The dual space of $C_c(X, E)$ coincides with $M_c(X, E')$.
Recall that a zero-dimensional Hausdorff topological space X is called a μ_o-space (see [2]) if every bounding subset of X is relatively compact. We

denote by $\mu_o X$ the smallest of all μ_o-subspaces of $\beta_o X$ which contain X. Then $X \subset \mu_o X \subset \theta_o X$ and, for each bounding subset A of X, the set $\overline{A}^{\beta_o X}$ is contained in $\mu_o X$ (see [2]). Moreover, if Y is another Hausdorff zero-dimensional space and $f : X \to Y$, then $f^{\beta_o}(\mu_o X) \subset \mu_o Y$ and so there exists a continuous extension $f^{\mu_o} : \mu_o X \to \mu_o Y$ of f.

Theorem 4.2. *Assume that $E' \neq \{0\}$ and let $G = C_c(X, E)$. Then G is polarly barrelled iff X is a μ_o-space and E polarly barrelled.*

Proof: Assume that G is polarly barrelled.
I. E is polarly barrelled. Indeed, let Φ be a $w^\star$-bounded subset of E' and let $x \in X$. For $u \in E'$, let

$$u_x : G \to \mathbb{K}, \quad u_x(f) = u(f(x)).$$

Let $H = \{u_x : u \in \Phi\}$. For $f \in C(X, E)$, we have

$$\sup_{u \in \Phi} |u_x(f)| = \sup_{u \in \Phi} |u(f(x))| < \infty$$

and so H is a $w^\star$-bounded subset of G'. By our hypothesis, there exists $p \in cs(E)$ and Y a compact subset of X such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is equicontinuous.
II. X is a μ_o-space. In fact, let A be a bounding subset of X and let $x' \in E'$, $x' \neq 0$. Define p on E by $p(x) = |x'(s)|$. Then $p \in cs(E)$. The set

$$D = \{f \in G : \|f\|_{A,p} \leq 1\}$$

is a polar barrel in G and so D is a neighborhood of zero in G. Let Y a compact subset of X and $q \in cs(E)$ be such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset D.$$

But then $A \subset Y$ and so $\overline{A}$ is compact.
Conversely, suppose that E is polarly barrelled and X a μ_o-space. Let H be a $w^\star$-bounded subset of the dual space $M_c(X, E')$ of G. Let $s \in E$ and

$$D = \{ms : m \in H\} \subset M(X).$$

For $h \in C_{rc}(X)$, we have that

$$\sup_{m \in H} | < ms, h > | = \sup_{m \in H} | < m, hs > | < \infty.$$

Thus, considering $M(X)$ as the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, D is $w^\star$-bounded of F' and so $\sup_{m \in H} \|ms\| = d_s < \infty$. Hence, $|m(V)s| \leq d_s$ for all $V \in K(X)$. It follows that the set

$$M = \bigcup_{m \in H} m(K(X))$$

is a $w^\star$-bounded subset of E'. Since E is polarly barrelled, there exists $p \in cs(E)$ such that $|u(s)| \leq 1$ for all $u \in M$ and all $s \in E$ with $p(s) \leq 1$. Hence $\sup_{m \in H} \|m\|_p < \infty$. We may choose p so that $\|m\|_p \leq 1$ for all $m \in H$. Let

$$Z = S(H) = \overline{\bigcup_{m \in H} supp(m)}.$$

Then Z is bounding. In fact, assume that Z is not bounding. Then, by [6, Proposition 6.6], there exists a sequence (m_n) in H and $f \in C(X, E)$ such that $< m_n, f >= \lambda^n$, for all n, where $|\lambda| > 1$, which contradicts the fact that H is $w^\star$-bounded. By our hypothesis now, Z is compact. Since

$$\{f \in G : \|f\|_{Z,p} \leq 1\} \subset H^o,$$

the result follows.

Corollary 4.3. *$C_c(X)$ is polarly barrelled iff X is a μ_o-space.*

Let now G, E be Hausdorff locally convex spaces. We denote by $L_s(G, E)$ the space $L(G, E)$ of all continuous linear maps, from G to E, equipped with the topology of simple convergence.

Theorem 4.4. *Assume that E is polar and let G be polarly barrelled. If E is a μ_o-space (e.g. when E is metrizable or complete), then $L_s(G, E)$ is a μ_o-space.*

Proof: Let Φ be a bounding subset of $L_s(G, E)$. For $x \in G$, the set

$$\Phi(x) = \{\phi(x) : \phi \in \Phi\}$$

is a bounding subset of E and hence its closure M_x in E is compact. Φ is a topological subspace of E^G and it is contained in the compact set $M = \prod_{x \in G} M_x$. Since the closure of Φ in E^G is compact, it suffices to show that this closure is contained in $L(G, E)$. To this end, we prove first that, given a polar neighborhood W of zero in E, there exists a neighborhood

U of zero in G such that $\phi(U) \subset W$ for all $\phi \in \Phi$. In fact, for $\phi \in \Phi$, let ϕ' be the adjoint map. Let

$$Z = \bigcup_{\phi \in \Phi} \phi'(H),$$

where H is the polar of W in E'. If $x \in G$, then $\Phi(x)$ is a bounded subset of E and hence $\Phi(x) \subset \alpha W$, for some $\alpha \in \mathbb{K}$. If now $\phi \in \Phi$ and $u \in H$, then

$$| < \phi'(u), x > | = | < u, \phi(x > | \leq |\alpha|,$$

which proves that Z is a $w^\star$-bounded subset of G'. As G is polarly barrelled, the polar $U = Z^o$, of Z in G, is a neighborhood of zero and $\phi(U) \subset H^o = W$, for all $\phi \in \Phi$, which proves our claim. Let now $\phi \in E^G$ be in the closure of Φ. Then ϕ is linear. There exists a net (ϕ_δ) in Φ converging to ϕ in E^G. If $x \in U$, then $\phi(x) = \lim \phi_\delta(x) \in W$, which proves that ϕ is continuous. Hence the result follows.

Corollary 4.5. *If E is polarly barrelled, then the weak dual E'_σ of E is a μ_o-space.*

Theorem 4.6. *Suppose that E is polar and G polarly barrelled. For $f \in C(X, E)$, let $f^{\mu_o} : \mu_o X \to \hat{E}$ be its continuous extension. If $T : G \to C_c(X, E)$ is a continuous linear map, then the map*

$$\tilde{T} : G \to C_c(\mu_o X, \hat{E}), \quad s \mapsto (Ts)^{\mu_o},$$

is continuous

Proof: Note that $\hat{E}$ is θ_o-complete and hence a μ_o-space. Let

$$\phi : X \to L_s(G, E), \quad < \phi(x), s >= (Ts)(x).$$

Then ϕ is continuous. Since $L_s(G, \hat{E})$ is a μ_o-space, there exists a continuous extension

$$\phi^{\mu_o} : \mu_o X \to L_s(G, \hat{E}).$$

Let now A be a compact subset of $\mu_o X$ and p a polar continuous seminorm on E. We denote also by p the continuous extension of p to all of $\hat{E}$. Let

$$V = \{g \in C(\mu_o X, \hat{E}) : \|g\|_{A,p} \leq 1\}.$$

The set $\Phi = \phi^{\mu_o}(A)$ is compact in $L_s(G, \hat{E})$. As in the proof of Theorem 4.4, there exists a neighborhood U of zero in G such that

$$\psi(U) \subset W = \{s \in \hat{E} : p(s) \leq 1\},$$

for all $\psi \in \Phi$. Now, for $y \in A$ and $s \in U$, we have

$$p((\tilde{T}s)(y)) = p(< \phi^{\mu_o}(y), s >) \leq 1$$

and so $\tilde{T}s \in V$. This proves that $\tilde{T}$ is continuous and the result follows.

Theorem 4.7. *Assume that E is polar and polarly barrelled and let τ_o be the locally convex topology on $C(X,E)$ generated by the seminorms $f \mapsto \|f^{\mu_o}\|_{A,p}$, where A ranges over the family of all compact subsets of $\mu_o X$ and $p \in cs(E)$. Then :*

(1) *$(C(X,E), \tau_o)$ is polarly barrelled and τ_o is finer than τ_b (and hence finer than τ_c).*

(2) *If τ is any polarly barrelled topology on $C(X,E)$ which is finer than τ_c, then τ is finer than τ_o. Hence τ_o is the polarly barrelled topology associated with each of the topologies τ_b and τ_c.*

Proof: (1). Since E is polarly barrelled, the same is true for $\hat{E}$. The space
$F = C_c(\mu_o X, \hat{E})$ is polarly barrelled and the map

$$S : (C(X,E), \tau_o) \to F, \quad f \mapsto f^{\mu_o},$$

is a linear homeomorphism. Thus τ_o is polarly barrelled. Also, since for each bounding subset B of X, its closure $\overline{B}^{\mu_o X}$ is compact, it follows that τ_o is finer than τ_b.
(2). Let τ be a polarly barrelled topology on $C(X,E)$, which is finer than τ_c, and let $G = (C(X,E), \tau)$. The identity map

$$T : G \to C_c(X,E)$$

is continuous and hence the map

$$\tilde{T} : G \to C_c(\mu_o X, \hat{E}), \quad f \mapsto f^{\mu_o},$$

is continuous. This proves that τ_o is coarser than τ and the Theorem follows.

Theorem 4.8. *Suppose that E is polar. Then $G = (C(X,E), \tau_b)$ is polarly barrelled iff E is polarly barrelled and, for each compact subset A of $\mu_o X$, there exists a bounding subset B of X such that $A \subset \overline{B}^{\mu_o X}$.*

Proof: Assume that G is polarly barrelled. It is easy to see that E is polarly barrelled. In view of the preceding Theorem, $\tau_b = \tau_o$. Thus, for each compact subset A of $\mu_o X$ and each non-zero $p \in cs(E)$, there exist a bounding subset B of X and $q \in cs(E)$ such that

$$\{f \in C(X,E) : \|f\|_{B,q} \leq 1\} \subset \{f : \|f^{\mu_o}\|_{A,p} \leq 1\}.$$

It follows easily that $A \subset \overline{B}^{\mu_o X}$. Conversely, suppose that the condition is satisfied. The condition clearly implies that τ_o is coarser than τ_b and hence $\tau_b = \tau_o$, which implies that G is polarly barrelled by the preceding Theorem.

Let us say that a family $\mathcal{F}$ of subsets of a a set Z is finite on a subset F of Z if the family of all members of $\mathcal{F}$ which meet F is finite.

Definition 4.9. A subset D, of a topological space Z, is said to be w-bounded if every family $\mathcal{F}$ of open subsets of Z, which is finite on each compact subset of Z, is also finite on D. If this happens for families of clopen sets, then D is said to be w_o-bounded. We say that Z is a w-space (resp. a w_o-space) if every w-bounded (resp. w_o-bounded) subset is relatively compact.

Lemma 4.10. *A subset D, of a zero-dimensional topological space Z, is w-bounded iff it is w_o-bounded.*

Proof: Assume that D is not w-bounded. Then, there exists an infinite sequence (x_n) of distinct elements of D and a sequence (V_n) of open sets such that $x_n \in V_n$ and (V_n) is finite on each compact subset of X. By [5, Lemma 2.5], there exists a subsequence (x_{n_k}) and pairwise disjoint clopen sets W_k with $x_{n_k} \in W_k$. We may choose $W_k \subset V_{n_k}$. Now (W_k) is clearly finite on each compact subset of X, which implies that D is not w_o-bounded. Hence the Lemma follows.

We easily get the following

Lemma 4.11. *Every w_o-bounded subset of X is bounding.*

Theorem 4.12. *Assume that $E' \neq \{0\}$. Then $G = C_c(X,E)$ is polarly quasi-barrelled iff E is polarly quasi-barrelled and X a w_o-space.*

Proof: Suppose that E is polarly quasi barrelled and X a w_o-space. Let H be a strongly bounded subset of the dual space $M_c(X,E)$ of G. We show first that there exists $p \in cs(E)$ such that $\sup_{m\in H} \|m\|_p < \infty$. In fact, let B be a bounded subset of E and consider the set

$$D = \{ms : m \in H,\, s \in B\}.$$

If $h \in C_{rc}(X)$, then the set $\{hs : s \in B\}$ is a bounded subset of G and so

$$\sup_{m\in H} \left| \int hs\, dm \right| = \sup_{m\in H} \left| \int h\, d(ms) \right| < \infty.$$

Considering D a a subset of the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, we see that D is a $w^\star$-bounded subset of F' and hence equicontinuous. Thus

$$d = \sup_{m\in H, s\in B} \|ms\| < \infty.$$

Let

$$\Phi = \bigcup_{m\in H} m(K(X)).$$

Then for $A \in K(X)$, $s \in B, m \in H$, we have $|m(A)s| \leq \|ms\| \leq d$. Hence Φ is a strongly bounded subset of E'. By our hypothesis, Φ is an equicontinuous subset of E'. Thus, there exists $p \in cs(E)$ such that $|m(A)s| \leq 1$ for all $m \in H$ and all $s \in E$ with $p(s) \leq 1$. It follows from this that $\sup_{m\in H} \|m\|_p = r < \infty$. We may choose p so that $r \leq 1$. Let now

$$Y = S(H) = \overline{\bigcup_{m\in H} supp(m)}.$$

Then Y is w_o-bounded. Assume the contrary. Then, there exists a sequence (V_n) of distinct clopen subsets of X, such that $V_n \cap Y \neq \emptyset$ for all n and (V_n) is finite on each compact subset of X. . For each n there exists $m_n \in H$ with $V_n \cap supp(m_n) \neq \emptyset$. Then $(m_n)_p(V_n) > 0$. There are a clopen subset W_n of V_n and $s_n \in E$, with $p(s_n) \leq 1$, such that $m(W_n)s_n = \gamma_n \neq 0$. Let $|\lambda| > 1$ and take

$$M = \{\gamma_n^{-1} \lambda^n \chi_{W_n} s_n : n \in \mathbf{N}\}.$$

Since (W_n) is finite on each compact subset of X, it follows that M is a bounded subset of G and so M is absorbed by H^o. Let $\lambda_o \neq 0$ be such that $M \subset \lambda_o H^o$. But then

$$1 \geq |\lambda_o^{-1} \gamma_n^{-1} \lambda^n m_n(W_n) s_n| = |\lambda_o^{-1} \lambda^n|$$

for all n, which is a contradiction. So Y is w_o-bounded and hence compact by our hypothesis. Moreover

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

Indeed, let $\|f\|_{Y,p} \leq 1$. The set $V = \{x : p(f(x)) > 1\}$ is disjoint from Y and hence $m_p(V) = 0$ for all $m \in H$. Thus, for $m \in H$, we have

$$\left|\int_V f\,dm\right| \leq \|f\|_p \cdot m_p(V) = 0$$

and so

$$\left|\int f\,dm\right| = \left|\int_{V^c} f\,dm\right| \leq m_p(V^c) \leq 1.$$

Conversely, suppose that G is polarly quasi-barrelled. Let Φ be a strongly bounded subset of E' and let $x \in X$. For $u \in E'$, define u_x on G by $u_x(f) = u(f(x))$. Then $u_x \in G'$. The set $H = \{u_x : u \in \Phi\}$ is a strongly bounded subset of G'. Indeed, let D be a bounded subset of G. Since the set $\{f(x) : f \in D\}$ is a bounded subset of E, we have that

$$\sup_{f \in D, u \in \Phi} |u_x(f)| = \sup_{f \in D, u \in \Phi} |u(f(x))| < \infty.$$

By our hypothesis, H is an equicontinuous subset of G'. Thus, there exists a compact subset Y of X and $p \in cs(E)$ such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\}.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is an equicontinuous subset of E', which proves that E is polarly quasi-barrelled. Finally, let A be a w_o-bounded subset of X and choose a non-zero element x' of E'. Let $p(s) = |x'(s)|$ and consider the set

$$Z = \{f \in G : \|f\|_{A,p} \leq 1\}.$$

Then Z is a polar set. We will show that Z is bornivorous. So, suppose that there exists a bounded subset M of G which is not absorbed by Z. Then, there exists a sequence (f_n) in M, $\|f_n\|_{A,p} > n$. Let

$$V_n = \{x : p(f_n(x)) > n\}.$$

Then V_n intersects A. Since A is w_o-bounded, there exists a compact subset Y of X such that (V_n) is not finite on Y, which is a contradiction since $\sup_{f \in M} \|f\|_{Y,p} < \infty$. This contradiction shows that Z absorbs bounded subsets of G. In view of our hypothesis, there exist a compact subset Y of X and $q \in cs((E)$ such that

$$\{f \in G : \|f\|_{Y,q} \leq 1\},$$

which implies that $A \subset Y$ and so A is relatively compact. This clearly completes the proof.

Corollary 4.13. (1) *$C_c(X)$ is polarly quasi-barrelled iff X is a w_o-space.*

(2) *If $E' \neq \{0\}$, then $C_c(X, E)$ is polarly quasi-barrelled iff both E and $C_c(X)$ are polarly quasi- barrelled.*

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