ANNALES MATHÉMATIQUES

BLAISE PASCAL

Amel Dilmi

Groups whose proper subgroups are locally finite-by-nilpotent

Volume 14, n° 1 (2007), p. 29-35.

<http://ambp.cedram.org/item?id=AMBP_2007__14_1_29_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **14**, 29-35 (2007)

Groups whose proper subgroups are locally finite-by-nilpotent

Amel Dilmi

Abstract

If $\mathcal{X}$ is a class of groups, then a group G is said to be minimal non $\mathcal{X}$-group if all its proper subgroups are in the class $\mathcal{X}$, but G itself is not an $\mathcal{X}$-group. The main result of this note is that if $c > 0$ is an integer and if G is a minimal non $(\mathcal{LF})\mathcal{N}$ (respectively, $(\mathcal{LF})\mathcal{N}_c$)-group, then G is a finitely generated perfect group which has no non-trivial finite factor and such that $G/Frat(G)$ is an infinite simple group; where $\mathcal{N}$ (respectively, $\mathcal{N}_c$, $\mathcal{LF}$) denotes the class of nilpotent (respectively, nilpotent of class at most c, locally finite) groups and $Frat(G)$ stands for the Frattini subgroup of G.

Résumé

Si $\mathcal{X}$ est une classe de groupes, alors un groupe G est dit minimal non $\mathcal{X}$-groupe si tous ses sous-groupes propres sont dans la classe $\mathcal{X}$, alors que G lui-même n'est pas un $\mathcal{X}$-groupe. Le principal résultat de cette note affirme que si $c > 0$ est un entier et si G est un groupe minimal non $(\mathcal{LF})\mathcal{N}$ (respectivement, $(\mathcal{LF})\mathcal{N}_c$)-groupe, alors G est un groupe parfait, de type fini, n'ayant pas de facteur fini non trivial et tel que $G/Frat(G)$ est un groupe simple infini ; où $\mathcal{N}$ (respectivement, $\mathcal{N}_c$, $\mathcal{LF}$) désigne la classe des groupes nilpotents (respectivement, nilpotents de classe égale au plus à c, localement finis) et $Frat(G)$ est le sous-groupe de Frattini de G.

1. Introduction

If $\mathcal{X}$ is a class of groups, then a group G is said to be minimal non-$\mathcal{X}$ if all its proper subgroups are in the class $\mathcal{X}$, but G itself is not an $\mathcal{X}$-group. Many results have been obtained by many authors on minimal non $\mathcal{X}$-groups for various classes of groups $\mathcal{X}$, for example see [1], [2],

Keywords: Locally finite-by-nilpotent proper subgroups, Frattini factor group.
Math. classification: 20F99.

[3], [5], [6], [8], [11], [12], [13]. In particular, in [13] it is proved that if G is a finitely generated minimal non $\mathcal{FN}$-group, then G is a perfect group which has no non-trivial finite factor and such that $G/Frat(G)$ is an infinite simple group; where $\mathcal{N}$ (respectively, $\mathcal{F}$) denotes the class of nilpotent (respectively, finite) groups and $Frat(G)$ stands for the Frattini subgroup of G. The aim of the present note is to extend the above results to minimal non $(\mathcal{LF})\mathcal{N}$ (respectively, $(\mathcal{LF})\mathcal{N}_c$)-groups, and to prove that there are no minimal non $(\mathcal{LF})\mathcal{N}$ (respectively, $(\mathcal{LF})\mathcal{N}_c$)-groups which are not finitely generated; where $c > 0$ is an integer and $\mathcal{N}_c$ (respectively, $\mathcal{LF}$) denotes the class of nilpotent groups of class at most c (respectively, locally finite groups). More precisely we shall prove the following results.

Theorem 1.1. *If G is a minimal non $(\mathcal{LF})\mathcal{N}$-group, then G is a finitely generated perfect group which has no non-trivial finite factor and such that $G/Frat(G)$ is an infinite simple group.*

Using Theorem 1.1, we shall prove the following result on minimal non $(\mathcal{LF})\mathcal{N}_c$-groups.

Theorem 1.2. *Let $c > 0$ be an integer and let G be a minimal non $(\mathcal{LF})\mathcal{N}_c$-group. Then G is a finitely generated perfect group which has no non-trivial finite factor and such that $G/Frat(G)$ is an infinite simple group.*

Note that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are two classes of groups such that $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then a minimal non $\mathcal{X}_1$-group is either a minimal non $\mathcal{X}_2$-group or an $\mathcal{X}_2$-group. From Xu's results [13, Theorem 3.5], an infinitely generated minimal non $\mathcal{FN}$-group is a locally finite-by-nilpotent group. So one might expect, as we shall prove in Proposition 2.1, that there are no infinitely generated minimal non $(\mathcal{LF})\mathcal{N}$-group.

Note that minimal non $(\mathcal{LF})\mathcal{N}$ (respectively, non $(\mathcal{LF})\mathcal{N}_c$)-groups exist. Indeed, the group constructed by Ol'shanskii [7] is a simple torsion-free finitely generated group whose proper subgroups are cyclic.

2. Minimal non $(\mathcal{LF})\mathcal{N}$-groups

A part of Theorem 1.1 is an immediate consequence of the following Proposition:

Proposition 2.1. *Let G be a group whose proper subgroups are in the class $(\mathcal{LF})\mathcal{N}$. Then G belongs to $(\mathcal{LF})\mathcal{N}$ if it satisfies one of the following two conditions:*

(i) G is finitely generated and has a proper subgroup of finite index,

(ii) G is not finitely generated.

Proof. (*i*) Suppose that G is finitely generated and let N be a proper subgroup of finite index in G. By [10, Theorem 1.6.9] we may assume that N is normal in G. So N is in $(\mathcal{LF})\mathcal{N}$ and it is also finitely generated. Hence $\gamma_{k+1}(N)$ is locally finite for some integer $k \geq 0$. Since N is of finite index in G, $G/\gamma_{k+1}(N)$ is a finitely generated group in the class $\mathcal{NF}$, so that it satisfies the maximal condition on subgroups. Therefore every proper subgroups of $G/\gamma_{k+1}(N)$ is in $\mathcal{FN}$. Now Lemma 4 of [3] states that a finitely generated locally graded group whose proper subgroups are finite-by-nilpotent is itself finite-by-nilpotent. Since groups in the class $\mathcal{NF}$ are clearly locally graded, we deduce that $G/\gamma_{k+1}(N)$ is in $\mathcal{FN}$, so G is in $(\mathcal{LF})\mathcal{N}$.

(*ii*) Suppose that G is not finitely generated and let $x_1, ..., x_n$ be n elements of finite order in G. Since the subgroup $\langle x_1, ..., x_n\rangle$ is proper in G, it is in $(\mathcal{LF})\mathcal{N}$, hence it is finite. This means that the elements of finite order in G form a locally finite subgroup T. If G/T is not finitely generated, then it is locally nilpotent and its proper subgroups are nilpotent as G/T is torsion-free. Now Theorem 2.1 of [11] states that a torsion-free locally nilpotent group with all proper subgroups nilpotent is itself nilpotent. Therefore G/T is nilpotent, so G is in $(\mathcal{LF})\mathcal{N}$. Now if G/T is finitely generated, then there exists a finitely generated subgroup H such that $G = HT$. Since G is not finitely generated, H is proper in G, so H is in $(\mathcal{LF})\mathcal{N}$. Since $G/T \simeq H/H \cap T$, we deduce that G/T is in $(\mathcal{LF})\mathcal{N}$, hence G/T is nilpotent since it is torsion-free. Therefore G is in $(\mathcal{LF})\mathcal{N}$. □

Since finitely generated locally graded groups have proper subgroups of finite index, the previous Proposition admits the following consequence :

Corollary 2.2. *Let G be a locally graded group whose proper subgroups are in the class $(\mathcal{LF})\mathcal{N}$. Then G is in the class $(\mathcal{LF})\mathcal{N}$.*

Corollary 2.3. *Let G be a non perfect group whose proper subgroups are in the class $(\mathcal{LF})\mathcal{N}$. Then G is in the class $(\mathcal{LF})\mathcal{N}$.*

Proof. If G is not finitely generated, then G is in $(\mathcal{LF})\mathcal{N}$ from (*ii*) of Proposition 2.1. Now suppose that G is finitely generated. Therefore G/G', being a non trivial finitely generated locally graded group, has a non trivial finite image. So G has a proper subgroup of finite index. Thus we deduce from (i) of Proposition 2.1 that G is in $(\mathcal{LF})\mathcal{N}$. □

Proof of Theorem 1.1. Let G be a minimal non $(\mathcal{LF})\mathcal{N}$-group. It follows from Proposition 2.1 and Corollary 2.3 that G is a finitely generated perfect group which has no non trivial finite factor. Now we prove that $G/Frat(G)$ is an infinite simple group. Since finitely generated groups have maximal subgroups, $G/Frat(G)$ is non trivial and therefore infinite. Let N be a proper normal subgroup of G properly containing $Frat(G)$. Then N is in $(\mathcal{LF})\mathcal{N}$ and there is an $x \in N$ such that $x \notin Frat(G)$. Hence there is a maximal subgroup M of G such that $x \notin M$, so N is not contained in M. Then $G = NM$ and we have $\gamma_{k+1}(M)$ is locally finite for some integer $k \geq 0$. Since G is perfect, then

$$G = \gamma_{k+1}(G) = \gamma_{k+1}(NM).$$

We show by induction on k that $\gamma_{k+1}(NM) \subseteq N\gamma_{k+1}(M)$. If $k = 0$, then the result follows immediately. Now let $k > 0$, suppose inductively that $\gamma_k(NM) \subseteq N\gamma_k(M)$ and let g be an element of $\gamma_{k+1}(NM)$. Hence g can be written as a finite product of elements of the form $[x_1y_1, ..., x_{k+1}y_{k+1}]$ with $x_i \in N$ and $y_i \in M$ for every $1 \leq i \leq k+1$. It follows by the inductive hypothesis that the commutator $v = [x_1y_1, ..., x_ky_k]$ of weight k is in $N\gamma_k(M)$. So we get $[v, x_{k+1}y_{k+1}] = [xy, x_{k+1}y_{k+1}]$ with $x \in N$ and $y \in \gamma_k(M)$. Therefore

$$[xy, x_{k+1}y_{k+1}] = [x, y_{k+1}]^y [y, y_{k+1}] \left([x, x_{k+1}]^y [y, x_{k+1}]\right)^{y_{k+1}}.$$

We have that $[y, y_{k+1}]$ is in $\gamma_{k+1}(M)$ and since N is normal in G, we have that $[x, y_{k+1}]^y$ and $([x, x_{k+1}]^y [y, x_{k+1}])^{y_{k+1}}$ belong to N. Thus $[x_1y_1, ..., x_{k+1}y_{k+1}]$ is in $N\gamma_{k+1}(M)$ and consequently g belongs to $N\gamma_{k+1}(M)$. Hence the inclusion $\gamma_{k+1}(NM) \subseteq N\gamma_{k+1}(M)$ hold. We deduce $G = N\gamma_{k+1}(M)$. Thus $G/N' = (N/N')(\gamma_{k+1}(M)N'/N')$. Since $\gamma_{k+1}(M)N'/N'$ is locally finite, G/N' is in $\mathcal{A}(\mathcal{LF})$, where $\mathcal{A}$ denotes the class of abelian groups; so G/N' is a locally graded group. We deduce from Corollary 2.2 that G/N' is in $(\mathcal{LF})\mathcal{N}$. Now Theorem 1.2 of [4] states that if $N \lhd G$ such that N and G/N' are in $(\mathcal{LF})\mathcal{N}$, then G is in $(\mathcal{LF})\mathcal{N}$. So G is in $(\mathcal{LF})\mathcal{N}$, a contradiction. This means that $G/Frat(G)$ is a simple group. □

3. Minimal non $(\mathcal{LF})\mathcal{N}_c$-group

Lemma 3.1. *Let G be a group and let F be the locally finite radical of G. If G/F is nilpotent, then G/F is torsion-free.*

Proof. Put $\overline{G} = G/F$ and suppose that $\overline{G}$ is nilpotent and let $\overline{x}$ be an element of finite order in $\overline{G}$. First of all, we show that the normal closure $\overline{x}^{\overline{G}}$ is locally finite; to this end, let $\left\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \right\rangle$ be a finitely generated subgroup of $\overline{x}^{\overline{G}}$. Since every element $\overline{h}_i$, where $1 \leq i \leq n$, can be written as a finite product of elements $\overline{x}^{\overline{g}}$, there is an integer $s > 0$ such that $\left\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \right\rangle$ is a subgroup of $\left\langle \overline{x}^{\overline{g}_1}, \overline{x}^{\overline{g}_2}, ..., \overline{x}^{\overline{g}_s} \right\rangle$ for some $\overline{g}_j \in \overline{G}$, with $1 \leq j \leq s$. Moreover $\left\langle \overline{x}^{\overline{g}_1}, \overline{x}^{\overline{g}_2}, ..., \overline{x}^{\overline{g}_s} \right\rangle$ being nilpotent and generated by finitely many elements of finite order, is finite. So $\left\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \right\rangle$ is finite and consequently $\overline{x}^{\overline{G}}$ is locally finite. Now since $\overline{G}$ has no non trivial locally finite normal subgroup, then $\overline{x}^{\overline{G}}$ is trivial. Thus $\overline{x} = \overline{1}$, hence $\overline{G}$ is torsion-free. □

Proposition 3.2. *Let $c > 0$ be an integer and let G be a group whose proper subgroups are in the class $(\mathcal{LF})\mathcal{N}_c$. Then G is in the class $(\mathcal{LF})\mathcal{N}_c$ if it satisfies one of the following two conditions:*

(*i*) *G is finitely generated and has a proper subgroup of finite index,*

(*ii*) *G is not finitely generated.*

Proof. Suppose that G satisfies one of the conditions (i) or (ii). Since $(\mathcal{LF})\mathcal{N}_c$ is included in $(\mathcal{LF})\mathcal{N}$, it follows from Proposition 2.1 that G is in $(\mathcal{LF})\mathcal{N}$. So there is a normal subgroup N such that N is locally finite and G/N is nilpotent. Thus N is contained in F, the locally finite radical of G, and therefore G/F is nilpotent. Clearly, we may assume that G is not locally finite, so G/F is non trivial and by Lemma 3.1, it is torsion-free. If G/F is finitely generated, then as G/F is clearly locally graded, G/F has a proper normal subgroup of finite index. So G/F belongs to $((\mathcal{LF})\mathcal{N}_c)\mathcal{F}$, hence G/F is in $\mathcal{N}_c\mathcal{F}$ as it is torsion-free. Therefore G/F, being a nilpotent torsion-free group in the class $\mathcal{N}_c\mathcal{F}$, is in $\mathcal{N}_c$ by Lemma 6.33 of [9]. Consequently G is in $(\mathcal{LF})\mathcal{N}_c$. Now suppose that G/F is not finitely generated and let $a_1, ..., a_{c+1}$ be elements of G/F. Since $\langle a_1, ..., a_{c+1}\rangle$ is proper in G/F, it is in $(\mathcal{LF})\mathcal{N}_c$ and consequently it is in $\mathcal{N}_c$ because G/F is torsion-free. So $[a_1, ..., a_{c+1}] = 1$, hence G/F is in $\mathcal{N}_c$. Therefore G is in $(\mathcal{LF})\mathcal{N}_c$. □

Since finitely generated locally graded groups have proper subgroups of finite index, Proposition 3.2 admits the following consequence :

Corollary 3.3. *Let $c > 0$ be an integer and let G be a locally graded group whose proper subgroups are in the class $(\mathcal{LF})\mathcal{N}_c$. Then G is in the class $(\mathcal{LF})\mathcal{N}_c$.*

Proof of Theorem 1.2. Let G be a minimal non $(\mathcal{LF})\mathcal{N}_c$-group. It follows that every proper subgroup of G is in $(\mathcal{LF})\mathcal{N}$. Now suppose that G is in $(\mathcal{LF})\mathcal{N}$, so there exists a normal subgroup N of G such that N is locally finite and G/N is nilpotent. By Corollary 3.3, G/N is in $(\mathcal{LF})\mathcal{N}_c$, consequently we deduce that G is in $(\mathcal{LF})\mathcal{N}_c$ because N is locally finite; a contradiction. Hence G is a minimal non $(\mathcal{LF})\mathcal{N}$-group, and by Theorem 1.1, G is a finitely generated perfect group which has no non-trivial finite factor and such that $G/Frat(G)$ is an infinite simple group. □

Acknowlegments : The author would like to thank the referee for his comments and her supervisor Dr N. Trabelsi for his encouragements while doing this work.

References

[1] A. Asar – Nilpotent-by-Chernikov, *J. London Math.Soc* **61** (2000), no. 2, p. 412–422.

[2] V. Belyaev – Groups of the Miller-Moreno type, *Sibirsk. Mat. Z.* **19** (1978), no. 3, p. 509–514.

[3] B. Bruno & R. E. Phillips – On minimal conditions related to Miller-Moreno type groups, *Rend. Sem. Mat. Univ. Padova* **69** (1983), p. 153–168.

[4] G. Endimioni & G. Traustason – On torsion-by-nilpotent groups, *J. Algebra* **241** (2001), no. 2, p. 669–676.

[5] M. Kuzucuoglu & R. E. Phillips – Locally finite minimal non FC-groups, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), p. 417–420.

[6] M. F. Newman & J. Wiegold – Groups with many nilpotent subgroups, *Arch. Math.* **15** (1964), p. 241–250.

[7] A. Y. OLSHANSKI – An infinite simple torsion-free noetherian group, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), p. 1328–1393.

[8] J. OTAL & J. M. PENA – Groups in which every proper subgroup is Cernikov-by-nilpotent or nilpotent-by-Cernikov, *Arch.Math.* **51** (1988), p. 193–197.

[9] D. J. S. ROBINSON – *Finiteness conditions and generalized soluble groups*, Springer-Verlag, 1972.

[10] ———, *A course in the theory of groups*, Springer-Verlag, 1982.

[11] H. SMITH – Groups with few non-nilpotent subgroups, *Glasgow Math. J.* **39** (1997), p. 141–151.

[12] M. XU – Groups whose proper subgroups are Baer groups, *Acta. Math. Sinica* **40** (1996), p. 10–17.

[13] ———, Groups whose proper subgroups are finite-by-nilpotent, *Arch. Math.* **66** (1996), p. 353–359.

AMEL DILMI
Department of Mathematics
Faculty of Sciences
Ferhat Abbas University
Setif 19000
ALGERIA
di_amel@yahoo.fr