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Hyper–(Abelian–by–finite) groups with many subgroups of finite depth

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Abstract

The main result of this note is that a finitely generated hyper-(Abelian-by-finite) group G is finite-by-nilpotent if and only if every infinite subset contains two distinct elements x, y such that $\gamma_n(\langle x,\, x^y\rangle) = \gamma_{n+1}(\langle x,\, x^y\rangle)$ for some positive integer $n = n(x, y)$ (respectively, $\langle x, x^y\rangle$ is an extension of a group satisfying the minimal condition on normal subgroups by an Engel group).

Groupes hyper-(Abelien-par-fini) ayant beaucoup de sous-groupes de profondeur finie

Résumé

Le principal résultat de cet article est qu'un groupe G hyper-(Abélien-par-fini) de type fini est fini-par-nilpotent si, et seulement si, toute partie infinie de G contient deux éléments distincts x, y tels que $\gamma_n(\langle x, x^y\rangle) = \gamma_{n+1}(\langle x, x^y\rangle)$ pour un certain entier positif $n = n(x, y)$ (respectivement, $\langle x, x^y\rangle$ est une extension d'un groupe vérifiant la condition minimale sur les sous-groupes normaux par un groupe d'Engel).

1. **Introduction and results**

Let $\mathcal{X}$ be a class of groups. Denote by $(\mathcal{X}, \infty)$ (respectively, $(\mathcal{X}, \infty)^*$) the class of groups G such that for every infinite subset X of G, there exist distinct elements $x, y \in X$ such that $\langle x, y\rangle \in \mathcal{X}$ (respectively, $\langle x, x^y\rangle \in \mathcal{X}$). Note that if $\mathcal{X}$ is a subgroup closed class, then $(\mathcal{X}, \infty) \subseteq (\mathcal{X}, \infty)^*$.

In answer to a question of Erdös, B.H. Neumann proved in [16] that a group G is centre-by-finite if and only if G is in the class $(\mathcal{A}, \infty)$, where $\mathcal{A}$ denotes the class of Abelian groups. Lennox and Wiegold showed in [13]

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that a finitely generated soluble group is in the class $(\mathcal{N},\infty)$ (respectively, $(\mathcal{P},\infty)$) if and only if it is finite-by-nilpotent (respectively, polycyclic), where $\mathcal{N}$ (respectively, $\mathcal{P}$) denotes the class of nilpotent (respectively, polycyclic) groups. Other results of this type have been obtained, for example in [1]—[3], [4]—[6], [7], [8], [13], [14]—[16], [21], [22] and [23].

We say that a group G has finite depth if the lower central series of G stabilises after a finite number of steps. Thus if $\gamma_n(G)$ denotes the n^{th} term of the lower central series of G, then G has finite depth if and only if $\gamma_n(G) = \gamma_{n+1}(G)$ for some positive integer n. Denote by Ω the class of groups which has finite depth. Moreover, if k is a fixed positive integer, let Ω_k denotes the class of groups G such that $\gamma_k(G) = \gamma_{k+1}(G)$.

Clearly, any group in the class $\mathcal{FN}$ is of finite depth, where $\mathcal{F}$ denotes the class of finite groups. From this and the fact that $\mathcal{FN}$ is a subgroup closed class, we deduce that finite-by-nilpotent groups belong to $(\Omega,\infty)^*$. Here we shall be interested by the converse. In [5], Boukaroura has proved that a finitely generated soluble group in the class (Ω,∞) is finite-by-nilpotent. We obtain the same result when (Ω,∞) is replaced by $(\Omega,\infty)^*$and soluble by hyper-(Abelian-by-finite). More precisely we shall prove the following result.

Theorem 1.1. *Let G be a finitely generated hyper-(Abelian-by-finite) group. Then, G is in the class $(\Omega,\infty)^*$ if, and only if, G is finite-by-nilpotent.*

Note that Theorem 1.1 improves the result of [12] which asserts that a finitely generated soluble-by-finite group whose subgroups generated by two conjugates are of finite depth, is finite-by-nilpotent.

It is clear that an Abelian group G in the class $(\Omega_1,\infty)^*$ is finite. For if G is infinite, then it contains an infinite subset $X = G\backslash\{1\}$. Therefore there exist two distinct elements x, y $(\neq 1)$ in X such that $\gamma_1(\langle x, x^y\rangle) = \gamma_2(\langle x, x^y\rangle) = 1$; so $x = 1$, which is a contradiction. From this it follows that a hyper-(Abelian-by-finite) group G in the class $(\Omega_1,\infty)^*$ is hyper-(finite) as $(\Omega_1,\infty)^*$ is a subgroup and a quotient closed class. But it is not difficult to see that a hyper-(finite) group is locally finite [17, Part 1, page 36]. So G is locally finite. Now if G is infinite, then it contains an infinite Abelian subgroup A [17, Theorem 3.43]. Since A is in the class $(\Omega_1,\infty)^*$, it is finite; a contradiction and G, therefore, is finite. As consequence of Theorem 1.1, we shall prove other results on the class $(\Omega_k,\infty)^*$.

Corollary 1.2. *Let k be a positive integer and let G be a finitely generated hyper-(Abelian-by-finite) group. We have:*

(i) If G is in the class $(\Omega_k,\infty)^*$, then there exists a positive integer $c=c(k)$, depending only on k, such that $G/Z_c(G)$ is finite.

(ii) If G is in the class $(\Omega_2,\infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If G is in the class $(\Omega_3,\infty)^*$, then G is in the class $\mathcal{F}\mathcal{N}_3^{(2)}$, where $\mathcal{N}_3^{(2)}$ denotes the class of groups whose 2-generator subgroups are nilpotent of class at most 3.

Let k be a fixed positive integer, denote by $\mathcal{M}$, $\mathcal{E}_k$ and $\mathcal{E}$ respectively the class of groups satisfying the minimal condition on normal subgroups, the class of k-Engel groups and the class of Engel groups. Using Theorem 1.1, we will prove the following results concerning the classes $(\mathcal{M}\mathcal{E},\infty)^*$ and $(\mathcal{M}\mathcal{E}_k,\infty)^*$

Theorem 1.3. *Let G be a finitely generated hyper-(Abelian-by-finite) group. Then, G is in the class $(\mathcal{M}\mathcal{E},\infty)^*$ if, and only if, G is finite-by-nilpotent.*

Note that this theorem improves Theorem 3 of [23] (respectively, Corollary 3 of [5]) where it is proved that a finitely generated soluble group in the class $(\mathcal{C}\mathcal{N},\infty)^*$ (respectively, $(\mathcal{X}\mathcal{N},\infty)$) is finite-by-nilpotent, where $\mathcal{C}$ (respectively, $\mathcal{X}$) denotes the class of Chernikov groups (respectively, the class of groups satisfying the minimal condition on subgroups).

Corollary 1.4. *Let k be a positive integer and let G be a finitely generated hyper-(Abelian-by-finite) group. We have:*

(i) If G is in the class $(\mathcal{M}\mathcal{E}_k,\infty)^*$, then there exists a positive integer $c=c(k)$, depending only on k, such that $G/Z_c(G)$ is finite.

(ii) If G is in the class $(\mathcal{M}\mathcal{A},\infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If G is in the class $(\mathcal{M}\mathcal{E}_2,\infty)^*$, then G is in the class $\mathcal{F}\mathcal{N}_3^{(2)}$.

Note that these results are not true for arbitrary groups. Indeed, Golod [9] showed that for each integer $d>1$ and each prime p, there are infinite d-generator groups all of whose $(d-1)$-generator subgroups are finite p-groups. Clearly, for $d=3$, we obtain a group G which belongs to the class $(\mathcal{F},\infty)^*$. Therefore, G belongs to the classes $(\Omega,\infty)^*$, $(\Omega_k,\infty)^*$, $(\mathcal{M}\mathcal{E},\infty)^*$ and $(\mathcal{M}\mathcal{E}_k,\infty)^*$, but it is not finite-by-nilpotent.

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2. **Proofs of Theorem 1.1 and Corollary 1.2**

Let $\mathcal{E}(\infty)$ the class of groups in which every infinite subset contains two distinct elements x, y such that $[x,_n y] = 1$ for a positive integer $n = n(x, y)$. In [15], it is proved that a finitely generated soluble group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent. We will extend this result to finitely generated hyper-(Abelian-by-finite) groups (Proposition 2.5).

Our first lemma is a weaker version of Lemma 11 of [23], but we include a proof to keep our paper reasonably self contained.

Lemma 2.1. *Let G be a finitely generated Abelian-by-finite group. If G is in the class $(\mathcal{FN}, \infty)$, then it is finite-by-nilpotent.*

Proof. Let G be a finitely generated infinite Abelian-by-finite group in the class $(\mathcal{FN}, \infty)$. Hence there is a normal torsion-free Abelian subgroup A of finite index. Let x be a non trivial element in A and let g in G. Then the subset $\{x^i g : i \text{ a positive integer}\}$ is infinite, so there are two positive integers m, n such that $\langle x^m g, x^n g\rangle$ is finite-by-nilpotent, hence $\langle x^r, x^n g\rangle$ is finite-by-nilpotent where $r = m-n$. Thus there are two positive integers c and d such that $[x^r,_c x^n g]^d = 1$. The element x being in A which is Abelian and normal in G, we have $[x^r,_c x^n g] = [x^r,_c g] = [x,_c g]^r$; so $[x,_c g]^{r.d} = 1$. Now $[x,_c g]$ belongs to the torsion-free group A, so $[x,_c g] = 1$. It follows that x is a right Engel element of G. Since G is Abelian-by-finite and finitely generated, it satisfies the maximal condition on subgroups; so the set of right Engel elements of G coincides with its hypercentre which is equal to $Z_i(G)$, the $(i+1)$-th term of the upper central series of G, for some integer $i > 0$ [17, Theorem 7.21]. Hence, $A \leq Z_i(G)$; and since A is of finite index in G, $G/Z_i(G)$ is finite. Thus, by a result of Baer [10, Theorem 1], G is finite-by-nilpotent. □

Lemma 2.2. *Let G be a finitely generated Abelian-by-finite group. If G is in the class $\mathcal{E}(\infty)$, then it is finite-by-nilpotent.*

Proof. Let G be an infinite finitely generated Abelian-by-finite group in $\mathcal{E}(\infty)$, and let A be an Abelian normal subgroup of finite index in G. It is clear that all infinite subsets of G contains two different elements x, y such that $xA = yA$; so $y = xa$ for some a in A and $\langle x, y\rangle = \langle x, a\rangle$. Thus $\langle x, y\rangle$ is a finitely generated metabelian group in the class $\mathcal{E}(\infty)$. It follows by the result of Longobardi and Maj [15, Theorem 1], that $\langle x, y\rangle$

is finite-by-nilpotent. Hence G is in the class $(\mathcal{FN}, \infty)$. Now, by Lemma 2.1, G is finite-by-nilpotent; as required. □

Lemma 2.3. *A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is nilpotent-by-finite.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Since $\mathcal{E}(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that G is not nilpotent-by-finite but every proper homomorphic image of G is in the class $\mathcal{NF}$. Since G is hyper-(Abelian-by-finite), G contains a non-trivial normal subgroup H such that H is finite or Abelian; so we have G/H is in $\mathcal{NF}$. If H is finite then G is nilpotent-by-finite, a contradiction. Consequently H is Abelian and so G is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, G is a finite extension of a soluble group; there is therefore a normal soluble subgroup K of G of finite index. Now, K is a finitely generated soluble group in the class $\mathcal{E}(\infty)$; it follows, by the result of Longobardi and Maj [15, Theorem 1], that K is finite-by-nilpotent. By a result of P. Hall [10, Theorem 2], K is nilpotent-by-finite and so G is nilpotent-by-finite, a contradiction. Now, the Lemma is shown. □

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 2.3 has the following consequence:

Corollary 2.4. *Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Then G satisfies the maximal condition on subgroups.*

Proposition 2.5. *A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in $\mathcal{E}(\infty)$. According to Corollary 2.4, G satisfies the maximal condition on subgroups. Now, since $\mathcal{E}(\infty)$ is a quotient closed class, we may assume that every proper homomorphic image of G is in $\mathcal{FN}$, but G itself is not in $\mathcal{FN}$. Our group G being hyper-(Abelian-by-finite), contains a non-trivial normal subgroup H such that H is finite or Abelian; so by hypothesis G/H is in the class $\mathcal{FN}$. If H is finite, then G is finite-by-nilpotent, a contradiction. Consequently H is Abelian and so G is in the class $\mathcal{A}(\mathcal{FN})$, hence G is in $(\mathcal{AF})\mathcal{N}$. Now, since G satisfies the maximal condition on subgroups, it follows from Lemma 2.2, that G is in $(\mathcal{F}\mathcal{N})\mathcal{N}$, so it is in $\mathcal{F}(\mathcal{N}\mathcal{N})$. Consequently, there is a finite normal subgroup K of G such that G/K is soluble. The group G/K, being a finitely generated soluble group in the class $\mathcal{E}(\infty)$, is in $\mathcal{F}\mathcal{N}$, by the result of Longobardi and Maj [15, Theorem 1]. So G is in the class $\mathcal{F}\mathcal{N}$, which is a contradiction and the Proposition is shown. □

The remainder of the proof of Theorem 1.1 is adapted from that of Lennox's Theorem [11, Theorem 3]

Lemma 2.6. *Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega,\infty)^*$. If G is residually nilpotent, then G is in the class $\mathcal{F}\mathcal{N}$.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega,\infty)^*$ and assume that G is residually nilpotent. Let X be an infinite subset of G, there are two distinct elements x and y of X such that $\langle x, x^y\rangle \in \Omega$. It follows that there exists a positive integer k such that $\gamma_k(\langle x, x^y\rangle) = \gamma_{k+1}(\langle x, x^y\rangle)$. The group $\langle x, x^y\rangle$, being a subgroup of G, is residually nilpotent, so $\bigcap_{i\in\mathbb{N}}\gamma_i(\langle x, x^y\rangle) = 1$. Hence $\gamma_k(\langle x, x^y\rangle) = \bigcap_{i\in\mathbb{N}}\gamma_i(\langle x, x^y\rangle) = 1$. Since $\langle x, x^y\rangle = \langle [y,x], x\rangle$; $\gamma_k(\langle [y,x], x\rangle) = 1$, thus $[y,_k x] = 1$. We deduce that G is a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. It follows, by Proposition 2.5, that G is in the class $\mathcal{F}\mathcal{N}$, as required. □

Lemma 2.7. *If G is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega,\infty)^*$, then it is nilpotent-by-finite.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in $(\Omega,\infty)^*$. Since finitely generated nilpotent-by-finite groups are finitely presented and $(\Omega,\infty)^*$ is a quotient closed class of groups, by [17, Lemma 6.17], we may assume that every proper quotient of G is nilpotent-by-finite, but G itself is not nilpotent-by-finite. Since G is hyper-(Abelian-by-finite), it contains a non-trivial normal subgroup K such that K is finite or Abelian; so G/K is in $\mathcal{N}\mathcal{F}$. In this case, K is Abelian and so G is in the class $\mathcal{A}(\mathcal{N}\mathcal{F})$ and therefore it is in the class $(\mathcal{A}\mathcal{N})\mathcal{F}$. Consequently, G has a normal subgroup N of finite index such that N is Abelian-by-nilpotent. Moreover, N being a subgroup of finite index in a finitely generated group, is itself finitely generated, and so N is a finitely generated Abelian-by-nilpotent group. It follows, by a result of Segal [19, Corollary 1], that N has a residually nilpotent normal subgroup of finite index. Thus, G has a residually nilpotent normal subgroup H, of finite index. Therefore, H is residually nilpotent and it is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. So, by Lemma 2.6, H is in the class $\mathcal{FN}$, hence H is in the class $\mathcal{NF}$. Thus G is in the class $\mathcal{NF}$, a contradiction which completes the proof. □

Lemma 2.8. *Let G be a finitely generated group in the class $(\Omega, \infty)^*$ which has a normal nilpotent subgroup N such that G/N is a finite cyclic group. Then G is in the class $\mathcal{FN}$.*

Proof. We prove by induction on the order of G/N that G is in the class $\mathcal{FN}$. Let $n = |G/N|$; if $n = 1$, then $G = N$ and G is nilpotent. Now suppose that $n > 1$ and let q be a prime dividing n. Since G/N is cyclic, it has a normal subgroup of index q. Thus G has a normal subgroup H of index q containing N. Since $|H/N| < |G/N|$, then by the inductive hypothesis, H is in the class $\mathcal{FN}$. Let T be the torsion subgroup of H. Since H is finitely generated, T is finite. So H/T is a finitely generated torsion-free nilpotent group. Therefore, by Gruenberg [18, 5.2.21], H/T is residually a finite p-group for all primes p and hence, in particular, H/T is residually a finite q-group. But H has index q in G from which we get that G/T is residually a finite q-group [20, Exercise 10, page 17]. This means that G/T is residually nilpotent. It follows, by Lemma 2.6, that G/T is in the class $\mathcal{FN}$. So G itself is in $\mathcal{FN}$. □

Proof of Theorem 1.1. Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. Hence, by Lemma 2.7, G is in the class $\mathcal{NF}$. Let K be a normal nilpotent subgroup of G such that G/K is finite. Since K is a finitely generated nilpotent group, it has a normal torsion-free subgroup of finite index [18, 5.4.15 (i)]. Thus, G has a normal torsion-free nilpotent subgroup N of finite index. Let x be a non-trivial element of G. Since N is finitely generated, $\langle N, x\rangle$ is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. Furthermore, $\langle N, x\rangle / N$ is cyclic. Therefore, by Lemma 2.8, $\langle N, x\rangle$ is in the class $\mathcal{FN}$. Consequently, there is a finite normal subgroup H of $\langle N, x\rangle$ such that $\langle N, x\rangle / H$ is nilpotent. Therefore $\gamma_{k+1}(\langle N, x\rangle) \leq H$ for some positive integer k; so $\gamma_{k+1}(\langle N, x\rangle)$ is finite. Hence, there is a positive integer m such that $[g,_k x]^m = 1$, for all $g \in N$. Since $[g,_k x]$ is an element of the torsion-free group N, we get that $[g,_k x] = 1$. Thus, g is a right Engel element of G; so $N \subseteq R(G)$, where $R(G)$ denotes the set of right Engel elements of G. Moreover, since G is a finitely generated nilpotent-by-finite group, it satisfies the maximal condition on subgroups. Therefore, from Baer [17, Theorem 7.21], $R(G)$ coincides with the hypercentre of G which equal to $Z_n(G)$ for some positive integer n. Thus $N \leq Z_n(G)$, so $Z_n(G)$ is of finite index in G. It follows, by a result of Baer [10, Theorem 1], that G is in the class $\mathcal{FN}$. □

Proof of Corollary 1.2. (i) Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega_k,\infty)^*$; from Theorem 1.1, G is in the class $\mathcal{FN}$. Let H be a normal finite subgroup of G such that G/H is nilpotent. It is clear that G/H is in the class $(\Omega_k,\infty)^*$. Let $\bar{X}$ be an infinite subset of G/H; there are therefore two distinct elements $\bar{x}=xH$, $\bar{y}=yH$ $(x,y\in G)$ of $\bar{X}$ such that $\langle \bar{x},\bar{x}^{\bar{y}}\rangle \in \Omega_k$, so $\gamma_k(\langle \bar{x},\bar{x}^{\bar{y}}\rangle)=\gamma_{k+1}(\langle \bar{x},\bar{x}^{\bar{y}}\rangle)$. Now, since $\langle \bar{x},\bar{x}^{\bar{y}}\rangle$ is nilpotent, there is an integer i such that $\gamma_i(\langle \bar{x},\bar{x}^{\bar{y}}\rangle)=1$; so $\gamma_k(\langle \bar{x},\bar{x}^{\bar{y}}\rangle)=1$. Since $\langle \bar{x},\bar{x}^{\bar{y}}\rangle=\langle [\bar{y},\bar{x}],\bar{x}\rangle$, we have $\gamma_k(\langle [\bar{y},\bar{x}],\bar{x}\rangle)=1$ and thus $[\bar{y},_k\bar{x}]=1$. Consequently, G/H is in the class $\mathcal{E}_k(\infty)$ of groups in which every infinite subset contains two distinct elements g, h such that $[g,_k h]=1$. The group G/H, being a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$; it follows by a result of Abdollahi [2, Theorem 3], that there is an integer $c=c(k)$, depending only on k, such that $(G/H)/Z_c(G/H)$ is finite. By a result of Baer [10, Theorem 1], $\gamma_{c+1}(G/H)=\gamma_{c+1}(G)H/H$ is finite; and since H is finite, $\gamma_{c+1}(G)$ is finite. According to a result of P. Hall [10, 1.5], $G/Z_c(G)$ is finite.

(ii) If G is in the class $(\Omega_2,\infty)^*$, then by Theorem 1.1 G is finite-by-nilpotent. Therefore, G has a finite normal subgroup H such that G/H is nilpotent. Since G/H is in the class $(\Omega_2,\infty)^*$, it is in the class $\mathcal{E}_2(\infty)$. Hence, by Abdollahi [1, Theorem], $(G/H)/Z_2(G/H)$ is finite, so $\gamma_3(G/H)$ is finite. Since H is finite, $\gamma_3(G)$ is finite. It follows, by P. Hall [10, 1.5], that $G/Z_2(G)$ is finite.

(iii) Now if G is in the class $(\Omega_3,\infty)^*$, then by Theorem 1.1 G has a finite normal subgroup H such that G/H is nilpotent. Since G/H is in the class $(\Omega_3,\infty)^*$, it is in the class $\mathcal{E}_3(\infty)$. Hence, by Abdollahi [2, Theorem 1] G/H is in the class $\mathcal{FN}_3^{(2)}$; consequently G is in the class $\mathcal{FN}_3^{(2)}$. □

3. **Proofs of Theorem 1.3 and Corollary 1.4**

We start by showing a weaker version of Theorem 1.3:

Lemma 3.1. *A finitely generated hyper-(Abelian-by-finite) group in the class* $(\mathcal{MN}, \infty)^*$ *is finite-by-nilpotent.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{MN}, \infty)^*$, and let X be an infinite subset of G. There are therefore two distinct elements x, y of X such that $\langle x, x^y \rangle$ is in the class $\mathcal{MN}$, so there exists a normal subgroup N of $\langle x, x^y \rangle$ such that N is in $\mathcal{M}$ and $\langle x, x^y \rangle / N$ is nilpotent. Now, $\gamma_{i+1}(\langle x, x^y \rangle) \leq N$ for some positive integer i, therefore $\gamma_{i+1}(\langle x, x^y \rangle) \geq \gamma_{i+2}(\langle x, x^y \rangle) \geq ...$ is an infinite descending sequence of normal subgroups of N; however N is in $\mathcal{M}$, therefore there exists a positive integer $n \geq i+1$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$. Hence, G is in the class $(\Omega, \infty)^*$; it follows, by Theorem 1.1, that G is finite-by-nilpotent. □

Lemma 3.2. *A finitely generated hyper-(Abelian-by-finite) group in the class* $(\mathcal{ME}, \infty)^*$ *is nilpotent-by-finite.*

Proof. Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}, \infty)^*$. Since $(\mathcal{ME}, \infty)^*$ is a closed quotient class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that G is not nilpotent-by-finite, but every proper homomorphic image of G is nilpotent-by-finite. Since G is hyper-(Abelian-by-finite), there exists a non-trivial normal subgroup H of G such that H is finite or Abelian; so we have G/H is nilpotent-by-finite. If H is finite then G is nilpotent-by-finite, a contradiction. Consequently H is Abelian and so G is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, G is a finite extension of a soluble group. Let K be a normal soluble subgroup of G of finite index. Clearly, K is in $(\mathcal{ME}, \infty)^*$, and since all soluble Engel group coincides with its Hirsch-Plotkin radical which is locally nilpotent [17, Theorem 7.34], we deduce that K is in the class $(\mathcal{MN}, \infty)^*$; it follows by Lemma 3.1 that K is finite-by-nilpotent. According to a result of P. Hall [10, Theorem 2], K is nilpotent-by-finite. Thus, G is nilpotent-by-finite, a contradiction. The proof is now complete. □

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 3.2 has the following consequence:

Corollary 3.3. *Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}, \infty)^*$. Then G satisfies the maximal condition on subgroups.*

Proof of Theorem 1.3. It is clear that all finite-by-nilpotent groups are in the class $(\mathcal{ME},\infty)^*$. Conversely, let G be a finitely generated hyper-(Abelian-by-finite) group in $(\mathcal{ME},\infty)^*$. According to Corollary 3.3, G satisfies the maximal condition on subgroups. Since Engel groups satisfying the maximal condition on subgroups are nilpotent [18, 12.3.7], we deduce that G is in the class $(\mathcal{MN},\infty)^*$. It follows, by Lemma 3.1, that G is in the class $\mathcal{FN}$; as required. □

Proof of Corollary 1.4. (*i*) Let G be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}_k,\infty)^*$; from Theorem 1.3, G is in the class $\mathcal{FN}$. Let N be a normal finite subgroup of G such that G/N is nilpotent. Since G/N is nilpotent and finitely generated, its torsion subgroup T/N is finite, so T is finite and G/T is a torsion-free nilpotent group. Clearly, the property $(\mathcal{ME}_k,\infty)^*$ is inherited by G/T, and since G/T is torsion-free and soluble, it belongs to $(\mathcal{E}_k,\infty)^*$ [17, Theorem 5.25]. Let $\bar{X}$ be an infinite subset of G/T; there are therefore two distinct elements $\bar{x}=xT$, $\bar{y}=yT$ $(x,y\in G)$ of $\bar{X}$ such that $\langle\bar{x},\bar{x}^{\bar{y}}\rangle$ is a k-Engel group. Since $\langle\bar{x},\bar{x}^{\bar{y}}\rangle=\langle[\bar{y},\bar{x}],\bar{x}\rangle$, we have $[\bar{y},_{k+1}\bar{x}]=[[\bar{y},\bar{x}],_k\bar{x}]=1$. Hence, G/T is in the class $\mathcal{E}_{k+1}(\infty)$. The group G/T, being a finitely generated soluble group in the class $\mathcal{E}_{k+1}(\infty)$; it follows by a result of Abdollahi [2, Theorem 3], that there is an integer $c=c(k)$, depending only on k, such that $(G/T)/Z_c(G/T)$ is finite. By a result of Baer [10, Theorem 1], $\gamma_{c+1}(G/T)=\gamma_{c+1}(G)T/T$ is finite; and since T is finite, $\gamma_{c+1}(G)$ is finite. According to a result of P. Hall [10, 1.5], $G/Z_c(G)$ is finite.

(*ii*) If G is in the class $(\mathcal{MA},\infty)^*=(\mathcal{ME}_1,\infty)^*$, then by Theorem 1.3, G is finite-by-nilpotent. We proceed as in (i) until we obtain that G/T is in the class $\mathcal{E}_2(\infty)$. Hence, by Abdollahi [1, Theorem], $(G/T)/Z_2(G/T)$ is finite, so $\gamma_3(G/T)$ is finite. Since T is finite, $\gamma_3(G)$ is finite. It follows, by P. Hall [10, 1.5], that $G/Z_2(G)$ is finite.

(*iii*) Now if G is in the class $(\mathcal{ME}_2,\infty)^*$, we proceed as in (i) until we obtain that G/T is in the class $\mathcal{E}_3(\infty)$. Hence, by Abdollahi [2, Theorem 1] G/T is in the class $\mathcal{FN}_3^{(2)}$; consequently G is in the class $\mathcal{FN}_3^{(2)}$. □

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