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Didier Arnal,
Nadia Bel Baraka and Norman J. Wildberger

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Diamond representations of $\mathfrak{sl}(n)$

Didier Arnal
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Abstract

In [6], there is a graphic description of any irreducible, finite dimensional $\mathfrak{sl}(3)$ module. This construction, called diamond representation is very simple and can be easily extended to the space of irreducible finite dimensional $\mathcal{U}_q(\mathfrak{sl}(3))$-modules.

In the present work, we generalize this construction to $\mathfrak{sl}(n)$. We show it is in fact a description of the reduced shape algebra, a quotient of the shape algebra of $\mathfrak{sl}(n)$. The basis used in [6] is thus naturally parametrized with the so called quasi standard Young tableaux. To compute the matrix coefficients of the representation in this basis, it is possible to use Groebner basis for the ideal of reduced Plücker relations defining the reduced shape algebra.

1. Introduction

In this paper, we consider the irreducible finite dimensional representations of the Lie algebra $\mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$. Of course these representations are well known and there are very explicit descriptions for them, for instance in [2].

First, $\mathfrak{sl}(n)$ acts naturally on $\mathbb{C}^n$, its fundamental representations are the natural actions on $\mathbb{C}^n, \wedge^2\mathbb{C}^n, \dots, \wedge^{n-1}\mathbb{C}^n$, they have highest weights $\omega_1, \dots, \omega_{n-1}$. Each simple $\mathfrak{sl}(n)$-module has a highest weight λ and this highest weight characterizes the module. Note $\mathbb{S}^\lambda$ this module, it is a submodule of the tensor product

$$Sym^{a_1}(\mathbb{C}^n) \otimes Sym^{a_2}(\wedge^2\mathbb{C}^n) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1}\mathbb{C}^n),$$

if $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$.

The direct sum $\mathbb{S}^\bullet$ of all the simple modules has a natural realization as the shape algebra of $\mathfrak{sl}(n)$, *i.e.* as the algebra $\mathbb{C}[SL(n)]^{N^+}$ of polynomial functions on the group $SL(n)$, which are invariant under the right multiplication by upper triangular matrices. Let g be an element in $SL(n)$, denote $\delta^{(s)}_{i_1,\dots,i_s}(g)$ the determinant of the submatrix of g obtained by considering the s first columns of g and the rows $i_1 < \cdots < i_s$, then $\mathbb{S}^\bullet$ is generated as an algebra by the functions $\delta^{(s)}_{i_1,\dots,i_s}$. More precisely, it is the quotient of $\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]$ by the ideal $P(\delta)$ generated by the Plücker relations.

Generally a parametrization of a basis for $\mathbb{S}^\lambda$ is given by the set of semi-standard Young tableaux T of shape λ *i.e.* with a_{n-1} columns of size $n-1$, $\dots$, a_1 columns of size 1.

Using this description, we give here a natural ordering on the set of variables $\delta^{(s)}_{i_1,\dots,i_s}$, we determine the Groebner basis of $P(\delta)$ for this ordering, getting the corresponding basis of the quotient as monomials δ^T, for T semi-standard.

Thus the action of upper triangular matrices on this basis can be easily computed. (See for instance the description given in [4]).

On the other hand, in [6], N. Wildberger gave a really different presentation of the simple $\mathfrak{sl}(3)$-modules. This description is based on the construction of the diamond cone for $\mathfrak{sl}(3)$, it is an infinite dimensional indecomposable module for the Heisenberg Lie algebra with a very explicit basis. The matrix coefficients are integral numbers and fixing the highest weight λ, it is easy to build the corresponding representation of $\mathfrak{sl}(3)$, on the submodule generated by this vector in the diamond cone.

In this paper, we extend this presentation to $\mathfrak{sl}(n)$. In fact the diamond cone module is a quotient of the shape algebra. We call this quotient the reduced shape algebra. It is the quotient of $\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]$ by the ideal $P_{red}(\delta)$ sum of the ideal of Plücker relations and the ideal generated by $\delta^{(s)}_{1,\dots,s} - 1$.

With the same approach as above, we define a new ordering on the variables $\delta^{(s)}_{i_1,\dots,i_s}$, with this ordering, we can compute the Groebner basis for $P_{red}(\delta)$ and the corresponding basis for the quotient : the set of monomials δ^T, for some Young tableaux T called here quasi-standard. The action of the upper triangular matrices on this basis is easy to compute : this gives us the diamond cone for $\mathfrak{sl}(n)$.

In order to refind the complete $\mathfrak{sl}(n)$-modules, we have to define a symmetry on each $\mathbb{S}^\lambda$ and on the corresponding submodule in the reduced shape algebra. This symmetry exchanges the role of N^+ and N^- and we get the complete $\mathfrak{sl}(n)$ representation.

Unfortunately, this symmetry corresponds to a modification of the ordering on Young tableaux, thus, if $n > 3$ to a different basis in $\mathbb{S}^\lambda$. The $\mathfrak{n}^-$ action on the first base is not so simple as in [6].

2. Usual (algebraic) presentation of the $\mathfrak{sl}(n)$ simple modules

Let us consider the Lie algebra $\mathfrak{sl}(n) = \mathfrak{sl}(n,\mathbb{C})$: it is the set of $n \times n$ traceless matrices, *i.e.* the Lie algebra of the Lie group $SL(n)$ of $n \times n$ matrices, with determinant 1. The Cartan algebra $\mathfrak{h}$ is the space of diagonal matrices:

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix}, \quad \theta_j \in \mathbb{C}, \quad \theta_1 + \dots + \theta_n = 0 \right\}.$$

We put $\alpha_i(H) = \theta_i$. The root system of $\mathfrak{sl}(n)$ is the set of linear form on $\mathfrak{h}$ generated by the $\alpha_i - \alpha_j$, $(i \neq j)$.

The usual basis Δ for the root system is given by :

$$\Delta = \{\alpha_i - \alpha_{i+1},\ i = 1, 2, \dots, n-1\}$$

The root space corresponding to the positive root $\eta = \alpha_i - \alpha_j$ $(i < j)$ is generated by the upper triangular matrix:

$$X_\eta = \begin{bmatrix} 0 & & & & \\ & \ddots & 1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}.$$

The root space corresponding to $-\eta$ is generated by lower triangular matrix:

$$Y_\eta = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \ddots & \\ 1 & & & \ddots & \\ & & & & 0 \end{bmatrix} = {}^tX_\eta$$

these matrices generate $\mathfrak{sl}(n)$ as a Lie algebra.

A weight λ for $\mathfrak{sl}(n)$ is a linear form :

$$\lambda : \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} \mapsto \sum_{i=1}^{n-1} a_i\theta_1 + \sum_{i=2}^{n-1} a_i\theta_2 + \cdots + a_{n-1}\theta_{n-1}.$$

If $a_1, \ldots, a_{n-1}$ are positive integral numbers, we shall say that λ is a dominant integral weight. This is the case if and only if λ is a linear combination $\lambda = \sum_{j=1}^{n-1} a_j\omega_j$, with positive integral coefficients a_j, of the fundamental weights:

$$\omega_j = \alpha_1 + \cdots + \alpha_j : \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} \mapsto \theta_1 + \cdots + \theta_j \qquad (1 \leq j \leq n-1).$$

The set of simple $\mathfrak{sl}(n)$-modules up to equivalence is isomorphic to the set of dominant integral weights. More precisely, $\mathfrak{sl}(n)$ acts naturally on $V = \mathbb{C}^n$ (with canonical basis $e_1, \ldots, e_n$), thus also on the totally antisymmetric tensor products $\wedge^j V$ $(j = 1, \ldots, n-1)$ and on the symmetric tensor products $Sym^{a_j}(\wedge^j V)$ and finally on

$$Sym^{a_1}(V) \otimes Sym^{a_2}(\wedge^2 V) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1} V).$$

For each dominant integral weight $\lambda = \sum a_j\omega_j$, the corresponding simple module $\mathbb{S}^\lambda(V)$ is the submodule of

$$Sym^{a_1}(V) \otimes Sym^{a_2}(\wedge^2 V) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1} V).$$

generated by the vector:

$$v^\lambda = (e_1)^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})^{a_{n-1}}.$$

With this construction, we get each simple $\mathfrak{sl}(n)$-module, and two distinct weights λ, λ' give rise to inequivalent simple $\mathfrak{sl}(n)$-modules.

Of course, this action can be exponentiated to a representation of $SL(n)$. Let us thus put

$$\Omega = \begin{bmatrix} 0 & & & \varepsilon_n \\ & & \cdot & \\ & \cdot & & \\ \varepsilon_n & & & 0 \end{bmatrix}$$

where $\varepsilon_n = 1$ if $\left[\frac{n}{2}\right]$ is even and $\varepsilon_n = e^{\frac{i\pi}{n}}$ if $\left[\frac{n}{2}\right]$ is odd. Then Ω belongs to $SL(n)$. In fact, this matrix, acting by adjoint action generates the longest element of the Weyl group of $SL(n)$. It corresponds to a change in the choice of simple roots and nilpotent subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, if $X = [x_{ij}]$ is a strictly upper triangular matrix, $\Omega^{-1}X\Omega = \left[x_{(n+1-i)(n+1-j)}\right]$ is strictly lower triangular. Let us put:

$$v_-^\lambda = (e_n)^{a_1} \otimes (e_n \wedge e_{n-1})^{a_2} \otimes \cdots \otimes (e_n \wedge \cdots \wedge e_2)^{a_{n-1}} = \varepsilon_n^{-|\lambda|}\Omega.v^\lambda,$$

with $|\lambda| = a_1 + 2a_2 + \cdots + (n-1)a_{n-1}$. Then v_-^λ is a lowest weight vector in $\mathbb{S}^\lambda(V)$.

3. The shape algebra: abstract algebraic presentation

Let us put:

$$\mathbb{S}^\bullet(V) = \bigoplus_\lambda \mathbb{S}^\lambda(V).$$

Since we have an explicit realization of each highest weight vector, it is possible to define a natural comultiplication Δ on $\mathbb{S}^\bullet(V)$, just by defining

$$\Delta : \mathbb{S}^\lambda(V) \longrightarrow \bigoplus_{\mu+\nu=\lambda} \mathbb{S}^\mu(V) \otimes \mathbb{S}^\nu(V)$$

as the unique $\mathfrak{sl}(n)$-morphism sending v^λ on

$$\Delta(v^\lambda) = \sum_{\mu+\nu=\lambda} v^\mu \otimes v^\nu.$$

Δ is cocommutative. The contragredient module $(\mathbb{S}^\lambda)^*$ is naturally identified with $\mathbb{S}^{{}^t\lambda}$ where ${}^t\lambda = \sum a_{n-i}\omega_i$ if $\lambda = \sum a_i\omega_i$. By transposition, Δ defines a commutative multiplication m on $\mathbb{S}^\bullet(V)$:

$$m = {}^t\Delta : \mathbb{S}^{{}^t\mu}(V) \otimes \mathbb{S}^{{}^t\nu}(V) \longrightarrow \mathbb{S}^{{}^t\mu + {}^t\nu}(V).$$

By definition, if $\mu = \sum_j b_j\omega_j$, $\nu = \sum_j c_j\omega_j$,

$$m(v^\mu \otimes v^\nu) = v^\mu.v^\nu = v^{\mu+\nu} = e_1^{b_1+c_1} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})^{b_{n-1}+c_{n-1}}.$$

Since each isotypic component of the $SL(n)$ module $\mathbb{S}^\bullet(V)$ is simple the multiplication m is characterized by this relation and the condition

$$m\left(\mathbb{S}^\mu(V) \otimes \mathbb{S}^\nu(V)\right) \subset \mathbb{S}^{\mu+\nu}(V).$$

We shall call shape algebra of $SL(n)$ the algebra $\mathbb{S}^\bullet(V)$ equipped with the above multiplication.

By construction the shape algebra is generated as an algebra by the subspace $V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^{n-1}V$. Thus it is a quotient of the algebra denoted in [2]:

$$\begin{aligned} A^\bullet(V) &= Sym^\bullet\left(V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^{n-1}V\right) \\ &= \bigoplus_{a_1,\dots,a_{n-1}} Sym^{a_{n-1}}(\wedge^{n-1}V) \otimes \cdots \otimes Sym^{a_1}(V). \end{aligned}$$

We define now the ideal of Plücker relations: it is the ideal P of $A^\bullet(V)$ generated by the vectors in $Sym^2(\wedge^p V)$:

$$\begin{aligned} &(e_{i_1} \wedge \cdots \wedge e_{i_p}).(e_{j_1} \wedge \cdots \wedge e_{j_p}) + \\ &\qquad + \sum_{\ell=1}^{p} (-1)^\ell (e_{j_1} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\ell}} \wedge \cdots \wedge e_{i_p}).(e_{i_\ell} \wedge e_{j_2} \wedge \cdots \wedge e_{j_p}) \end{aligned}$$

and by the vectors in $\wedge^p V \otimes \wedge^q V$ $(p > q)$

$$\begin{aligned} &(e_{i_1} \wedge \cdots \wedge e_{i_p}).(e_{j_1} \wedge \cdots \wedge e_{j_q}) + \\ &\qquad + \sum_{\ell=1}^{p} (-1)^\ell (e_{j_1} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\ell}} \wedge \cdots \wedge e_{i_p}).(e_{i_\ell} \wedge e_{j_2} \wedge \cdots \wedge e_{j_q}). \end{aligned}$$

Theorem 3.1. (Characterization of $\mathbb{S}^\bullet(V)$) *The shape algebra $\mathbb{S}^\bullet(V)$ is the quotient of $A^\bullet(V)$ by the ideal P.*

This theorem is well known. There is a complete proof in [2] p. 235, this result is cited by Towber in [4] as a theorem due to Kostant.

We define a symmetry τ in $\mathbb{S}^\bullet(V)$ just by putting:

$$\tau(v) = \Omega.v \quad \text{if} \quad v \in \mathbb{S}^\bullet(V).$$

Since the multiplication is a morphism of $\mathfrak{sl}(n)$ and $SL(n)$ modules, $\tau(vv') = \tau(v)\tau(v')$. Especially, we can define the multiplication just as above by fixing $v_-^\lambda.v_-^\mu = v_-^{(\lambda+\mu)}$.

Now for each matrix A in $\mathfrak{sl}(n)$, $\Omega A\Omega = {}^\tau A$ is the matrix defined by a central symmetry on the entries of A:

$${}^\tau A = [a_{n+1-i,n+1-j}] \text{ if } A = [a_{i,j}]$$

If A is the matrix X_η for a positive root $\eta = \alpha_i - \alpha_j$, ${}^\tau X_\eta = \Omega X_\eta \Omega$ is the matrix Y_{τ_η} if ${}^\tau\eta$ is the positive root ${}^\tau\eta = \alpha_{n+1-j} - \alpha_{n+1-i}$ Then for each v in $\mathbb{S}^\bullet$:

$$(\tau \circ X_\eta \circ \tau)(v) = \Omega X_\eta \Omega \ v = Y_{\tau_\eta} v$$

4. The shape algebra: geometric presentation

The shape algebra can also be viewed as an algebra of functions on a quotient $SL(n)/N^+$ of the Lie group $SL(n)$. Denote N^+ the subgroup of matrices $n^+ = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

Let us consider the space $\mathbb{C}[SL(n)] = \mathbb{C}[g_{ij}]/(det - 1)$ of all polynomial functions f with respect to the entries g_{ij} of the matrix $g \in SL(n)$. There is a $SL(n) \times SL(n)$ action on this space, defined as follows:

$$((g_1, g_2).f)(g) = f({}^t g_1 g g_2).$$

Since this space is generated by the invariant finite dimensional subspaces of class of functions with degree less than N ($N = 0, 1 \dots$), this action is completely reducible in a sum of finite dimensional simple $SL(n) \times SL(n)$ modules. The highest vector for these modules are class of functions f such that:

$$f({}^t n_1^+ g n_2^+) = f(g), \qquad n_1^+ \in N^+, \ n_2^+ \in N^+.$$

But, let us consider the restriction of f to the dense set of the elements g such that, for $s = 1, \dots, n$, $\delta^{(s)}_{1,2,\dots,s}(g) \neq 0$. On this set, using the Gauss method, we can reduce g to a diagonal matrix, getting:

$$g = {}^t n_1^+ \begin{bmatrix} \delta^{(1)}_1(g) & & & & 0 \\ & \dfrac{\delta^{(2)}_{1,2}(g)}{\delta^{(1)}_1(g)} & & & \\ & & \ddots & & \\ & & & \dfrac{\delta^{(n-1)}_{1,2,\dots,n-1}(g)}{\delta^{(n-2)}_{1,\dots,n-2}(g)} & \\ 0 & & & & \dfrac{1}{\delta^{(n-1)}_{1,2,\dots,n-1}(g)} \end{bmatrix} n_2^+.$$

If f is a highest weight vector, its weight is (λ, λ) ($\lambda = \sum a_i \omega_i$), then f is a polynomial function in the variables

$$\delta^{(1)}_1(g), \frac{\delta^{(2)}_{1,2}(g)}{\delta^{(1)}_1(g)}, \dots, \frac{\delta^{(n-1)}_{1,2,\dots,n-1}(g)}{\delta^{(n-2)}_{1,\dots,n-2}(g)}, \frac{1}{\delta^{(n-1)}_{1,2,\dots,n-1}(g)},$$

homogeneous with degree $a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \dots, a_{n-1}, 0$, *i.e.* the function f is a multiple of the function:

$$\delta^\lambda = \left(\delta^{(1)}_1\right)^{a_1} \left(\delta^{(2)}_{1,2}\right)^{a_2} \cdots \left(\delta^{(n-1)}_{1,2,\dots,n-1}\right)^{a_{n-1}}.$$

Acting with only the first factor $SL(n)$ on these functions, we get all the N^+ right invariant polynomial functions on $SL(n)$. Due to the form of the bi-invariant functions f, these functions are polynomial functions in the δ-variables :

$$\mathbb{C}[SL(n)]^{N^+} \simeq \mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]/P(\delta),$$

where $P(\delta)$ is an ideal.

Acting on δ^λ ($\lambda = \sum_i a_i \omega_i$) on the left by $N^- = {}^t(N^+)$, we get polynomial functions which contains only monomials of the form:

$$\prod_{s=1}^{n-1} \prod_{k=1}^{a_s} \delta^{(s)}_{i_1^k,\dots,i_s^k}.$$

Let us call $V^{a_1,\dots,a_{n-1}}$ the space of such functions. In view of our description, it is a simple module and the isotypic component of type λ in $\mathbb{C}[SL(n)]^{N^+}$.

Finally the usual pointwise multiplication of polynomial functions send $V^{a_1,\dots,a_{n-1}} \otimes V^{b_1,\dots,b_{n-1}}$ into $V^{(a_1+b_1),\dots,(a_{n-1}+b_{n-1})}$. Thus the above identification

$$\mathbb{S}^\bullet(V) \simeq \mathbb{C}[SL(n)]^{N^+},$$

characterized by $v^\lambda \mapsto \delta^\lambda$ is a morphism of algebra.

Proposition 4.1. (Geometric description of $\mathbb{S}^\bullet(V)$) *The shape algebra is isomorphic to the algebra $\mathcal{O}(SL(n)/N^+)$ of the regular functions on the homogeneous space $SL(n)/N^+$.*

The ideal $P(\delta)$ is the ideal generated by the Plücker relations written on the δ functions.

Remark 4.2. In this presentation of $\mathbb{S}^\bullet(V)$, the $SL(n)$ action on the elements of the shape algebra, viewed as a polynomial function f is very natural since it is just:

$$(g.f)(g') = f({}^t g g'),\ g \in SL(n),\ g' \in SL(n).$$

The symmetry τ can be directly implemented in $\mathbb{C}[SL(n)]^{N^+}$. Indeed τ is up to conjugation by Ω, a morphism of $SL(n)$ modules and the formula

$$\tau(e_1 \wedge \dots \wedge e_s) = \varepsilon_n^s e_n \wedge \dots \wedge e_{n+1-s}$$

becomes here

$$\tau(\delta^{(s)}_{1,2,\dots,s}) = \varepsilon_n^s \delta^{(s)}_{n,(n-1),\dots,(n+1-s)}.$$

But, if we put for any regular function f on $SL(n)$, $(\theta f)(g) = f(\Omega g)$, we define a bijection from $\mathbb{C}[SL(n)]^{N^+}$ into itself such that

$$g.\theta(f) = \theta(\Omega^{-1} g \Omega.f) \quad \text{and} \quad \theta(\delta^{(s)}_{1,\dots,s}) = \varepsilon_n^s \delta^{(s)}_{n,n-1,\dots,n+1-s}.$$

Thus $\tau = \theta$ or:

$$(\tau f)(g) = f(\Omega g).$$

5. **The shape algebra : Combinatorial presentation**

The usual basis of $\mathbb{S}^\lambda(V)$ are parametrized by the semi standard Young tableaux with shape λ. Let us be more precize:

We can naturally associate to each δ variable a column C:

$$\delta^C = \delta^{(p)}_{i_1,\dots,i_p} \longrightarrow \begin{array}{|c|}\hline i_1\\\hline i_2\\\hline \vdots\\\hline i_p\\\hline\end{array}.$$

Then if we identify two Young tableaux which differ only by a permutation of their columns, the set of Young tableaux defines a linear basis for the algebra $\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]$:

$$\delta^T = \delta^{(p_1)}_{i_1,\dots,i_{p_1}}\delta^{(p_2)}_{j_1,\dots,j_{p_2}}\dots\delta^{(p_k)}_{\ell_1,\dots,\ell_{p_k}} \longrightarrow \begin{array}{|c|c|c|c|}\hline \ & \cdots & \ & \ \\\hline \ & \vdots & \ \\\cline{1-3} \ & \cdots \\\cline{1-2} \ \\\cline{1-1}\end{array}$$

$(p_1 \leq p_2 \leq \dots \leq p_k)$. That means, we read the Young tableau from right to left, using the following convention: if two different columns C and C' have the same height p, we put in the first place in T the column C if

$$i_p = i'_p, i_{p-1} = i'_{p-1}, \dots, i_{r+1} = i'_{r+1}, \text{ and } i_r < i'_r.$$

The Plücker relations are quadratic in the δ variables, they correspond to linear combination of Young tableaux with two columns, for instance, we get for $\mathfrak{sl}(3)$ the following relation between tableaux:

$$\delta^{(2)}_{12}\delta^{(1)}_3 - \delta^{(2)}_{23}\delta^{(1)}_1 + \delta^{(2)}_{13}\delta^{(1)}_2 \longrightarrow \begin{array}{|c|c|}\hline 1 & 3\\\hline 2\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 2 & 1\\\hline 3\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}$$

In order to describe a basis for the quotient space:

$$\mathbb{S}^\bullet(V) = \mathbb{C}[\delta^{(j)}_{i_1,\dots,i_j}]/_{P(\delta)},$$

we will use the notion of Groebner basis [1].

Let us consider the algebra $\mathbb{C}[X_1,\dots,X_k]$ of polynomials in the variables X_i and an ideal I of $\mathbb{C}[X_1,\dots,X_k]$.

An ordering on the set $\{X^a = X_1^{a_1}\dots X_k^{a_k},\ a \in \mathbb{N}^k\}$ of monomials is a monomial ordering if it is a well-ordering and if for all $c \in \mathbb{N}^k$, $X^{a+c} > X^{b+c}$ if $X^a > X^b$. For instance, the lexicographic ordering on the words $a_1\dots a_k$, which corresponds to the variables ordering $X_1 > \dots > X_k$, the graded lexicographic ordering, taking into account a degree on some variables, are monomial orderings (see [1]).

Suppose we fix a monomial ordering on the set of monomials, then any polynomial g has an unique leading term $LT(g)$; the greatest monomial happening in g for this ordering.

Definition 5.1. A finite subset $\{g_1, \dots, g_k\}$ of an ideal I is said to be a reduced Groebner basis for I if and only if the leading term of any element of I is divisible by one of the leading term of g_i, if the coefficient of $LT(g_i)$ is 1 for every i and if for all g_i no monomial of g_i is divisible by the leading term of some g_j $j \neq i$.

For each monomial ordering and each ideal I, there is an unique reduced Groebner basis for I [1]. Moreover, if $\{g_1, \dots, g_k\}$ is a reduced Groebner basis for I, then the set of (classes of) monomials which are not divisible by any monomials $LT(g_i)$ $(i = 1, \dots, k)$ is a basis for the quotient $\mathbb{C}[X_1, \dots, X_k]/I$.

Following [2], we know there is in the ideal $P(\delta)$ the following elements for any $p \geq q \geq r$:

$$\delta^{(p)}_{i_1,i_2,\dots,i_p}\delta^{(q)}_{j_1,j_2,\dots,j_q} + \sum_{\substack{A \subset \{i_1,\dots,i_p\} \\ \#A = r}} \pm\delta^{(p)}_{(\{i_1,\dots,i_p\}\setminus A)\cup\{j_1,\dots,j_r\}}\delta^{(p)}_{A\cup\{j_{r+1},\dots,j_q\}} \quad (*)$$

where $\delta^{(p)}_{(\{i_1,\dots,i_p\}\setminus A)\cup\{j_1,\dots,j_r\}} = 0$ if there is a repetition of some index and, if $\{k_1, \dots, k_p\} = (\{i_1, \dots, i_p\} \setminus A) \cup \{j_1, \dots, j_r\}$ and $k_1 < \dots < k_p$, then

$$\delta^{(p)}_{(\{i_1,\dots,i_p\}\setminus A)\cup\{j_1,\dots,j_r\}} = \delta^{(p)}_{k_1,\dots,k_p}.$$

Now if T is a tableau, if T contains ℓ_i columns with height i $(i = 1, \dots, n-1)$, we call shape of T the $(n-1)$-uplet

$$\lambda(T) = (\ell_1, \dots, \ell_{n-1}).$$

We first consider the (total) lexicographic ordering on the family of shapes $\lambda = (\ell_1, \dots, \ell_{n-1})) < \mu = (m_1, \dots, m_{n-1}))$ if and only if:

$$\begin{cases} \ell_1 < m_1 \\ \quad \text{or} \\ \ell_1 = m_1 \text{ and } \ell_2 < m_2 \\ \qquad \dots \\ \quad \text{or} \\ \ell_1 = m_1, \ \ell_2 = m_2, \dots, \ell_{n-2} = m_{n-2} \text{ and } \ell_{n-1} < m_{n-1}. \end{cases}$$

For later use, let us now say that a shape $\lambda = (\ell_1, \dots, \ell_{n-1})$ is included in a shape $\mu = (m_1, \dots, m_{n-1})$ ($\lambda \subset \mu$) if for each i, $\ell_i \leq m_i$. This defines a partial ordering on shapes and of course $\lambda \leq \mu$ if $\lambda \subset \mu$.

If we identify the shape λ of a Young tableau with the highest weight of the representation V^λ containing δ^T, then the ordering $\lambda \subset \mu$ coincides with the usual (partial) ordering on the dual $\mathfrak{h}^*$ of the Cartan algebra $\mathfrak{h}$ defined by our choice of positive roots.

Moreover, we put an ordering on the variables $\delta^{(p)}_{i_1,\dots,i_p}$ by the following relations:

$$\delta^{(1)}_{\dots} > \delta^{(2)}_{\dots} > \delta^{(n-1)}_{\dots}$$

and $\delta^{(p)}_{i_1,\dots,i_p} > \delta^{(p)}_{j_1,\dots,j_p}$ if $i_p = j_p, \dots, i_{r+1} = j_{r+1}$ and $i_r < j_r$.

We then put the following weighted lexicographic ordering on the monomials δ^T: $\delta^T < \delta^{T'}$ if and only if $\lambda(T) < \lambda(T')$ or $\lambda(T) = \lambda(T')$ and $\delta^T < \delta^{T'}$ for the lexicographic ordering induced by the ordering of their variables. Since lexicographic ordering is a monomial ordering, our ordering is also a monomial ordering.

Remark 5.2. In [2], an ordering $<<$ on Young tableaux having the same shape is defined,in fact our ordering is the reverse ordering since:

$$\delta^T < \delta^{T'} \text{ if and only if } T' << T.$$

Later on, we shall simply write $T < T'$ instead of $\delta^T < \delta^{T'}$.

Recall that a Young tableau is semi standard if its entries are increasing along each row (and strictly increasing along each column). It is well known that the set of semi standard Young tableau gives a basis for $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}]/P(\delta)$ (see [2] for instance).

Our ordering defines an unique Groebner basis for $P(\delta)$. We shall now build this basis.

For any non semi standard Young tableau T with 2 columns, there exists an element Q_T in $P(\delta)$ of the form $(*)$. This relation can be written as:

$$Q_T = \delta^T + \sum_{j=1}^{n} \pm\delta^{T_j} \text{ with } \delta^{T_j} < \delta^T \ \forall j.$$

Each T_j has the same shape as T ($\lambda(T_j) = \lambda(T)$) but some of them can be non semi standard. We repeat the construction for each non semi standard T_j and finally we get, for each non semi standard T with 2 columns, an element Q_T^{red} in $P(\delta)$ such that the leading term of Q_T^{red} is δ^T and all the monomials of Q_T^{red} have the form $a\delta^{T'}$ with T' semi standard and $\delta^{T'} < \delta^T$.

Theorem 5.3. (The non semi standard Groebner basis)
The set

$$G = \left\{Q_S^{red},\ S \text{ non semi standard with 2 columns}\right\}$$

is the reduced Groebner basis of $P(\delta)$ for our ordering.

Proof. First denote by $\mathcal{NS}$ the set of all monomials δ^T with T non semi standard. Since each non semi standard T has 2 consecutive columns such that the sub tableau defined by these 2 columns is non semi standard, δ^T is divisible by one of the δ^S, *i.e.* by one of the leading term of Q_S^{red}.

Thus the ideal $< \delta^S >$ generated by the leading terms of G contains the vector space $span(\mathcal{NS})$.

Conversely let T be a semi standard Young tableau. Suppose T belongs to the ideal $< LT(P(\delta)) >$ generated by the leading terms of all the Q in $P(\delta)$. That means:

$$\delta^T = Q - \sum_{T'<T} a_{T'}\delta^{T'}.$$

If any T' is semi standard we keep this relation. If some of the T' are non semi standard, then $\delta^{T'}$ is in $< \delta^S >$ thus in $< LT(P(\delta)) >$ and we repeat the construction for $\delta^{T'}$. We get finally:

$$\delta^T = Q_0 - \sum_{\substack{T''<T\\ T''\text{semi standard}}} a_{T''}\delta^{T''},\ Q_0 \in P(\delta).$$

This implies that

$$\delta^T + \sum_{T''} a_{T''}\delta^{T''} \in P(\delta).$$

But this is impossible, since the set $\{\delta^T, T \text{ semi standard}\}$ is a basis for $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}]/P(\delta)$ Thus:

$$< LT(P(\delta)) >= span(\mathcal{NS}).$$

Moreover, since any monomial in Q_S^{red} is either δ^S or $a_T\delta^T$ with T semi standard, it cannot be divisible by a $\delta^{S'}$ with $S' \neq S$, S' non semi standard with two columns. This proves our theorem. $\square$

The usual basis of the shape algebra $\mathbb{S}^\bullet(V)$ by semi standard Young tableaux can thus be described as a natural basis of the quotient of the polynomial algebra $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}]$ by the ideal of Plücker relations, if we put the ordering $<$ on the monomials δ^T.

Especially, we can write the action of any element of the Lie algebra $\mathfrak{sl}(n)$ on any polynomial function with variables $\delta^{(s)}_{i_1,\dots,i_p}$, for instance, if $X_\alpha = E_{ij}$ $i \neq j$ then X_α acts on $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}]$ as the derivation:

$$X_\alpha f = \frac{d}{ds}|_{s=0} f(exp\ s^t X_\alpha.) = \sum_{\{i_1,\dots,i_p\}\cap\{i,j\}=\{j\}} \pm\delta^{(p)}_{(\{i_1,\dots,i_p\}\setminus\{j\})\cup\{i\}} \frac{\partial f}{\partial \delta^{(p)}_{i_1,\dots,i_p}}.$$

Finally, the Cartan algebra acts on f as the derivation

$$Hf = \frac{d}{ds}|_{s=0} f(exp\ s^t H.) = \sum (\theta_{i_1} + \cdots + \theta_{i_p})\delta^{(p)}_{i_1,\dots,i_p} \frac{\partial f}{\partial \delta^{(p)}_{i_1,\dots,i_p}},$$

if $H = \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix}$. This action defines the action on the quotient by $P(\delta)$, since we have a Groebner basis for the ideal $P(\delta)$, the quotient action on the basis of semi standard Young tableaux reduces to compute the canonical form of the polynomial $X_\alpha f$ or Hf, this is easy to do with some usual computer software.

As an illustration, we give on Figure 1 a graphic description of the N^+ part of the adjoint representation $\mathbb{S}^{\omega_1+\omega_2}(\mathbb{C}^3)$ of $\mathfrak{sl}(3)$ (see [3] for similar presentation).

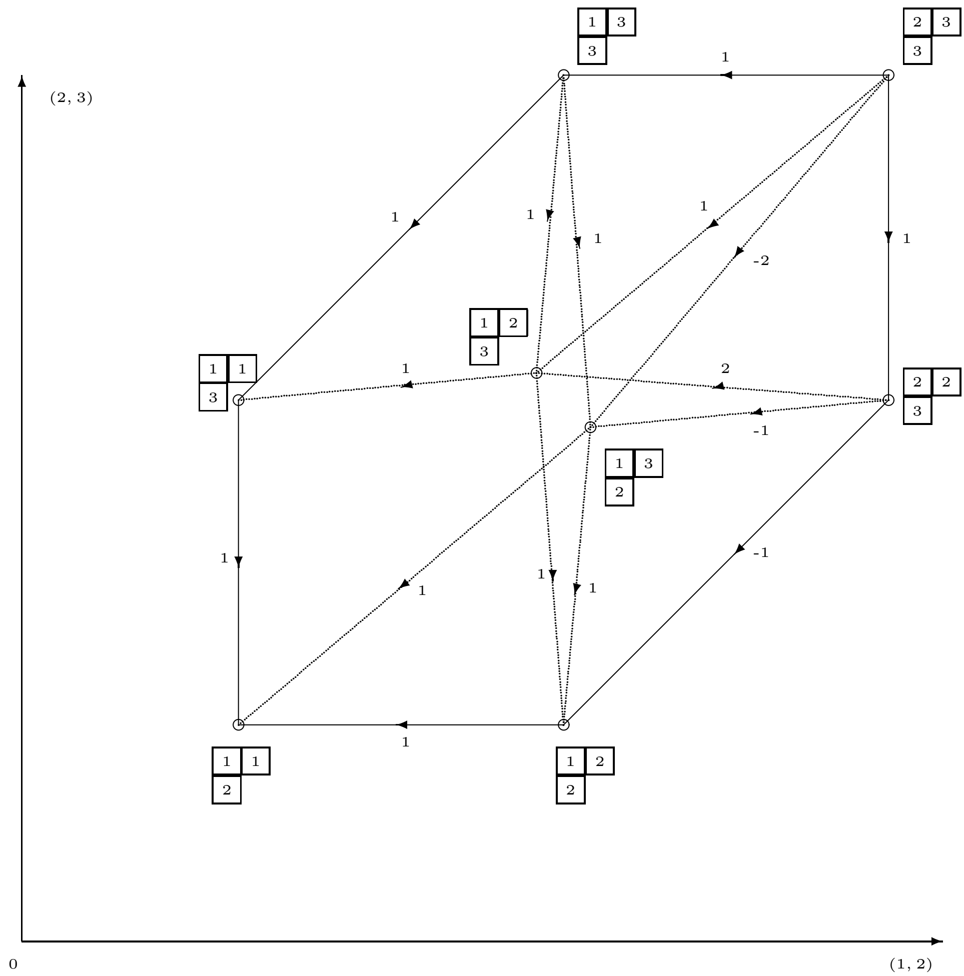

FIGURE 1. The N^+ part of the adjoint representation $\mathbb{S}^{\omega_1+\omega_2}(\mathbb{C}^3)$ of $\mathfrak{sl}(3)$

If we change our Weyl chamber, we can repeat this construction, defining first anti semi standard tableaux as Young tableaux with entries strictly decreasing in each column and decreasing in each row. Then we define an ordering on the set of variables $\delta^{(s)}_{i_1,\dots,i_s}$ (now with $i_1 > i_2 > \cdots > i_s$)

by putting:

$$\delta^{(1)}_{\dots} > \delta^{(2)}_{\dots} > \dots > \delta^{n-1}_{\dots}$$

and

$$\delta^{(p)}_{i_1,\dots,i_p} > \delta^{(p)}_{j_1,\dots,j_p} \text{ if } i_p = j_p \dots i_{r+1} = j_{r+1} \text{ and } i_r > j_r.$$

Let T be any anti semi standard tableau. We can associate to T a monomial:

$$\begin{aligned}\delta^T &= \delta^{(c_1)}_{a^1_1\dots a^1_{c_1}} \delta^{(c_2)}_{a^2_1\dots a^2_{c_2}} \cdots \\ &= \pm\delta^{(c_1)}_{a^1_{c_1}\dots a^1_1} \delta^{(c_2)}_{a^2_{c_2}\dots a^2_1} \cdots\end{aligned}$$

and exchange the variables corresponding to columns with equal height, then we get another Young tableau T' such that $\delta^T = \delta^{T'}$.
For instance:

$$T = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline\end{array}\,, \qquad \delta^{(2)}_{43}\delta^{(2)}_{21} = \delta^{(2)}_{12}\delta^{(2)}_{34}, \qquad T' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

or:

$$T = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\,, \qquad \delta^{(3)}_{432}\delta^{(1)}_{1} = -\delta^{(3)}_{234}\delta^{(1)}_{1}, \qquad T' = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\,.$$

As this example shows, if $n > 2$, T' is generally not semi standard thus our change of ordering on the variables δ defines a new Groebner basis on the shape algebra if $n > 2$.

Now, the symmetry τ corresponds to the following operation on tableaux since:

$$\tau(\delta^{(s)}_{i_1,\dots,i_s}) = \varepsilon^s_n \delta^{(s)}_{n+1-i_1,\dots,n+1-i_s}$$

We can define τ directly on Young tableaux by replacing each entry a^i_j of T by $n+1-a^i_j$. The anti semi standard tableaux are exactly the image by τ of the semi standard ones.

6. **The reduced shape algebra : Algebraic presentation**

Let us keep our notations: $V = \mathbb{C}^n$ is a complex vector space with dimension n. From now one, we shall study a quotient of the shape algebra $\mathbb{S}^\bullet(V)$.

Definition 6.1. Let R^+ be the ideal in the shape algebra generated by the $v^\lambda - 1$:

$$\begin{aligned} R^+ &= \langle v^{\sum a_j\omega_j} - 1 = (e_1)^{a_1}(e_1\wedge e_2)^{a_2}\dots(e_1\wedge\dots\wedge e_{n-1})^{a_n} - 1\rangle \\ &= \langle e_1 - 1, e_1\wedge e_2 - 1, \dots, e_1\wedge\dots\wedge e_{n-1} - 1\rangle. \end{aligned}$$

We call <u>reduced shape algebra</u> the quotient

$$\mathbb{S}^\bullet_{red}(V) = {\mathbb{S}^\bullet(V)} \big/ {R^+}.$$

This reduced shape algebra is not a natural $\mathfrak{sl}(n)$ module. Since the ideal R^+ is invariant under the action of the solvable group HN^+ consisting of upper triangular matrices in $SL(n)$, the quotient is only a HN^+ module. The action of the Cartan group H is still diagonal, let us study the N^+ (or $\mathfrak{n}^+$) action on $\mathbb{S}^\bullet_{red}(V)^+$.

Proposition 6.2. ($\mathbb{S}^\bullet_{red}(V)^+$ is an indecomposable module) *Denote by π^+ the canonical projection from $\mathbb{S}^\bullet(V)$ to $\mathbb{S}^\bullet_{red}(V)^+$. Then*

- *i) The space of vectors $u \in \mathbb{S}^\bullet_{red}(V)^+$ such that $\mathfrak{n}^+u = 0$ is $\mathbb{C}1$.*
- *ii) $\mathbb{S}^\bullet_{red}(V)^+$ is an indecomposable module.*
- *iii) For any λ, the $\mathfrak{n}^+$ module $\mathbb{S}^\lambda(V)$ is equivalent to the submodule $\pi^+\left(\mathbb{S}^\lambda(V)\right)$ of $\mathbb{S}^\bullet_{red}(V)^+$.*
- *iv) For any $\mu \subset \lambda$, $\pi^+\left(\mathbb{S}^\mu(V)\right)$ is a submodule of $\pi^+\left(\mathbb{S}^\lambda(V)\right)$.*

Proof. i) We know ([5] p. 317 for instance) that, in each $\mathbb{S}^\lambda(V)$, the space of vectors u such that $\mathfrak{n}^+u = 0$ is exactly $\mathbb{C}v^\lambda$. This gives *i)* in the quotient $\mathbb{S}^\bullet_{red}(V)^+$.

ii) Let u be a non zero vector in $\mathbb{S}^\bullet_{red}(V)^+$, the $\mathfrak{n}^+$ module W generated by u is finite dimensional since u is a finite sum of image through π^+ of weights vectors. The $\mathfrak{n}^+$ action is locally nilpotent on $\mathbb{S}^\bullet(V)$, thus it is also locally nilpotent on $\mathbb{S}^\bullet_{red}(V)^+$, as a consequence W contains a non trivial vector annhilated by $\mathfrak{n}^+$. This vector is a multiple of 1. Thus any $\mathfrak{n}^+$ submodule of $\mathbb{S}^\bullet_{red}(V)^+$ contains 1, $\mathbb{S}^\bullet_{red}(V)^+$ is an indecomposable $\mathfrak{n}^+$ module.

iii) Let π^+_λ be the restriction of π^+ to $\mathbb{S}^\lambda(V)$. It is a morphism of $\mathfrak{n}^+$ modules. If its kernel is not vanishing, thanks to Lie theorem, the $\mathfrak{n}^+$ module $Ker(\pi^+_\lambda)$ contains a non zero vector annihilated by $\mathfrak{n}^+$, this vector is a multiple of v^λ, but $\pi^+(v^\lambda) = 1 \neq 0$. Thus π^+_λ is an isomorphism of $\mathfrak{n}^+$ modules.

iv) The relation $\mu \subset \lambda$ is equivalent to say there is a dominant integral weight ν such that $\lambda = \mu + \nu$. In $\mathbb{S}^\bullet(V)$, the multiplication by v^ν sends $\mathbb{S}^\mu(V)$ into $\mathbb{S}^\lambda(V)$. In the quotient, this operation becomes the identity mapping: $\pi^+(uv^\nu) = \pi^+(u)$ for any u in $\mathbb{S}^\mu(V)$. $\square$

Similarly, we define $\mathbb{S}^\bullet_{red}(V)^-$ as the quotient of $\mathbb{S}^\bullet(V)$ by the ideal R^- generated by $\{e_n \wedge \cdots \wedge e_{n+1-s} - 1,\ s = 1, \ldots, n-1\}$. It is a HN^- module. If we denote π^- the canonical morphism, we get the same proposition with $''-''$ instead of $''+''$ everywhere.

7. The reduced shape algebra, Geometrical presentation

As above, we can write everything in term of the functions $\delta^{(p)}_{i_1,\ldots,i_p}$. If $R(\delta)^+$ is the ideal generated by $\delta^{(p)}_{1,\ldots,p} - 1$, we get:

$$\mathbb{S}^\bullet_{red}(V)^+ \simeq \mathbb{C}[SL(n,\mathbb{C})]^{N^+} \big/ R(\delta)^+ = \mathbb{C}[\delta^{(p)}_{i_1,\ldots,i_p}] \big/ (R(\delta)^+ + P(\delta)).$$

Suppose now f is a polynomial function, invariant with respect to the right multiplication by N^+. Then f is characterized by its restriction to the dense open subset of $SL(n)$ whose elements are the matrices g such that $\delta^{(p)}_{1,\ldots,p}(g) \neq 0$ for all p. On this set, by the use of the Gauss method, we can write:

$$g = \begin{bmatrix} g'_{11} & 0 & 0 & \ldots & 0 \\ g'_{21} & g'_{22} & 0 & \ldots & 0 \\ g'_{31} & g'_{32} & g'_{33} & \ldots & 0 \\ & & & \ddots & \\ g'_{n1} & g'_{n2} & g'_{n3} & \ldots & g'_{nn} \end{bmatrix} \begin{bmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & 1 & a_{23} & \ldots & a_{2n} \\ 0 & 0 & 1 & \ldots & a_{3n} \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

With, for all $k<j$:

$$g'_{jk}=\frac{\delta^{(k)}_{1,2,\dots,k-1,j}(g)}{\delta^{(k-1)}_{1,2,\dots,k-1}(g)},\quad g'_{jj}=\frac{\delta^{(j)}_{1,2,\dots,j}(g)}{\delta^{(j-1)}_{1,2,\dots,j-1}(g)}.$$

By N^+ right invariance, we get

$$\begin{aligned}
&f(g)=\\
&=\frac{1}{\prod(\delta^{(j)}_{1,\dots,j}(g))^{b_j}}\Phi(\delta^{(k)}_{1,2,\dots,k-1,j}(g),\delta^{(j)}_{1,\dots,j}(g))\\
&=\frac{1}{\prod(\delta^{(j)}_{1,\dots,j}(g))^{b_j}}\sum_{(c_1,\dots,c_{n-1})}\Phi_{c_1,\dots,c_{n-1}}(\delta^{(k)}_{1,2,\dots,k-1,j}(g))\prod_j\left(\delta^{(j)}_{1,\dots,j}(g)-1\right)^{c_j}\\
&=\frac{1}{\prod(\delta^{(j)}_{1,2,\dots,j}(g))^{b_j}}\sum_{(c_1,\dots,c_{n-1})}F_{c_1,\dots,c_{n-1}}(g)\prod_j\left(\delta^{(j)}_{1,\dots,j}(g)-1\right)^{c_j}.
\end{aligned}$$

By definition, the functions $\Phi_{c_1,\dots,c_{n-1}}$ and $F_{c_1,\dots,c_{n-1}}$ are polynomial, $F_{c_1,\dots,c_{n-1}}$ is right invariant by N^+ and

$$F_{0,\dots,0}-f=\left(\prod(\delta^{(j)}_{1,\dots,j})^{b_j}-1\right)f-\sum_{c_1+\dots+c_{n-1}>0}F_{c_1,\dots,c_{n-1}}\prod_j\left(\delta^{(j)}_{1,\dots,j}-1\right)^{c_j}$$

belongs to $R(\delta)^+$. For any g in N^-, any $k<j$, we have $\delta^{(k)}_{1,\dots,k-1,j}(g)=g_{jk}$ and $f(g)=F_{0,\dots,0}(g)$. The restriction of f to N^- characterizes the function $F_{0,\dots,0}$. $F_{0,\dots,0}$ and f are in the same class modulo $R(\delta)^+$. Conversely, any polynomial function $F(g_{jk})$ on N^- defines an unique function

$$f(g)=F(\delta^{(k)}_{1,\dots,k-1,j}(g))$$

in $\mathbb{C}[SL(n,\mathbb{C})]^{N^+}$. The restriction mapping is an isomorphism of algebra between $\mathbb{S}^\bullet_{red}(V)^+$ and $\mathbb{C}[N^-]$.

Remark 7.1. In this presentation of $\mathbb{S}^\bullet_{red}(V)^+$, the N^+ action on the elements of the reduced shape algebra is very natural since it is:

$$(g.f)(g')=f({}^tgg'),\ g\in N^+,\ g'\in N^-,\ f\in\mathbb{C}[N^-].$$

But since ${}^{\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}]}\big/_{R(\delta)^+}$ is simply $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}(i_p>p)]$, we have also:

$$\mathbb{S}^\bullet_{red}(V)^+\simeq{}^{\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}(i_p>p)]}\big/_{P_{red}(\delta)^+}$$

where $P_{red}(\delta)^+$ is the ideal generated by the reduced Plücker relations *i.e.* the Plücker relations in which we replace the functions $\delta^{(p)}_{1,\dots,p}$ by 1.

Especially, if $X_\alpha = E_{ij}$ $i<j$ then X_α acts on $\mathbb{C}[\delta^{(p)}_{i_1,\dots,i_p}(i_p>p)]$ as the derivation:

$$\begin{aligned} X_\alpha f &= \frac{d}{ds}|_{s=0} f(exp\ s^t X_\alpha .) \\ &= \pm \frac{\partial f}{\partial \delta^{(p)}_{(\{1,\dots,p\}\setminus\{i\})\cup\{j\}}} + \sum_{\substack{\{i_1,\dots,i_p\}\cap\{i,j\}=\{j\} \\ \{i_1,\dots,i_p\}\setminus\{j\})\cup\{i\}\neq\{1,\dots,p\}}} \pm \delta^{(p)}_{(\{i_1,\dots,i_p\}\setminus\{j\})\cup\{i\}} \frac{\partial f}{\partial \delta^{(p)}_{i_1,\dots,i_p}}. \end{aligned}$$

The same construction for $\mathbb{S}^\bullet_{red}(V)^-$ gives:

$$R^-(\delta) = \theta\left(R^+(\delta)\right)$$

is the ideal generated by the set $\{\delta^{(p)}_{n,\dots,(n-p+1)} - \varepsilon^p_n\}$, $\mathbb{S}^\bullet_{red}(V)^-$ is the quotient of $\mathbb{C}[SL(n)]^{N^+}$ (which is stabilized by θ) by $R^-(\delta)$. The Gauss formula allows to write:

$$g = \begin{bmatrix} g'_{11} & & & g'_{1n} \\ & & \cdot & \\ & \cdot & & \\ g'_{n1} & & & 0 \end{bmatrix} \begin{bmatrix} 1 & & & a_{1n} \\ & \cdot & & \\ & & \cdot & \\ 0 & & & 1 \end{bmatrix}$$

if $\delta^{(p)}_{n,\dots,(n-p+1)}(g) \neq 0$ for any p.

And any f is characterized modulo $R(\delta)^-$ by its restriction to:

$$\left\{\begin{bmatrix} g'_{ij} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix}\right\} = \varepsilon_n N^+\Omega = \left\{\begin{bmatrix} 1 & & & g'_{ij} \\ & \cdot & & \\ & & \cdot & \\ 0 & & & 1 \end{bmatrix}\begin{bmatrix} 0 & & & \varepsilon_n \\ & & \cdot & \\ & \cdot & & \\ \varepsilon_n & & & 0 \end{bmatrix}\right\}$$

Finally, if we put $f(\varepsilon_n n^+\Omega) = h(n^+)$, we get $\mathbb{S}^\bullet_{red}(V)^- \simeq \mathbb{C}[N^+]$ with the natural N^- action:

$$(g.h)(g_1) = h({}^t g g_1).$$

Theorem 7.2. (The reduced shape algebras) *The reduced shape algebras are isomorphic to the algebra of polynomial functions on N^- (respextively N^+):*

$$\mathbb{S}^\bullet_{red}(V)^+ \simeq \mathbb{C}[N^-] = \mathbb{C}[\mathfrak{n}^-]$$
$$\mathbb{S}^\bullet_{red}(V)^- \simeq \mathbb{C}[N^+] = \mathbb{C}[\mathfrak{n}^+].$$

The last assertions of the theorem come from the observation that the exponential mapping from the Lie algebra $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$) onto the Lie group N^- (resp. N^+) is a polynomial bijection with a polynomial inverse mapping.

8. The reduced shape algebra: Combinatorial presentation

8.1. Super and quasi standard Young tableaux

In order to describe the restricted shape algebra and the restricted Plücker relations, we have to perform the quotient of the preceding construction by the ideal generated by $\{\delta^{(s)}_{12\dots s} - 1\}$. On the Young tableaux this operation can be viewed as an 'extraction of trivial columns'.

A column whose height is c in a tableau is trivial if its entries are $1, 2, \dots, c$, a Young tableau T is trivial if each column of T is trivial. Now let T be a Young tableau (semi standard or not), we define the extraction of trivial columns in T in the following manner:

Denote a_{ij} the entries of T (a_{ij} is in the row i and the column j, for any j, $a_{ij} < a_{(i+1)j}$ and the heights $c_1, \dots, c_t$ of the columns in T are decreasing). We say that the tableau T is reducible if

- there is a column j whose the s top entries are $1, 2, \dots, s$ ($a_{i,j} = i$ for $1 \leq i \leq s$),
- on the right of the column j, there is a column j' with height s in T (there is $j' \geq j$ such that $c_{j'} = s$),
- for any $k > j$, if $c_{k-1} > s$ and $c_k \geq s$, $a_{s+1,k-1} > a_{s,k}$.

Let T be a reducible Young tableau, let j the smallest index and s the largest index for which the above conditions hold. Let us suppress the trivial top part of the column j and shift to the left the right parts of the

s first rows (*i.e.* we shift to the left every a_{ik} with $1 \leq i \leq s$ and $j < k$), then we get a Young tableau R_1: the entries of R_1 are $b_{k\ell}$ with

$$b_{k\ell} = \begin{cases} a_{k(\ell+1)} & \text{if} \quad 1 \leq k \leq s \quad \text{and} \quad j \leq \ell \leq t-1 \\ \\ a_{k\ell} & \text{if} \quad s < k \quad \text{or} \quad \ell < j. \end{cases}$$

If the number of column of T was t, then R_1 has $t-1$ column, more precisely if the heights of the columns of T were: $(c_1, \dots, c_t)$ and the columns of heights s had the number $j', \dots, j''$, then the heights of the columns of R_1 are $(c'_1, \dots, c'_{t-1})$ with

$$\begin{cases} c'_k = c_k & \text{if} \quad 1 \leq k < j'' \\ \\ c'_k = c_{k+1} & \text{if} \quad j'' \leq k \leq t-1. \end{cases}$$

Simultaneously, we define L_1 as the Young tableau with only one trivial column with entries $1, \dots, s$.

Now if R_1 is reducible, we repeat the above operation, extracting a second trivial column from R_1, getting two Young tableaux a trivial one with two columns L_2 and a Young tableau R_2 with $t-2$ columns.

Repeating this construction, after m steps, we get a trivial Young tableau L_m with m columns and a Young tableau R_m with $t-m$ columns.

This construction stops when the Young tableau R_m is not reducible we say R_m is irreducible and call R_m the residue of T.

Definition 8.1. (Super, left and right Young tableaux) A super Young tableau is a pair $S = (L, R)$ of two Young tableaux, the left one L is a trivial Young tableau, the right one, R is an irreducible Young tableau. L or R can be the empty tableau without any column.

Our construction defines a mapping f (the extraction mapping) from the set $\mathcal{Y}$ of Young tableaux into the set $\mathcal{SUY}$ of super Young tableaux

$$f(T) = S = (L, R).$$

If λ is the shape of T and μ, ν the shapes of L and R, they corresponds to some highest weights still denoted λ, μ and ν. We have:

$$\lambda = \mu + \nu.$$

Lemma 8.2. (f is surjective) *The map f is a surjective mapping from $\mathcal{Y}$ onto $\mathcal{SUY}$.*

Proof. Starting with an irreducible Young tableau R, we can insert to it any family of trivial columns, say $L = \{D_1, \dots, D_\ell\}$, getting a new tableau T. We insert these columns in the following way: if the height of D_i is d_i, we insert $D_1, \dots, D_i$ such that any column of T, after D_i has height strictly less then d_i, the columns of T before D_i are the columns of R with length at least d_i, with their ordering and the column D_j ($j < i$). Then T is a Young tableau. Of course, if $\ell > 0$, T is reducible.

If (L, R) is a super Young tableau, if $\{D_1, \dots, D_\ell\}$ are the columns of L, we shall write:

$$T = h(L, R).$$

Let us now try to extract a trivial column from this T. Among the columns of L, the first one is D_1 with height d_1. In T, this column is the column p. Suppose the first trivial column extracted from T is the s top elements of the column j, with $j < p$. Since R is irreducible, there is a $k > j$ such that $c^R_{k-1} > s$, $c^R_k \geq s$ and $a^R_{s+1,k-1} \leq a^R_{s,k}$ (we denote c^R_k the height of the column k and $a^R_{i,j}$ the i,j-entry in R). We choose the smallest such k. Since we can now extract the trivial column from T, there is, in T, at least one new column, say D between the two columns $k-1$, k in R, which are now columns k_1, k_2 in T. We choose for D the last one: D is the column $k_2 - 1$ in T. The height of D is $c^T_{k_2-1} = d > c^T_{k_2} = c^R_k \geq s$ and we get:

$$a^T_{s+1,k_2-1} = s + 1 \leq a^R_{s+1,k-1} \leq a^R_{s,k} = a^T_{s,k_2}.$$

We cannot extract the trivial column consisting of the s top elements of the column j, with $j < p$. Of course, we can extract all the column p of T. Thus, in the computing of $f(T)$, the first step is just to eliminate the column D_1 from T, repeating this construction, we get $f(T) = (L, R)$ where L is the trivial tableau $(D_1, \dots, D_\ell)$. Thus

$$f \circ h(L, R) = (L, R)$$

and f is a surjective mapping. □

Definition 8.3. (Quasi standard tableaux) A super Young tableau $S = (L, R)$ is said <u>quasi-standard</u> if its right tableau R is semistandard.

A Young tableau T is said quasi-standard if it is irreducible and semistandard.

Let us denote by $\mathcal{QSY}$ (resp. $\mathcal{QY}$) the set of quasi standard super Young tableaux (resp. quasi standard Young tableaux). Denote $\mathcal{SY}$ the set of semistandard Young tableaux.

Lemma 8.4. *(f is a bijection from $\mathcal{SY}$ onto $\mathcal{QSY}$) The mapping f, when restricted to $\mathcal{SY}$ is a one-to-one onto mapping from $\mathcal{SY}$ onto $\mathcal{QSY}$.*

Proof. First it is clear that if T is semistandard, then each tableau in the sequence $R_1, \dots, R_m$ defined above is still semistandard, then f is a map from $\mathcal{SY}$ to $\mathcal{QSY}$.

Now let $S = (L, R)$ be an element of $\mathcal{QSY}$. Denote the rows of L by $(L'_1, \dots, L'_u)$, their lengths being $\ell'_1, \dots, \ell'_u$. Similarly, denote $(L''_1, \dots, L''_v)$ the rows of R, their lengths being $\ell''_1, \dots, \ell''_v$. We define the new tableau $T = g(S)$ as the tableau with the row i contains (from left to right) ℓ'_i entries i, then the ℓ''_i entries of the row i of R. In fact, T is a Young tableau since if $a^T_{i,j}$ is an entry of T, it is either i or an entry of R ($a^T_{i,j} = a^R_{i,j-\ell'_i}$ if $a^R_{r,s}$ are the entries of R). In any case, $a^T_{i,j} \geq i$.

If $a^T_{i,j} = i$, then $a^T_{i,j} = i < i+1 \leq a^T_{i+1,j}$. If $a^T_{i,j} = a^R_{i,j-\ell'_i}$, since $\ell'_i \geq \ell'_{i+1}$, $a^T_{i+1,j} = a^R_{i+1,j-\ell'_{i+1}}$ and $a^T_{i,j} = a^R_{i,j-\ell'_i} < a^R_{i+1,j-\ell'_i} \leq a^R_{i+1,j-\ell'_{i+1}} = a^T_{i+1,j}$.

T is semistandard: by construction each row in T is a increasing sequence of entries. g is a map from $\mathcal{QSY}$ to $\mathcal{SY}$.

The map g is the inverse mapping of $f|_{\mathcal{S}Y}$. Indeed if T is semistandard, if a column C of T begins by a trivial part, then all the columns before C begin with the same trivial part and suppressing the top of the first column or the top of C is the same operation, thus to construct the sequence $R_1, \dots, R_m$, we just have to consider the first column at each step.

Starting with $T = g(S)$, we can extract at each step a trivial column having the height of the corresponding column of L, but no more, since R is irreducible. Thus $f \circ g(S) = S$, for any $S \in \mathcal{QSY}$.

Conversely, starting with a semistandard T, we build first $f(T) = (L, R)$ and by construction the rows of L are the left part of the rows of T, thus $g \circ f(T) = T$. □

8.2. Quasi standard Young tableaux and Groebner basis

In this section we shall repeat the construction of section 5 but for the ideal $R(\delta)^+$ and the quasi standard Young tableaux.

First, we choose the following elimination order on the variables δ: defining the degree $deg(\delta^{(s)}_{i_1\dots i_s})$ as 1 if $i_s > s$ ($\delta^{(s)}_{i_1\dots i_s}$ is not trivial) and 0 if $i_s = s$ ($\delta^{(s)}_{i_1\dots i_s}$ is trivial), the degree of δ^T is the sum of degree of each variables and $T > T'$ if and only if:

$$\begin{cases} \quad deg(\delta^T) > deg(\delta^{T'}) \\ \text{or} \\ \quad deg(\delta^T) = deg(\delta^{T'}) \quad \text{and} \quad T > T' \text{ for the preceding ordering.} \end{cases}$$

This ordering is a monomial ordering. Now we look for the leading terms of elements in $R(\delta)^+$, for this ordering. We saw that the leading terms of elements in $P(\delta)$ for the preceding ordering were non semistandard monomials.

Lemma 8.5. (The set $\langle LT(R(\delta)^+)\rangle$) *For this ordering, the set of leading terms for elements in $R(\delta)^+$ is exactly the set:*

$$\langle LT(R(\delta)^+)\rangle = \{\delta^T,\ T \textit{ non quasi standard}\}.$$

Proof. Let T be a non quasi standard tableau.

<u>**Case 1**</u>: T is non semi standard.

Then T contains a non semistandard tableau with two columns T^0: $\delta^T = \delta^U \delta^{T^0}$. For T^0, we saw there is a Plücker relation P_{T^0} in $P(\delta)$ whose leading term for the ordering of section 5 was T^0.

<u>Case 1.1</u>: T^0 contains a trivial column C_i, since T^0 is non semistandard, it is its second column. $\delta^{T^0} = \delta^{(s)}_{1,\dots,s}\delta^{(c)}_{a_1,\dots,a_c}$. But $\delta^{(s)}_{1,\dots,s}$ is the leading term of the element $V_s = \delta^{(s)}_{1,\dots,s} - 1$ in $R(\delta)^+$. δ^T is the leading term of $P_T = \delta^U\delta^{(c)}_{a_1,\dots,a_c}V_s$ which is in $R(\delta)^+$.

<u>Case 1.2</u>: T^0 does not contain any trivial column. $\delta^{T^0} = \delta^{(s)}_{b_1,\dots,b_s}\delta^{(c)}_{a_1,\dots,a_c}$ with $c \geq s$, there is j such that $a_j > b_j$, we choose the largest such j, due to our conventions of writing, if $c = s$ then $j < s$ and $a_c > c$, $b_s > s$.

Thus the relation P_{T^0} has the following form:

$$\begin{aligned} P_{T^0} &= \delta^{T^0} - \sum_{\substack{A\subset\{a_1,\dots,a_c\}\\ \#A=j}} \pm\delta^{(s)}_{A\cup\{b_{j+1},\dots,b_s\}}\delta^{(c)}_{(\{a_1,\dots,a_c\}\setminus A)\cup\{b_1,\dots,b_j\}} \\ &= \delta^{T^0} - \sum_{\substack{S<T^0\\ S \text{ semi standard}}} \pm\delta^{(S)}. \end{aligned}$$

If a tableau S in this relation contains a trivial column, *i.e* $S = C_1C_2$ with C_1 trivial, we replace S by C_2 since

$$\delta^S - \delta^{C_2} = V_s.\delta^{C_2}.$$

The expression P_{T^0} becomes $P^{red}_{T^0}$ and δ^T is the leading term of

$$P_T = \delta^U P^{red}_{T^0}$$

which is in $R(\delta)^+$. Let us remark that the non leading monomials $a_S\delta^S$ in P_T satisfy $\delta^S < \delta^T$ and $\lambda(S) \subset \lambda(T)$ if $\lambda(S)$ is the shape of the tableau S.

Case 2: T is semi standard.

If T has only one column, this column is trivial. T is the leading term of some $P_T = \delta^T - 1$ in $R(\delta)^+$.

Since T is semi standard the construction of the super Young tableau $f(T)$ begins with the extraction of the top s elements $1,\dots,s$ of the first column of T. Let us look to the two first columns of T, C_1^T and C_2^T. By hypothesis, $\delta^{C_1^T} = \delta^{(c_1)}_{1,\dots,s,a_{s+1},\dots,a_{c_1}}$, $\delta^{C_2^T} = \delta^{(c_2)}_{b_1,\dots,b_s,b_{s+1},\dots,b_{c_2}}$ and $b_s < a_{s+1}$.

Let us define ∂T as the tableau with the following first columns $C_1^{\partial T}$ and $C_2^{\partial T}$:

$$\delta^{C_1^{\partial T}} = \delta^{(c_1)}_{b_1,\dots,b_s,a_{s+1},\dots,a_{c_1}},\ \delta^{C_2^{\partial T}} = \delta^{(c_2)}_{1,\dots,s,b_{s+1},\dots,b_{c_2}},$$

the other columns of ∂T being $C_i^{\partial T} = C_i^T$ $(i \geq 3)$. Let us write the Plücker relation corresponding to these two columns and s:

$$\delta^T - \delta^{\partial T} - \sum_{\substack{A \subset \{1,\dots,s,a_{s+1},\dots,a_{c_1}\} \\ A \neq \{1,\dots,s\} \\ \#A = s}} \pm \prod_{i \geq 3} \delta^{C_i^T} \delta^{(c_2)}_{A \cup \{b_{s+1},\dots,b_{c_2}\}} \delta^{(c_1)}_{(\{1,\dots,s,a_{s+1},\dots,a_{c_1}\} \setminus A) \cup \{b_1,\dots,b_s\}}$$
$$= \delta^T - \delta^{\partial T} - \sum_A \pm \delta^{T_A}.$$

Each term δ^{T_A} in the sum has a second column containing a_i with $i > s$, thus $a_i \geq a_{s+1} > b_s$ and $\delta^{(c_2)}_{A \cup \{b_{s+1},\dots,b_{c_2}\}} < \delta^{C_2^T}$, $\delta^{T_A} < \delta^T$.

If $c_2 = s$, $deg(\delta^{\partial T}) < deg(\delta^T)$, δ^T is the leading term of an element in $R(\delta)^+$. If $c_2 > s$, we repeat this construction for ∂T, forgotting its first column. We get the following element of $R(\delta)^+$:

$$\delta^{\partial T} - \delta^{\partial^2 T} - \sum_{\substack{B \subset \{1,\dots,s,b_{s+1},\dots,b_{c_2}\} \\ B \neq \{1,\dots,s\} \\ \#B = s}} \pm \prod_{i \geq 4} \delta^{C_i^T} \delta^{(c_3)}_{B \cup \{c_{s+1},\dots,c_{c_3}\}} \delta^{(c_2)}_{(\{1,\dots,s,b_{s+1},\dots,b_{c_2}\} \setminus B) \cup \{c_1,\dots,c_s\}} \delta^{C_1^{\partial T}}$$
$$= \delta^{\partial T} - \delta^{\partial^2 T} - \sum_B \delta^{T_B}.$$

Each term δ^{T_B} in the sum has a third column containing b_i with $i > s$, thus $b_i \geq b_{s+1} > c_s$ and $\delta^{(c_3)}_{B \cup \{c_{s+1},\dots,c_{c_3}\}} < \delta^{C_3^T}$, $\delta^{T_B} < \delta^T$.

Repeating this operation we finally get an element in $R(\delta)^+$ of the form:

$$\delta^T - \delta^{\partial^k T} - \sum_j \delta^{T_j}$$

with $\delta^{T_j} < \delta^T$, $\lambda(T_j) \subset \lambda(T)$ for all j, the column $k+1$ of $\partial^k T$ is trivial, $deg(\delta^{\partial^k T}) < deg(\delta^T)$ and δ^T is the leading term of an element of $R(\delta)^+$. $\square$

Remark 8.6. The tableau $\partial^k T$ considered here is (perhaps up to a re-ordering of the columns with height s) the tableau $h(C, R_1)$ if C is the first trivial column:

$$\delta^C = \delta^{(s)}_{1\dots s}$$

and R_1 the first step in the extraction process for T.

We got an element of $R(\delta)^+$:

$$\delta^T - \delta^{R_1} - \sum \pm\delta^{T_j} \qquad (R_1 < T,\ T_j < T,\ \lambda(T_j) \subset \lambda(T))$$

If R_1 is quasi standard, we stop the process. If it is not the case, we continue the extraction, getting new tableaux $T'_k < R_1 < T$. Finally we get: $T = g(L,R)$ with $L \neq \emptyset$ and $\delta^T - \delta^R - \sum_k a_k\delta_k^T$ belonging to $R(\delta)^+$, R is quasi standard $R < T$, $T_k < T$ and $\lambda(T_k) \subset \lambda(T)$. We repeat this operation for each non quasi standard T_k; getting an element $P_T = \delta^T - \delta^R - \sum a_k\delta^{T_k}$ with $T_k < T$, $\lambda(T_k) \subset \lambda(T)$ and T_k quasi standard. This element P_T is in $R(\delta)^+$.

We proved that each non quasi standard Young tableau is the leading term of an explicit element P_T of $R(\delta)^+$. Let us now prove that any quasi standard Young tableau is not a leading term of an element in $R(\delta)^+$.

Let λ be a highest weight for $\mathfrak{sl}(n)$ and V^λ the corresponding simple module. We saw that V^λ is naturally a sub-module of $\mathbb{S}^\bullet_{red}(V)$. More precisely, V^λ is the space spanned by the classes modulo $R(\delta)^+$ of the monomials δ^T for all Young tableaux T of shape λ. A basis for V^λ is given by the classes of the monomials δ^T for T semi standard with shape λ in the quotient $\mathbb{C}[\delta]/R(\delta)^+$. Let us consider the subspace W^λ of V^λ spanned by the quasi standard and semi standard Young tableaux of shape λ.

Let us start with the usual basis of V^λ: the set of classes modulo $R(\delta)^+$ of δ^T, with T semi standard, with shape λ. This basis contains the basis of W^λ: the set of classes modulo $R(\delta)^+$ of δ^T, with T semi standard and quasi standard, with shape λ. The other tableaux are $T = g(L,R)$, $L \neq \emptyset$. We saw $\delta^T - \delta^R = \sum a_k\delta^{T_k}$ modulo $R(\delta)^+$ with T_k quasi standard $T_k < T$ and R is semi standard and quasi standard with shape $\mu \subset \lambda$. This proves that V^λ is a subspace of $\sum\limits_{\mu\subset\lambda} W^\mu$. But since g is injective, $dimV^\lambda = \sum\limits_{\mu\subset\lambda} dim(W^\mu)$ thus

$$V^\lambda = \bigoplus_{\mu\subset\lambda} W^\mu.$$

Let now T be a quasi standard Young tableau of shape λ. Suppose δ^T is the leading term of an element $T + \sum_k a_k\delta^{T_k}$ in $R(\delta)^+$, then using the first part of the proof, we can replace each δ^{T_k} with a non quasi standard T_k, by a linear combination of δ^{T_j} with quasi standard T_j modulo $R(\delta)^+$. Finally we get an element in $R(\delta)^+$ of the form $T + \sum_j b_j \delta^{T_j}$ with any T_j quasi standard and strictly smaller than T and $\lambda(T_j) \subset \lambda$. But this is impossible since the sum $\sum\limits_{\mu\subset\lambda} W^\mu$ is direct.

Finally, as in section 5, for each non quasi standard Young tableau, we got an element in $R(\delta)^+$ of the form:

$$P_T^{red} = \delta^T - \sum_j a_j \delta^{T_j}$$

with δ^{T_j} strictly smaller than δ^T and quasi standard.

Let T be a non quasi standard tableau with shape λ. We shall say that T is <u>minimal</u> if it does not contain any non quasi standard tableau with shape $\mu \subsetneqq \lambda$. For instance a semi standard non quasi standard tableau with one column or with 2 columns without trivial column are minimal.

If $n \leq 3$ there are no other semi standard, minimal, non quasi standard tableaux, but if $n \geq 4$ there is semi standard, minimal, non quasi standard tableau with at least 3 columns for instance:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$
$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}$$

Theorem 8.7. (The non quasi standard Groebner basis) *The set*

$$G = \{P_S^{red},\ S \textit{ semi standard minimal non quasi standard or } S \textit{ non semi standard with 2 columns, without any trivial column}\}$$

is the reduced Groebner basis of $R(\delta)^+$ for our ordering.

Proof. Let T be a non quasi standard tableau.

If T contains a trivial column C, δ^T is divisible by δ^C and C is minimal semi standard non quasi standard.

Suppose now T does not contain any trivial column.

If T is non semi standard, it contains a non semi standard tableau S with 2 columns, without any trivial column.

If T is semi standard then by definition it contains a minimal non quasi standard tableau S but thus S is semi standard.

Then:

$$\begin{aligned} < LT(R(\delta)^+) > &= \{\delta^T, T \text{ non quasi standard}\} \\ &= \{\text{monomial divisible by an element of } G\}. \end{aligned}$$

Now each monomial in any P_S^{red} of G which are not the leading term, has the form $a_{T'}\delta^{T'}$ with T' quasi standard.

But if $U \subset T'$, then U is also quasi standard. Indeed, U is semi standard, suppose U non quasi standard then U contains a first column

$$C_1 = (1, 2, \dots, s, a_{s+1}, \dots, a_{C_1}),$$

other columns

$$C_i = (b_1, b_2, \dots, b_s, b_{s+1}, \dots)$$

a last column

$$C_t = (c_1, c_2, \dots, c_{c_t}) \text{ with } t \leq s.$$

We can extract $(1, 2, \dots, s)$ from U.

Now, we can refind T' from U by adding some columns before C_1, between columns of U or after C_t. But T' is semi standard. By considering each case for these new columns, we directly see that the top $(1, 2, \dots, s)$ of columns C_1 can still be extracted from T' which is impossible since T' is quasi standard.

Thus any monomial of P_S^{red} is not divisible by the leading term of another $P_{S'}^{red}$.

This means that G is the reduced Groebner basis of $R(\delta)^+$ for our ordering. □

The same result holds with the anti standard tableau, image by τ of the quasi standard tableaux.

The anti quasi standard tableaux can be defined exactly as the quasi standard tableaux by extracting 'trivial' top of columns like:

$$\begin{array}{|c|}\hline n \\ \hline n-1 \\ \hline \vdots \\ \hline n-s \\ \hline\end{array}$$

They are still the image by τ of the quasi standard tableaux.

Remark 8.8. In fact, if $n \leq 3$, the quasi satndard Groebner basis is invariant under the action of θ. Similarly, with the symmetry τ, if we identify $\tau(T)$ with $\pm T'$ with T' the Young tableau such that $\delta^{\tau(T)} = \delta^{T'}$, then T quasi standard implies T' quasi standard. In the study of $\mathfrak{sl}(4)$ below, we shall see this is no more true for $n > 3$.

On Figure 2 we picture the adjoint representation of $\mathfrak{sl}(3)$ in $\mathbb{S}^+_{red}$ equipped with its Groebner basis.

Our choice of basis gives rise to a more symmetric graph than the usual choice described in section 5.

The same representation in $\mathbb{S}^-_{red}$ equipped with its Groebner basis is given on Figure 3. We resume our constructions by the two following diagrams of Figures 4 and 5, where we denote by $\mathcal{SY}$ (resp. $\mathcal{QY}$, $\mathcal{ASY}$, $\mathcal{AQY}$) the set of semi standard (resp. quasi standard, anti standard, anti quasistandard) tableaux.

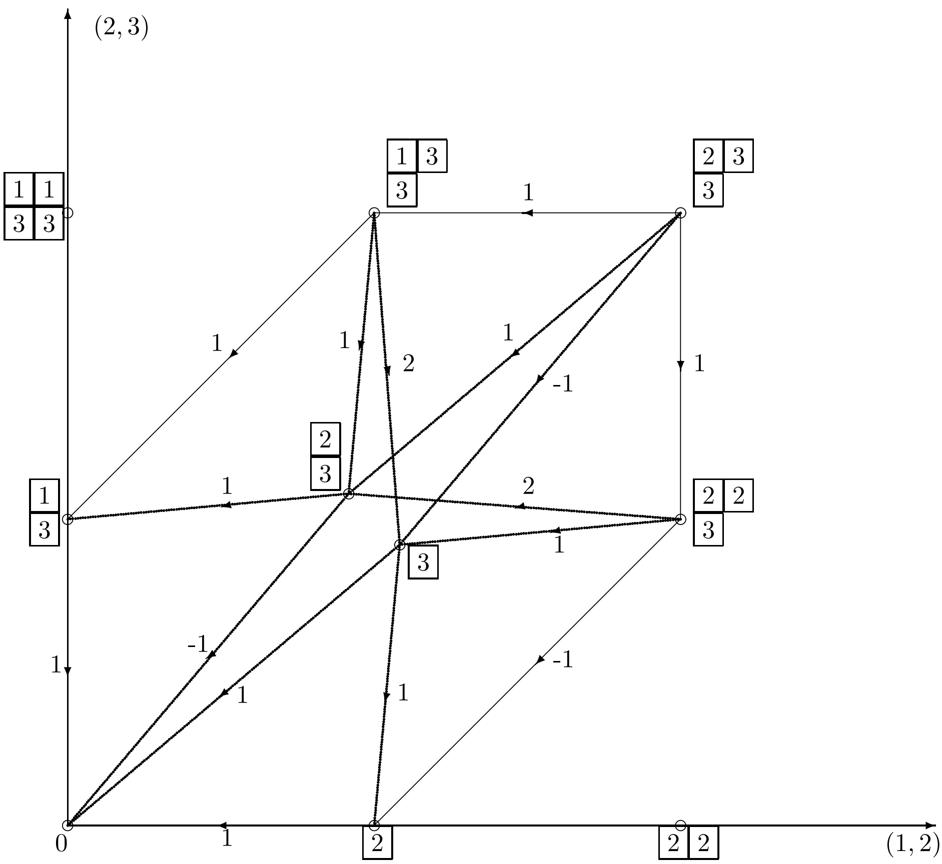

Figure 2. The adjoint representation of $\mathfrak{sl}(3)$ in $\mathbb{S}^+_{red}$ equipped with its Groebner basis.

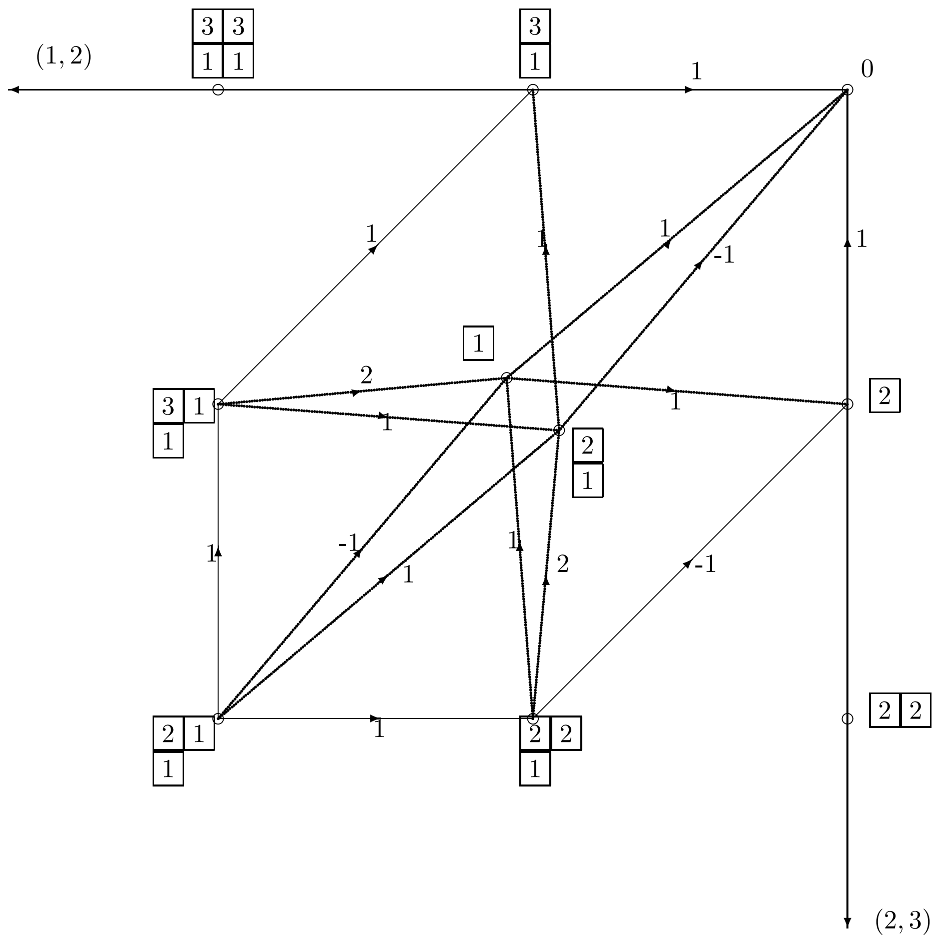

Figure 3. The adjoint representation of $\mathfrak{sl}(3)$ in $\mathbb{S}^{-}_{red}$ equipped with its Groebner basis.

$$
\begin{tikzcd}[column sep=small]
 & A^\bullet(V)=\mathbb{C}[V\oplus(V\wedge V)\oplus\cdots\oplus(V\wedge\cdots\wedge V)] \arrow[d] & \\
 & \mathbb{S}^\bullet(V)={}^{\mathbb{C}[V\oplus\cdots\oplus\wedge^{n-1}V]}/_{P} \arrow[loop below, "\tau"] \arrow[dl, "\pi^-"'] \arrow[dr, "\pi^+"] & \\
\mathbb{S}^-_{red} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & \mathbb{S}^+_{red} \arrow[d, leftrightarrow] \\
{}_{(P+R^-)}\backslash^{\mathbb{C}[V\oplus\cdots\oplus\wedge^{n-1}V]} & & {}^{\mathbb{C}[V\oplus\cdots\oplus\wedge^{n-1}V]}/_{(P+R^+)}
\end{tikzcd}
$$

Figure 4.

$$
\begin{tikzcd}[column sep=small]
{}^{\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]_{i_1>\cdots>i_s}}/_{P(\delta)} \arrow[r, phantom, "="] \arrow[d, leftrightarrow] & \mathbb{C}[SL(n)]^{N^+} \arrow[loop above, "\theta"] \arrow[r, phantom, "="] \arrow[ddl] \arrow[ddr] & {}^{\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]_{i_1>\cdots>i_s}}/_{P(\delta)} \arrow[d, leftrightarrow] \\
Vect(\mathcal{ASY}) & & Vect(\mathcal{SY}) \\
\mathbb{C}[N^+] \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & \mathbb{C}[N^-] \arrow[d, leftrightarrow] \\
{}^{\mathbb{C}[SL(n)]^{N^-}}/_{(P(\delta)+R(\delta)^-)} \arrow[d, leftrightarrow] & & {}^{\mathbb{C}[SL(n)]^{N^+}}/_{(P(\delta)+R(\delta)^+)} \arrow[d, leftrightarrow] \\
{}^{\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]_{i_s<n+1-s}}/_{P_{red}(\delta)^-} \arrow[d, leftrightarrow] & & {}^{\mathbb{C}[\delta^{(s)}_{i_1,\dots,i_s}]_{i_s>s}}/_{P_{red}(\delta)^+} \arrow[d, leftrightarrow] \\
Vect(\mathcal{AQY}) & & Vect(\mathcal{QY})
\end{tikzcd}
$$

Figure 5.

9. The $\mathfrak{sl}(2)$ case

9.1. Representations of $\mathfrak{sl}(2)$

The $\mathfrak{sl}(2)$-simple modules are characterized by a highest weight a. More precesely, the basis of $\mathfrak{sl}(2)$ is:

$$X_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y_\alpha = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If a is a positive integer, the simple module π^a acting on the space V^a is $a+1$-dimensional, with a basis v_n $(0 \leq n \leq a)$ and the matrices of the action are:

$$\pi^a(X_\alpha) = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & \ddots & \ddots & \\ & & & & a \\ 0 & & & & 0 \end{bmatrix},$$

$$\pi^a(H_\alpha) = \begin{bmatrix} a & 0 & & 0 \\ 0 & a-2 & & 0 \\ & & \ddots & \\ 0 & & & -a \end{bmatrix},$$

$$\pi^a(Y_\alpha) = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ a & 0 & & 0 & 0 \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{bmatrix}.$$

There is only one fundamental representation, associated to the weight ω_1. We realize it in the space generated by the functions $\delta_1^{(1)}(g) = g_{11}$, $\delta_2^{(1)}(g) = g_{21}$. The other representations are realized on the space of homogeneous polynomial functions of degree a in these variables.

9.2. Shape and reduced shape algebra

There are no Plücker relation between g_{11} and g_{21}, thus the shape algebra is isomorphic to the algebra

$$A^\bullet(V) = \mathbb{C}[g_{11}, g_{21}] \simeq \mathbb{S}(V).$$

The reduced shape algebra is the quotient by the ideal generated by $g_{11}-1$. Let us put:

$$\mathfrak{n}^- = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \right\}, \quad N^- = exp\left(\mathfrak{n}^-\right) = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \right\}.$$

Then:

$$\mathbb{S}^\bullet_{red}(V)^+ = \mathbb{C}[\delta_2^1] = \mathbb{C}[X],$$

The X_α acts on a polynomial function as the operator:

$$X_\alpha = \frac{\partial}{\partial X}.$$

We realize the $\mathfrak{sl}(2)$-diamond cone as the half line of the entire nodes $0, 1, \dots, a, a+1, \dots$, at each node n, we put the quasi standard Young tableau $\begin{array}{|c|c|c|}\hline 2 & \dots & 2 \\ \hline\end{array}$ or the monomial X^n. We have an explicit basis for the representation of N^+ on the diamond cone defined by the action of X_α, pictured by the graph:

For any $a \geq 0$, we define the diamond D_a as the graph generated by X^a, the vector space V^a as the vector space with basis the nodes of D_a.

We saw that the anti semi standard (resp. the anti quasi standard) basis can be identified with the semi standard (resp. the quasi standard) basis. More precisely, a being fixed, the action of τ on V^a, denoted by $\tau^{(a)}$ is defined as:

$$\tau^{(a)}(X^n) = X^{a-n}$$

$$\tau^{(a)}\left(\begin{array}{|c|c|c|}\hline 2 & \dots & 2 \\ \hline\end{array} \right) = \begin{array}{|c|c|c|}\hline 2 & \dots & 2 \\ \hline\end{array}.$$

We can see $\tau^{(a)}$ as the succession of four operations:
completion of the tableau T,
action of τ on the complete tableau,
reordering the new tableau,
cancelling the trivial columns $\boxed{1}$.

For instance if $a = 5$, we get

$$compl\Big(\boxed{2\,|\,2\,|\,2}\Big) = \boxed{1\,|\,1\,|\,2\,|\,2\,|\,2}$$

$$\tau\Big(\boxed{1\,|\,1\,|\,2\,|\,2\,|\,2}\Big) = \boxed{2\,|\,2\,|\,1\,|\,1\,|\,1}$$

$$ord\Big(\boxed{2\,|\,2\,|\,1\,|\,1\,|\,1}\Big) = +\boxed{1\,|\,1\,|\,1\,|\,2\,|\,2}$$

$$cancell\Big(\boxed{1\,|\,1\,|\,1\,|\,2\,|\,2}\Big) = \boxed{2\,|\,2}.$$

We put:

$$Y_\alpha(X^n) = (\tau^{(a)} \circ X_\alpha \circ \tau^{(a)})(X^n) = (a-n)X^{n+1}$$

and $H_\alpha = [X_\alpha, Y_\alpha]$ or:

$$H_\alpha(X^n) = [(n+1)(a-n) - n(a-n+1)]X^n = (a-2n)X^n.$$

We finally complete the diamond D_a by adding the edges corresponding to the Y_α-action.

10. The $\mathfrak{sl}(3)$ case

10.1. Representations of $\mathfrak{sl}(3)$

The $\mathfrak{sl}(3)$-simple modules are characterized by their highest weight. More precisely, the basis of $\mathfrak{sl}(3)$ is:

$$X_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{\alpha+\beta} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$Y_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_{\alpha+\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The simple modules have non multiplicity free weights. We can describe then by using the reduced shape algebra. The fundamental modules are three dimensional, they are realized on the space $V^{\omega_1} = \mathbb{C}^3$ and $V^{\omega_2} = \wedge^2\mathbb{C}^3$.

For each pair of natural integers, there is an unique irreducible representation $\pi(a,b)$ with highest weight $a\omega_1 + b\omega_2$.

10.2. Shape and reduced shape algebra

Now we have just one Plücker relation: let us put as above:

$$\begin{array}{ccc} \delta_1^{(1)} = g_{11}, & \delta_2^{(1)} = g_{21}, & \delta_3^{(1)} = g_{31} \\ \delta_{12}^{(2)} = g_{11}g_{22} - g_{12}g_{21}, & \delta_{13}^{(2)} = g_{11}g_{32} - g_{12}g_{31}, & \delta_{23}^{(2)} = g_{21}g_{32} - g_{22}g_{31}. \end{array}$$

Then the unique Plücker relation is:

$$\delta_1^{(1)}\delta_{23}^{(2)} - \delta_2^{(1)}\delta_{13}^{(2)} + \delta_3^{(1)}\delta_{12}^{(2)} = 0.$$

The shape algebra is the quotient of the algebra of polynomial functions in these 6 variables by the above relation.

The reduced shape algebra is obtained by imposing $\delta_1^{(1)} = 1$ and $\delta_{12}^{(2)} = 1$.

An explicit basis for this module $V^{(a,b)}$ and the X_η, Y_η, H_η actions on this basis can be found in [6] for instance. Let us briefly recall the construction of [6].

One defines a diamond cone D in $\mathbb{R}^3$ and a infinite dimensional vector space V with basis:

$$\begin{aligned} \mathcal{B} &= \{e_{m,n,\ell},\ (m,n,\ell) \in D \subset \mathbb{R}^3\} \\ &= \{e_{m,n,\ell},\ m,n \geq 0,\ -n \leq \ell \leq 2m-n,\ m-2n \leq \ell \leq m, \\ &\qquad \ell \equiv\ max(m,n)\ mod\ 2\}. \end{aligned}$$

The action of any X_η on these vectors $e_{m,n,\ell}$ is thus explicitely given in [6], we shall refind and present this explicit form below. Now the irreducible module $V^{(a,b)}$ with highest weight $a\omega_1 + b\omega_2$ is the module generated by the X_η action on the highest weight vector $e_{a+b,a+b,a-b}$.

A basis for this module is an explicit subset $\mathcal{B}^{(a,b)}$ of $\mathcal{B}$. There is a symmetry $\tau_{(a,b)}$ on $V^{(a,b)}$, $\tau_{(a,b)}(\mathcal{B}^{(a,b)}) = \mathcal{B}^{(a,b)}$ and the Y_η, H_η actions are defined as

$$Y_\eta = \tau_{(a,b)} \circ X_\eta \circ \tau_{(a,b)},\ H_\eta = [X_\eta, Y_\eta].$$

See [6] for explicit formulas.

With our notations, we have:

$$\mathfrak{n}^- = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ u & y & 0 \end{bmatrix} \right\}, \quad N^- = exp\left(\mathfrak{n}^-\right) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ u + \frac{xy}{2} & y & 1 \end{bmatrix} \right\}.$$

Then:

$$\delta_2^{(1)} = X, \quad \delta_3^{(1)} = \frac{xy}{2} + u = U, \quad \delta_{13}^{(2)} = Y, \quad \delta_{23}^{(2)} = \frac{xy}{2} - u = E$$

and

$$\begin{aligned} \mathbb{S}^\bullet_{red}(V)^+ &\simeq \mathbb{C}[x,y,u] \\ &= \mathbb{C}[\delta_2^{(1)}, \delta_3^{(1)}, \delta_{13}^{(2)}, \delta_{23}^{(2)}]\Big/_{\langle \delta_3^{(1)} + \delta_{23}^{(2)} - \delta_2^{(1)}\delta_{13}^{(2)} \rangle} \\ &= \mathbb{C}[X,Y,U,E]\Big/_{\langle U + E - XY \rangle}. \end{aligned}$$

The quasi standard ordering on variables is:

$$\delta_3^{(1)} > \delta_2^{(1)} > \delta_{23}^{(2)} > \delta_{13}^{(2)}, \qquad \text{or} \qquad U > X > E > Y.$$

Then the leading term for this basis is $\delta_2^{(1)}\delta_{13}^{(2)} = XY$, thus we get the basis:

$$\begin{aligned} &\Big\{(\delta_3^{(1)})^u(\delta_{23}^{(2)})^e(\delta_2^{(1)})^x = U^uE^eX^x, \quad u,\ e,\ x \in \mathbb{N}\Big\} \\ &\qquad \bigcup \Big\{(\delta_3^{(1)})^u(\delta_{23}^{(2)})^e(\delta_2^{(1)})^y = U^uE^eY^y, \quad u,\ e,\ y \in \mathbb{N},\ y > 0\Big\}. \end{aligned}$$

Now the action of X_α, X_β and $X_{\alpha+\beta}$ on these polynomials are the following:

$$X_\alpha = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial u}, \quad X_\beta = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial u}, \quad X_{\alpha+\beta} = \frac{\partial}{\partial u},$$

thus

$$\begin{array}{llll} X_\alpha(X) = 1, & X_\alpha(Y) = 0, & X_\alpha(U) = 0, & X_\alpha(E) = Y, \\ X_\beta(X) = 0, & X_\beta(Y) = 1, & X_\beta(U) = X, & X_\beta(E) = 0, \\ X_{\alpha+\beta}(X) = 0, & X_{\alpha+\beta}(Y) = 0, & X_{\alpha+\beta}(U) = 1, & X_{\alpha+\beta}(E) = -1. \end{array}$$

Then the X_η are acting by derivations, we refind the diamond cone, the diamond $D^{(a,b)}$, the vector space $V^{(a,b)}$, the symmetry $\tau_{(a,b)}$ and the complete diamond graphs on $D^{(a,b)}$ described in [6] with the identification:

$$\begin{array}{ll} e_{m,n,\ell} = U^{n-\frac{m-\ell}{2}}E^{\frac{m-\ell}{2}}X^{m-n} & \text{if } m > n \\ e_{m,n,\ell} = U^{\frac{m+\ell}{2}}E^{\frac{m-\ell}{2}} & \text{if } m = n \\ e_{m,n,\ell} = U^{\frac{m+\ell}{2}}E^{m-\frac{n+\ell}{2}}Y^{n-m} & \text{if } m < n, \end{array}$$

our Groebner basis for $\mathbb{S}^\bullet_{red}(V)^+$ coincides exactly with the basis $\mathcal{B}$ given in [6].

10.3. X_η action, symmetry and Y_η action

With our notations, we have the following identification between column and variables X, U, Y, E:

$$X = \delta_2^{(1)}(g) \quad \longrightarrow \quad \boxed{2}$$

$$U = \delta_3^{(1)}(g) \quad \longrightarrow \quad \boxed{3}$$

$$Y = \delta_{13}^{(2)}(g) \quad \longrightarrow \quad \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$$

$$E = \delta_{23}^{(2)}(g) \quad \longrightarrow \quad \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$$

The unique reduced Plücker relation is:

$$\boxed{3} \quad - \quad \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array} \quad + \quad \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} \quad = \quad 0$$

For instance, with these notations, the X_α action on our basis is:

$U^uE^eX^x \qquad (x > 0)$	$eU^{u+1}E^{e-1}X^{x-1} + (e+x)U^uE^eX^{x-1}$
$U^uE^eY^y \qquad (y \geq 0)$	$eU^uE^{e-1}Y^{y+1}$

or

$e_{m,n,\ell} \qquad (m > n)$	$\frac{m-\ell}{2}e_{m-1,n,\ell+1} + (m - n + \frac{m-\ell}{2})e_{m-1,n,\ell-1}$
$e_{m,n,\ell} \qquad (n \geq m)$	$(m - \frac{n+\ell}{2})\ e_{m-1,n,\ell}$

.

And the X_β action is:

$U^uE^eX^x$	$(x>0)$	$uU^{u-1}E^eX^{x-1}$
$U^uE^eY^y$	$(y\geq 0)$	$uU^{u-1}E^eXY^y+yU^uE^eY^{y-1}$

or

$e_{m,n,\ell}$	$(m>n)$	$(n-\frac{m-\ell}{2})e_{m,n-1,\ell}$
$e_{m,n,\ell}$	$(n\geq m)$	$(n-m+\frac{n+\ell}{2})e_{m,n-1,\ell+1}+(\frac{n+\ell}{2})e_{m,n-1,\ell-1}$

.

For $\mathfrak{sl}(3)$, our symmetry τ on quasi standard Young tableaux induces a very simple transformation on $V^{(a,b)}$.

Starting with a quasi standard Young tableau T with a' columns of height 1 and b' columns of height 2, $a'\leq a$ and $b'\leq b$, as for $\mathfrak{sl}(2)$, we complete T by adding $a-a'$ trivial columns $\begin{array}{|c|}\hline 1\\\hline\end{array}$ and $b'-b$ trivial columns $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$. Then we act with τ, we reorder the entries of each column and finally we cancell the trivial columns. The resulting quasi standard tableau will be denoted $\tau^{(a,b)}(T)$.

For instance if $a=5$ and $b=3$ and $T=\begin{array}{|c|c|c|c|c|}\hline 2&2&2&2&3\\\hline 3&3\\\cline{1-2}\end{array}$, we get:

$$\begin{aligned}
\tau^{(5,3)}(T) &= \tau\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&1&1&2&2&3\\\hline 2&3&3\\\cline{1-3}\end{array}\right)\\
&= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 3&2&2&3&3&2&2&1\\\hline 2&1&1\\\cline{1-3}\end{array}\\
&= -\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&2&1&2&2&3&3\\\hline 2&2&3\\\cline{1-3}\end{array}\\
&\simeq -\begin{array}{|c|c|c|c|c|}\hline 2&2&2&3&3\\\hline 3\\\cline{1-1}\end{array}\ .
\end{aligned}$$

With the polynomial notations, we get:

$$\tau^{(a,b)}(U^uE^eX^x) = U^{a-(x+u)}E^{b-e}X^x$$
$$\tau^{(a,b)}(U^uE^eX^x) = U^{a-u}E^{b-(y+e)}Y^y.$$

And with the notations of [6]:

$$\tau^{(a,b)}(e_{m,n,\ell}) = e_{a+b-n,a+b-m,a-b+m-n-\ell}.$$

11. The $\mathfrak{sl}(4)$ case

11.1. Representations of $\mathfrak{sl}(4)$

As above, we have simple roots α, β and γ, with:

$$X_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover we have positive roots $\alpha+\beta$, $\beta+\gamma$ and $\alpha+\beta+\gamma$, with:

$$X_{\alpha+\beta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{\beta+\gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_{\alpha+\beta+\gamma} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We put $Y_\eta = {}^tX_\eta$ and

$$H_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The fundamental representations are 4 and 6 dimensional, they are associated to the fundamental highest weight ω_1 for the canonical representation on $V = \mathbb{C}^4$, ω_2 for the representation on $\wedge^2V$ and ω_3 for the representation on $\wedge^3V$. These fundamental representations are easy to describe, the reduction of the tensor product of any two of them is completely described in [2]. Especially, we can get the Plücker relations thanks to this decomposition.

11.2. Shape and reduced shape algebra

Now we have 10 Plücker relations: let us put as above:

$$\delta_i^{(1)} = g_{i1}, \quad \delta_{ij}^{(2)} = \begin{vmatrix} g_{i1} & g_{i2} \\ g_{j1} & g_{j2} \end{vmatrix}, \quad \delta_{ijk}^{(3)} = \begin{vmatrix} g_{i1} & g_{i2} & g_{i3} \\ g_{j1} & g_{j2} & g_{j3} \\ g_{k1} & g_{k2} & g_{k3} \end{vmatrix}$$

Then we have 4 Plücker relations between the $\delta^{(1)}$ and $\delta^{(2)}$:

$$\begin{aligned} \delta_1^{(1)}\delta_{23}^{(2)} - \delta_2^{(1)}\delta_{13}^{(2)} + \delta_3^{(1)}\delta_{12}^{(2)} &= 0, \\ \delta_2^{(1)}\delta_{34}^{(2)} - \delta_3^{(1)}\delta_{24}^{(2)} + \delta_4^{(1)}\delta_{23}^{(2)} &= 0, \\ \delta_1^{(1)}\delta_{34}^{(2)} - \delta_3^{(1)}\delta_{14}^{(2)} + \delta_4^{(1)}\delta_{13}^{(2)} &= 0, \\ \delta_1^{(1)}\delta_{24}^{(2)} - \delta_2^{(1)}\delta_{14}^{(2)} + \delta_4^{(1)}\delta_{12}^{(2)} &= 0. \end{aligned}$$

There are also 4 relations between the $\delta^{(2)}$ and $\delta^{(3)}$:

$$\begin{aligned} \delta_{14}^{(2)}\delta_{234}^{(3)} - \delta_{24}^{(2)}\delta_{134}^{(3)} + \delta_{34}^{(2)}\delta_{124}^{(3)} &= 0, \\ \delta_{12}^{(2)}\delta_{134}^{(3)} - \delta_{13}^{(2)}\delta_{124}^{(3)} + \delta_{14}^{(2)}\delta_{123}^{(3)} &= 0, \\ \delta_{12}^{(2)}\delta_{234}^{(3)} - \delta_{23}^{(2)}\delta_{124}^{(3)} + \delta_{24}^{(2)}\delta_{123}^{(3)} &= 0, \\ \delta_{13}^{(2)}\delta_{234}^{(3)} - \delta_{23}^{(2)}\delta_{134}^{(3)} + \delta_{34}^{(2)}\delta_{123}^{(3)} &= 0. \end{aligned}$$

And one between the $\delta^{(2)}$:

$$\delta_{12}^{(2)}\delta_{34}^{(2)} - \delta_{13}^{(2)}\delta_{24}^{(2)} + \delta_{14}^{(2)}\delta_{23}^{(2)} = 0.$$

And finally one between the $\delta^{(1)}$ and the $\delta^{(3)}$:

$$\delta_1^{(1)}\delta_{234}^{(3)} - \delta_2^{(1)}\delta_{134}^{(3)} + \delta_3^{(1)}\delta_{124}^{(3)} - \delta_4^{(1)}\delta_{123}^{(3)} = 0.$$

The shape algebra is the quotient of the algebra of polynomial functions in these 14 variables by the 10 above relations.

The reduced shape algebra is obtained by imposing $\delta_1^{(1)} = 1$, $\delta_{12}^{(2)} = 1$ and $\delta_{123}^{(3)} = 1$.

With our notations:

$$\mathfrak{n}^- = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ u & y & 0 & 0 \\ w & v & z & 0 \end{bmatrix} \right\}$$

and

$$N^- = exp(\mathfrak{n}^-) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ u + \frac{xy}{2} & y & 1 & 0 \\ w + \frac{xv}{2} + \frac{zu}{2} + \frac{xyz}{6} & v + \frac{yz}{2} & z & 1 \end{bmatrix} \right\}.$$

Then we get:

$$\delta_1^{(1)} = 1,\ \delta_2^{(1)} = x = X,\ \delta_3^{(1)} = \frac{xy}{2} + u = U,\ \delta_4^{(1)} = w + \frac{xv}{2} + \frac{zu}{2} + \frac{xyz}{6} = A$$

and

$$\delta_{123}^{(3)} = 1,\ \delta_{124}^{(3)} = z = Z,\ \delta_{134}^{(3)} = \frac{yz}{2} - v = W,\ \delta_{234}^{(3)} = \frac{xyz}{6} - \frac{xv}{2} - \frac{zu}{2} + w = C$$

and

$$\begin{array}{cc} \delta_{12}^{(2)} = 1, & \delta_{13}^{(2)} = y = Y, \\ \delta_{14}^{(2)} = v + \dfrac{yz}{2} = V, & \delta_{23}^{(2)} = \dfrac{xy}{2} - u = E, \\ \delta_{24}^{(2)} = \dfrac{xyz}{3} + \dfrac{xv}{2} - \dfrac{zu}{2} - w = D, & \delta_{34}^{(2)} = \dfrac{xy^2z}{12} + uv - yw = B. \end{array}$$

Now:

$$\begin{aligned} \mathbb{S}^\bullet_{red}(V)^+ &\simeq \mathbb{C}[x,y,z,u,v,w] \\ &= \mathbb{C}[\delta_2^{(1)}, \ldots, \delta_4^{(1)}, \delta_{13}^{(2)}, \ldots, \delta_{34}^{(2)}, \delta_{124}^{(3)}, \ldots, \delta_{234}^{(3)}] \big/ P_{red}(\delta)^+ \\ &= \mathbb{C}[X,Y,Z,U,E,W,V,A,C,D,B] \big/ Pluck \end{aligned}$$

where $Pluck$ is the ideal generated by the 10 polynomials:

$$\begin{aligned} &Pluck = \\ &\quad \langle U - XY + E,\ D - XV + A,\ B - UV + YA,\ XB - UD + AE, \\ &\quad B - YD + EV,\ C - XW + UZ - A, \\ &\quad VC - DW + BZ,\ W - YZ + V,\ C - EZ + D,\ YC - EW + B\rangle. \end{aligned}$$

Then we get the Groebner basis G:

$$\begin{aligned}
&\{ -B + UV - YA, XB - UD + AE, -D + XV - A,\\
&\ -U + XY - E, -C + XW - UZ + A, B - YD + EV,\\
&VC - DW + BZ, -C + EZ - D, YC - EW + B,\\
&\ -W + YZ - V, -EB - EYA - UB + UYD,\\
&UDW - UBZ - AEW - CB + AB\}.
\end{aligned}$$

The leading terms of this basis are:

$$XY,\ XV,\ UV,\ BX,\ YZ,\ EZ,\ YC,\ VC,\ XW,\ EV,\ UDW,\ UDY.$$

Now the basis of our space, *i.e.* the nodes of the $\mathfrak{sl}(4)$-diamond are monomials

$$X^xY^yZ^zW^wV^vU^uE^eA^aC^cD^dB^b$$

with:

$$\begin{aligned}
0 &= xy = xv = uv = bx = yz = ez = yc = vc = xw = ev\\
&= udw = udy.
\end{aligned}$$

The action of our generators X_α, X_β and X_γ on these polynomials are:

$$\begin{aligned}
X_\alpha &= \partial_x - \frac{y}{2}\partial_u + \left(\frac{yz}{12} - \frac{v}{2}\right)\partial_w,\\
X_\beta &= \partial_y + \frac{x}{2}\partial_u - \frac{z}{2}\partial_v - \frac{xz}{6}\partial_w,\\
X_\gamma &= \partial_z + \frac{y}{2}\partial_v + \left(\frac{xy}{12} + \frac{u}{2}\right)\partial_w.
\end{aligned}$$

Then we get:

$$\begin{aligned}
&X_\alpha(X) = 1, && X_\beta(X) = 0, && X_\gamma(X) = 0,\\
&X_\alpha(Y) = 0, && X_\beta(Y) = 1, && X_\gamma(Y) = 0,\\
&X_\alpha(Z) = 0, && X_\beta(Z) = 0, && X_\gamma(Z) = 1,\\
&X_\alpha(U) = 0, && X_\beta(U) = X, && X_\gamma(U) = 0,\\
&X_\alpha(E) = Y, && X_\beta(E) = 0, && X_\gamma(E) = 0,\\
&X_\alpha(W) = 0, && X_\beta(W) = Z, && X_\gamma(W) = 0,
\end{aligned}$$

$$\begin{array}{lll} X_\alpha(V)=0, & X_\beta(V)=0, & X_\gamma(V)=Y,\\ X_\alpha(A)=0, & X_\beta(A)=0, & X_\gamma(A)=U,\\ X_\alpha(C)=W, & X_\beta(C)=0, & X_\gamma(C)=0,\\ X_\alpha(D)=V, & X_\beta(D)=0, & X_\gamma(D)=V,\\ X_\alpha(B)=0, & X_\beta(B)=D, & X_\gamma(B)=0. \end{array}$$

Thus the X_η for η simple are acting on our basis of the reduced shape algebra by giving linear combination with integral coefficients, indeed, we find first such a linear combination on $\mathbb{Z}$ (even $\mathbb{Z}^+$) coefficients but on monomials which are perhaps not all admissible, then we come back to admissible monomials, using the reduced Plücker relations, but these relations are with coefficients ± 1, thus we finally get a combination of monomials in the basis with coefficients in $\mathbb{Z}$.

11.3. Symmetry

Now the symmetry τ on Young tableaux does not induce a simple operation $\tau^{(a,b,c)}$ on the basis of the simple module $V^{(a,b,c)}$.

For instance the tableau $\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array} = ZU$ is an element of the basis of $V^{(1,0,1)}$ (see the figure below). Repeating the operation performed for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, we get:

$$compl\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}$$

and

$$\tau\left(compl\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\right)\right) = \begin{array}{|c|c|}\hline 4 & 2\\ \hline 3 & \multicolumn{1}{c}{}\\ \cline{1-1} 1 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array} = -\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}$$

but this tableau is not quasi standard: the extraction of the trivial top $\boxed{1}$ of the first column is not trivial. Thus:

$$\tau^{(1,0,1)}\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\right) = -\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1} 4 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array} + \boxed{4}$$

or:

$$\tau^{(1,0,1)}(ZU) = -WX = -C - ZU + A.$$

We prefer to keep the new anti quasi standard Groebner basis to see τ as a global change of basis τ_{compl} inside the reduced shape algebra and to realize $Y_{\tau_\eta} = \tau_{compl} X_\eta \tau_{compl}$ by using the two basis. For instance in $V^{(1,0,1)}$ the basis is:

$$\{1, X, U, A, Z, W, C, WU, WA, CU, CA, CX, ZU, ZA, ZX\},$$

the image by τ_{compl} of this basis is:

$$\{AC, UC, XC, C, WA, ZA, A, ZX, Z, X, 1, WX, W, WU\}$$

The matrix of Y_{τ_η} on this new basis is exactly the matrix of X_η in the old one.

Figure 6 gives the presentation for the adjoint representation $V^{(1,0,1)}$ of $SL(4)$.

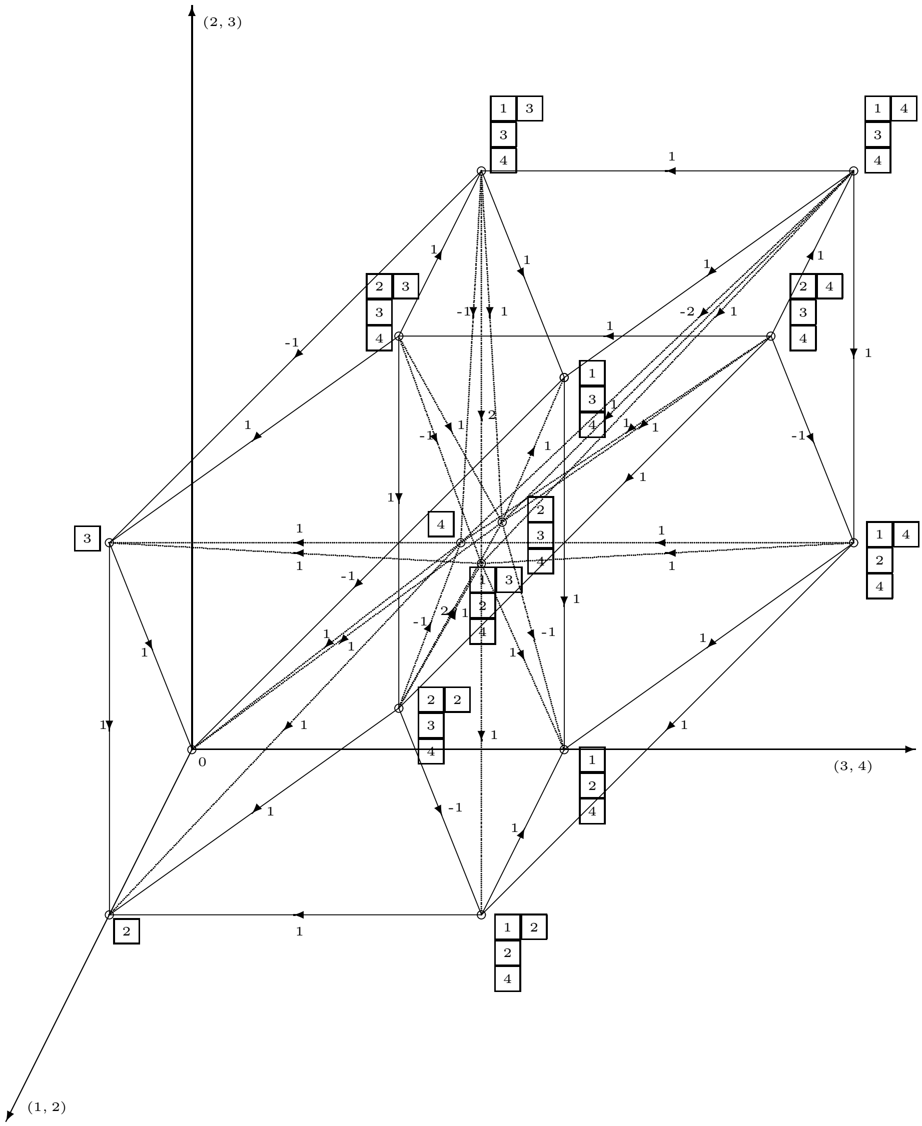

Figure 6. The adjoint representation $V^{(1,0,1)}$ of $SL(4)$

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Didier Arnal
Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne
U.F.R. Sciences et Techniques B.P. 47870
F-21078 Dijon Cedex
France
Didier.Arnal@u-bourgogne.fr

Nadia Bel Baraka
Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne
U.F.R. Sciences et Techniques B.P. 47870
F-21078 Dijon Cedex
France
Nadia.Bel-Baraka@u-bourgogne.fr

Norman J. Wildberger
School of Mathematics
University of New South Wales
Sydney 2052
Australia
n.wildberger@unsw.edu.au