ANNALES MATHÉMATIQUES

BLAISE PASCAL

Gilles Halbout

Formality theorems: from associators to a global formulation

Volume 13, n° 2 (2006), p. 313-348.

<http://ambp.cedram.org/item?id=AMBP_2006__13_2_313_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **13**, 313-348 (2006)

Formality theorems: from associators to a global formulation

Gilles Halbout

Abstract

Let M be a differential manifold. Let Φ be a Drinfeld associator. In this paper we explain how to construct a global formality morphism starting from Φ. More precisely, following Tamarkin's proof, we construct a Lie homomorphism "up to homotopy" between the Lie algebra of Hochschild cochains on $C^\infty(M)$ and its cohomology $(\Gamma(M, \Lambda TM),\ [-,-]_S)$. This paper is an extended version of a course given 8 - 12 March 2004 on Tamarkin's works. The reader will find explicit examples, recollections on G_∞-structures, explanation of the Etingof-Kazhdan quantization-dequantization theorem, of Tamarkin's cohomological obstruction and of globalization process needed to get the formality theorem. Finally, we prove here that Tamarkin's formality maps can be globalized.

1. Introduction

Let M be a differential manifold and $A = C^\infty(M)$ the algebra of smooth differential functions over M. Formality theorems link commutative objects with their non commutative analogs. More precisely, one has two graded Lie algebra structures:

- The space $T_{\text{poly}} = \Gamma(M, \Lambda TM)$ of multivector fields on M. It is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [20]), extending the Lie bracket of vector fields (see Example 2.3 in section 1).

- The space $D_{\text{poly}} = C(A, A) = \bigoplus_{k\geq 0} C^k(A, A)$, of regular Hochschild cochains (generated by differential k-linear maps from A^k to A and support preserving). This vector space D_{poly} is also endowed with a differential graded Lie algebra structure given by the Gerstenhaber bracket $[-,-]_G$ [9] and coHochschild differential b (see Example 2.4 in section 1).

We have:

Theorem 1.1. *The cohomology $H^*(D_{\text{poly}}, b)$ of D_{poly} with respect to b is isomorphic to the space T_{poly} ([15]).*

More precisely, one can construct a quasi-isomorphism of complexes

$$\varphi^1 : \ (T_{\rm poly}, 0) \to (D_{\rm poly}, b),$$

called the Hochschild-Kostant-Rosenberg quasi-isomorphism ([15]); it is defined, for $\alpha \in T_{\rm poly}$, $f_1, \ldots, f_n \in A$, by

$$\varphi^1 : \alpha \mapsto \big((f_1, \ldots, f_n) \mapsto \ \langle \alpha, df_1 \wedge \cdots \wedge df_n \rangle\big).$$

This map φ^1 is not a differential Lie algebra morphism but it is "up to (higher) homotopy". Formality maps are the collection of those homotopies: they are maps, $\varphi^{1,\ldots,1}\colon \Lambda^n T_{\rm poly} \to D_{\rm poly}$, for $n \geq 0$, such that

$$(d_T^1 + d_T^{1,1}) \circ \varphi = \varphi \circ d_T^{1,1}, \tag{1.1}$$

where we have "extended" the Lie bracket $[-,-]_S$ to a coderivation $d_T^{1,1}$: $\Lambda^{\cdot} T_{\rm poly} \to \Lambda^{\cdot} T_{\rm poly}$, the Lie bracket $[-,-]_G$ and the differential b to coderivations $d_D^{1,1}$ and d_D^1: $\Lambda^{\cdot} D_{\rm poly} \to \Lambda^{\cdot} D_{\rm poly}$ and the maps $\varphi^{1,\ldots,1}$ to morphisms of coalgebras : $\Lambda^{\cdot} T_{\rm poly} \to \Lambda^{\cdot} D_{\rm poly}$ on the corresponding cofree cocommutative coalgebras. In the first section of this paper we will recall precise definitions of $\Lambda^{\cdot} E$ for E a graded vector spaces, of the above maps and of their "extension".

Existence of such homotopies was proven for $M = \mathbb{R}^d$ by Kontsevich (see [18] and [19]) and Tamarkin (see [22]). They use different methods in their proofs. Kontsevich proved also that those maps can be globalized on a general manifold. When M is a Poisson manifold equipped with a Poisson bracket corresponding to a Poisson 2-tensor field π (such that $[\pi, \pi]_S = 0$), one can deduce the existence of a star-product $m_\star$ on M, *i.e.* an associative product on $A[[\hbar]]$ for $\hbar$ a formal parameter:

$$m_\star = m + \hbar\varphi^1(\pi) + \sum_{n\geq 2} \frac{\hbar^n}{n!} \varphi^n(\pi\Lambda \cdots \Lambda\pi).$$

Notice that until the end of the paper, we will use the notation Λ for the product on the exterior algebra $\Lambda^{\cdot} E$ and $\wedge$ for the exterior product on $T_{\rm poly}$.

The fact that $m_\star$ is associative, *i.e.* $[m_\star, m_\star]_G = 0$, follows from equation (1.1) and one has $\varphi^1(\pi) = \{-,-\}$, the Poisson bracket.

We will follow Tamarkin's proof and show how to build such homotopies. In the first three sections, we will suppose that $M = \mathbb{R}^d$.

The paper is organized as follows:

- In Section 2, we will make precise definitions of L_∞ and G_∞-structures and morphisms used to define the formality maps. Explicit formulas will be given.

- In Section 3, we will show that the space $D_{\rm poly}$ can be endowed with a G_∞-structure. This is where associators and Etingof-Kazhdan theorem will be needed. We will outline proofs by Etingof and Kazhdan and also by Enriquez.

- In Section 4, we will construct the formality maps when the manifold $M = \mathbb{R}^d$. To do so, we will describe obstructions to such a construction and show that they vanish when $M = \mathbb{R}^d$.

- In Section 5, we will prove that those maps can be globalized when M is an arbitrary manifold. To do so, we will follow Dolgushev's approach ([2]) where the globalization process was done to local Kontsevich's maps.

I would like to thank D. Manchon and D. Arnal for their invitation in Dijon, D. Calaque, V. Dolgushev, G. Ginot and B. Keller for many useful suggestions and B. Enriquez and P. Etingof for helping better understand the Etingof-Kazhdan dequantization theorem.

2. G_∞-structures

The first aim of this section is to give a precise meaning to Equation (1.1) and to explain what we mean by "canonical extension" on $\Lambda\, T_{\rm poly}$ or $\Lambda\, D_{\rm poly}$. To do so, let us reformulate the definition of a Lie algebra and more generally of a L_∞-algebra. For a graded vector space E, let us denote $TE = T(E[1])$ the free tensor algebra of E which, equipped with the coshuffle coproduct, is a bialgebra. The coshuffle coproduct Δ is defined on the generators x of TE by $\Delta(x) = x\otimes 1 + 1\otimes x$. Let us denote $\Lambda E = S(E[1])$ the free graded commutative algebra generated by $E[1]$, seen as a quotient of TE. The coshuffle coproduct is still well defined on ΛE which becomes a cofree cocommutative coalgebra. One can write an explicit formula for the coproduct $\Delta : \Lambda E \to (\Lambda E)\Lambda(\Lambda E)$,

$$\Delta(\gamma_1\Lambda\cdots\Lambda\gamma_n) = \frac{1}{2}\sum_{k=1}^{n-1}\sum_{\varepsilon\in S_n} \mathrm{sgn}(\varepsilon)(\gamma_{\varepsilon(1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(k)})\Lambda(\gamma_{\varepsilon(k+1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(n)}),$$

where $\mathrm{sgn}(\varepsilon)$ corresponds to the graded signature of the permutation defined, for any permutation ε of $\{1,...,n\}$ and any graded variables $\gamma_1,\ldots,\gamma_n$ in E (with degree shifted by minus one), by the identity

$$\gamma_1\cdots\gamma_n = \mathrm{sgn}(\varepsilon)\gamma_{\varepsilon^{-1}(1)}\cdots\gamma_{\sigma^{-1}(n)}$$

which holds in the free graded commutative algebra generated by $\gamma_1,\ldots,\gamma_n$. For $E_1, E_2 \in E$, $E_1 \Lambda E_2$ will stand for the corresponding quotient of $E_1[1]\otimes E_2[1]$ in ΛE. We will use the notations T^nE and Λ^nE for the elements of degree n. We have now

Definition 2.1. A vector space E is endowed with a L_∞-algebra (Lie algebra "up to homotopy") structure if there are degree one linear maps $d^{1,\ldots,1}\colon \Lambda^kE\to E[1]$ such that the asociated coderivations (extended with respect to the cofree cocommutative structure on ΛE) d: $\Lambda E\to\Lambda E$, satisfy $d\circ d=0$ where d is the coderivation

$$d = d^1 + d^{1,1} + \cdots + d^{1,\ldots,1} + \cdots .$$

One can again write explicit formulas for the extensions of the maps as coderivations ($\Delta\circ d^{1,\ldots,1} = (d^{1,\ldots,1}\otimes\mathrm{Id}+\mathrm{Id}\otimes d^{1,\ldots,1})\circ\Delta$):

$$\begin{aligned}d(\gamma_1\Lambda\cdots\Lambda\gamma_n) &= d^{1,\ldots,1}(\gamma_1\Lambda\cdots\Lambda\gamma_n)\\ &\quad+\sum_{k=1}^{n-1}\sum_{\varepsilon\in S_n}\mathrm{sgn}(\varepsilon)d^{1,\ldots,1}(\gamma_{\varepsilon(1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(k)})\Lambda\gamma_{\varepsilon(k+1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(n)}.\end{aligned}$$

In particular, we have

Remark 2.2. A differential Lie algebra $(E, d, [-,-])$ is a L_∞-algebra with structure maps $d^1 = d[1]$, $d^{1,1} = [-,-][1]$ and $d^{1,\ldots,1}\colon \Lambda^kE\to E[1]$ are 0 for $k\geq 3$.

Let us recall the two examples T_{poly} and D_{poly}:

Example 2.3. The space T_{poly} is a graded Lie algebra (and so a L_∞-algebra) with 0 differential and Schouten bracket $[-,-]_S$defined as follows

$$[\alpha,\beta\wedge\gamma]_S = [\alpha,\beta]_S\wedge\gamma + (-1)^{|\alpha|(|\beta|+1)}\beta\wedge[\alpha,\gamma]_S \tag{2.1}$$

for $\alpha,\beta,\gamma\in T_{\mathrm{poly}}$. For $f\in\Gamma(M,\Lambda^0TM)=C^\infty(M)$ and $\alpha\in\Gamma(M,\Lambda^1TM)$ we set $[\alpha,f]_S=\alpha\cdot f$, the action of the vector field α on f. The grading on $T_{\rm poly}$ is defined by $|\alpha| = n \Leftrightarrow \alpha \in \Gamma(M, \Lambda^{n+1}TM)$ and the exterior product is graded commutative:

$$\forall \alpha,\ \beta \in \Gamma(M, \Lambda TM),\ \alpha \wedge \beta = (-1)^{(|\alpha|+1)(|\beta|+1)} \beta \wedge \alpha.$$

Let us denote d_T the associated coderivation ($d_T^{1,1}$ is corresponding to $[-,-]_S[1]$). One can check that the Jacobi identity for $[-,-]_S$ is equivalent to $d_T^{1,1} \circ d_T^{1,1} = 0$.

Example 2.4. Similarly, $D_{\rm poly}$ is a differential graded Lie algebra (and so a L_∞-algebra). Its bracket, the Gerstenhaber bracket $[-,-]_G$, is defined, for $D, E \in D_{\rm poly}$, by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \ldots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E|\cdot i} D(x_1, \ldots, x_i, E(x_{i+1}, \ldots, x_{i+e}), \ldots).$$

The space $D_{\rm poly}$ has a grading defined by $\mid D \mid = k \Leftrightarrow D \in C^{k+1}(A, A)$. Finally, its differential is the coHochschild differential $b = [m, -]_G$, where $m \in C^2(A, A)$ is the commutative multiplication on A. Let us denote d_D the associated coderivation ($d_D^{1,1}$ corresponding to $[-,-]_G[1]$ and d_D^1 to $b[1]$). One can check that Jacobi identity for $[-,-]_G$, $b^2 = 0$ and compatibility between b and $[-,-]_G$ are equivalent to $(d_D^1 + d_D^{1,1}) \circ (d_D^1 + d_D^{1,1}) = 0$.

One can now define the generalization of Lie algebra morphisms:

Definition 2.5. A L_∞-morphism between two L_∞-algebras $(E_1, d_1 = d_1^1 + \cdots)$ and $(E_2, d_2 = d_2^1 + \cdots)$ is a morphism of differential cofree coalgebras, of degree 0,

$$\varphi :\ (\Lambda E_1, d_1) \to (\Lambda E_2, d_2).$$

In particular $\varphi \circ d_1 = d_2 \circ \varphi$. As φ is a morphism of cofree cocommutative coalgebras (i.e. $\Delta_2 \varphi = (\varphi \otimes \varphi)\Delta_1$ where Δ_1 and Δ_2 are the coproducts on E_1 and E_2), φ is determined by its image on the cogenerators, i.e., by its components: $\varphi^{1,\ldots,1} : \Lambda^k E_1 \to E_2[1]$. Again one gets a general formula for φ:

$$\varphi(\gamma_1\Lambda\cdots\Lambda\gamma_n) = \varphi^{1,\dots,1}(\gamma_1\Lambda\cdots\Lambda\gamma_n) + \sum_{p=1}^{n-1}\frac{1}{p!}\sum_{k_1,\dots,k_p\geq 1}^{k_1+\cdots+k_p=n}$$
$$\sum_{\varepsilon\in S_n}\operatorname{sgn}(\varepsilon)\varphi^{1,\dots,1}(\gamma_{\varepsilon(1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(k)})\Lambda\cdots\Lambda\varphi^{1,\dots,1}(\gamma_{\varepsilon(n-k+1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(n)})$$

where the signs are Quillen's signs corresponding to permutations of odd elements. Now equation (1.1) can be rewritten as follows: let d_T and d_D correspond respectively to the Lie algebra structure on T_{poly} and to the differential Lie algebra structure on D_{poly}. We want to construct a L_∞-morphism φ such that φ^1 is the Hochschild-Kostant-Rosenberg map and:

$$\varphi\circ d_T = d_D\circ\varphi.$$

If one tries to construct the maps $\varphi^{1,\dots,1} : \Lambda^n T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ by induction on n, one will find obstructions in the non acyclic Chevalley Eilenberg complex $\mathrm{Hom}(\Lambda T_{\mathrm{poly}}, T_{\mathrm{poly}}, [d_T, -])$.

Tamarkin's idea was then to extend the structure (or increase the constrains) to reduce the obstructions. Indeed, T_{poly} has a Gerstenhaber structure. It would be convenient to find such a structure on D_{poly} (we will see that D_{poly} has actually a G_∞-structure *i.e.* an "up to homotopy" Gerstenhaber structure) and to construct a G_∞-morphism between them (that restricts to a L_∞-morphism on the corresponding Lie algebra structures). Thanks to the addition of those extra operations, we will see that obstructions to the construction of G_∞-morphisms will vanish in the case $M = \mathbb{R}^d$. Let us end this section by some recollections on G_∞-structures. We will follow works of Ginot ([10]).

To define a G_∞-structure on E, we will need a bigger space than ΛE. Let us denote ${}^cT(E)$ the cofree tensor coalgebra of E (with coproduct Δ'). We will sometimes use the notation $E^{\otimes\cdot}$. Equipped with the shuffle product $\bullet$ (defined on the cogenerators ${}^cT(E)\otimes{}^cT(E)\to E$ as $\mathrm{pr}\otimes\varepsilon + \varepsilon\otimes\mathrm{pr}$, where $\mathrm{pr} : {}^cT(E)\to E$ is the projection and ε is the counit), it is a bialgebra. Let ${}^cT(E)^+$ be the augmentation ideal. We note $\underline{{}^cT(E)} = {}^cT(E)^+/({}^cT(E)^+\bullet{}^cT(E)^+)$ the quotient by the shuffles. It has a graded cofree coLie coalgebra structure (with coproduct $\delta = \Delta' - \Delta'^{\mathrm{op}}$), see [12] for example. Then $S(\underline{{}^cT(E)}[1])$ has a structure of cofree coGerstenhaber algebra (i.e., equipped with cofree coLie and cofree cocommutative coproducts δ and Δ satisfying compatibility condition). One can write δ explicitly: for $\gamma_i \in E^{\otimes p_i}$,

$$\delta(\gamma_1\Lambda\cdots\Lambda\gamma_n) =$$
$$\sum \operatorname{sgn}(\varepsilon) s_k \gamma_{\varepsilon(1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(i)}\Lambda(\alpha_1^k\cdots\alpha_j^k)\otimes(\alpha_{j+1}^k\cdots\alpha_{p_k}^k)\Lambda\gamma_{\varepsilon(i+1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(n)},$$

where the sum is over all integers $1 \le k \le n$, $1 \le j \le p_k$ and all permutations ε fixing k which are $(i, n-1-i)$-shuffles on $\{1,\dots,n\}-\{k\}$. We have denoted $\gamma_k = \alpha_1^k\cdots\alpha_j^k\alpha_{j+1}^k\cdots\alpha_{p_k}^k$ and the sign $s_k = (-1)^{(|\alpha_1|+\cdots+|\alpha_j|)(p_k-j)}$. Moreover, we still have:

$$\Delta(\gamma_1\Lambda\cdots\Lambda\gamma_n) = \sum \operatorname{sgn}(\varepsilon)(\gamma_{\varepsilon(1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(i)})\Lambda(\gamma_{\varepsilon(i+1)}\Lambda\cdots\Lambda\gamma_{\varepsilon(n)}),$$

where the sum is over $(i, n-i)$-shuffles. We use the notation $\underline{{}^cT^m(E)}$ for the elements of degree m, and, for $\gamma_1^1,\dots,\gamma_n^{p_n} \in E$, we have

$$|\underline{\gamma_1^1\otimes\cdots\otimes\gamma_1^{p_1}}\Lambda\cdots\Lambda\underline{\gamma_n^1\otimes\cdots\otimes\gamma_n^{p_n}}| = \sum_{i_1}^{p_1}|\gamma_1^{i_1}| + \cdots + \sum_{i_n}^{p_n}|\gamma_n^{i_n}| - n.$$

Definition 2.6. A vector space E is endowed with a G_∞-algebra (Gerstenhaber algebra "up to homotopy") structure if there are degree one linear maps $d^{p_1,\dots,p_k}\colon \underline{{}^cT^{p_1}(E)}\Lambda\cdots\Lambda\underline{{}^cT^{p_k}(E)} \subset \Lambda^k\underline{{}^cTE} \to E[1]$ such that the associated coderivations (extended with respect to the cofree coGerstenhaber structure on $\Lambda^c\underline{T(E)}$) d: $\Lambda^c\underline{T(E)} \to \Lambda^c\underline{T(E)}$ satisfies $d\circ d = 0$ where d is the coderivation

$$d = d^1 + d^{1,1} + \cdots + d^{p_1,\dots,p_k} + \cdots .$$

More details on G_∞-structures are given in [10].

In particular we have

Remark 2.7. If $(E, d, [-,-], \wedge)$ is a differential Gerstenhaber algebra, then $E[1]$ is a G_∞-algebra with structure maps $d^1 = d[1]$, $d^{1,1} = [-,-][1]$, $d^2 = \wedge[1]$ and other $d^{p_1,\dots,p_k}\colon \underline{{}^cT^{p_1}(E[1])}\,\Lambda\cdots\Lambda\underline{{}^cT^{p_k}(E[1])} \to E[2]$ are 0.

Applying this remark to the spaces T_{poly} and D_{poly} we get

Example 2.8. The space T_{poly} is a graded Gerstenhaber algebra and so a G_∞-algebra with maps $d_T^{1,1} = [-,-]_S[1]$ and $d_T^2 = \wedge[1]$ the exterior product. It is clear that d_T^2 is a well defined map $\underline{T_{\text{poly}}{}^{\otimes 2}} \to T_{\text{poly}}$ (because it is graded commutative).

Let us, as an exercise, extend maps $d_T^{1,1}$ and d_T^2 for degree 0 elements α, β, γ in T_{poly}.

$d_T^2(\alpha \otimes \beta \otimes \gamma) = d_T^2(\alpha \otimes \beta) \otimes \gamma + \alpha \otimes d_T^2(\beta \otimes \gamma) = (\alpha \wedge \beta) \otimes \gamma - \alpha \otimes (\beta \wedge \gamma)$.

and so the condition $d_T^2 \circ d_T^2(\alpha \otimes \beta \otimes \gamma) = (\alpha \wedge \beta) \wedge \gamma - \alpha \wedge (\beta \wedge \gamma) = 0$ is equivalent to the associativity of the map d_T^2.

In the same way, we have:

$$
\begin{aligned}
(d_T^{1,1} \circ d_T^2 + d_T^2 \circ d_T^{1,1}((\alpha \otimes \beta) \wedge \gamma)& \\
&= [\alpha \wedge \beta, \gamma]_S + d_T^2(\alpha \otimes d_T^{1,1}(\beta \wedge \gamma)) - d_T^2(d_T^{1,1}(\gamma \wedge \alpha) \otimes \beta) \\
&= [\alpha \wedge \beta, \gamma]_S + \alpha \wedge [\beta, \gamma]_S - [\gamma, \alpha]_S \wedge \beta = 0,
\end{aligned}
$$

by compatibility between $[-,-]_S$ and $\wedge$. So all the identities defining the Gerstenhaber algebra structure on T_{poly} can be summarized into the unique relation $(d_T^{1,1} + d_T^2) \circ (d_T^{1,1} + d_T^2) = 0$.

Example 2.9. The space D_{poly} is not a (graded) Gerstenhaber algebra when equiped with the product of cochains $\cup$ defined, for $D, E \in D_{\mathrm{poly}}$ and $x_1, \ldots, x_{|D|+|E|+2} \in A$, by

$(D \cup E)(x_1, \ldots, x_{|D|+|E|+2}) = (-1)^\gamma D(x_1, \ldots, x_{|D|+1}) E(x_{|D|+2}, \ldots, x_{|D|+|E|+2})$

where $\gamma = (|E|+1)(|D|+1)$. The projection of this product on the cohomology of (D_{poly}, b) is the exterior product $\wedge$, but unfortunately $(D_{\mathrm{poly}}, [-,-]_G, \cup, b)$ is not a Gerstenhaber algebra: one can see, for example, that $\cup$ is not a graded commutative product and thus can not be defined as a map $\underline{D_{\mathrm{poly}}{}^{\otimes 2}} \to D_{\mathrm{poly}}$. More generally, Gerstenhaber's cachain structure have the same "failure", only the cohomology behaves well.

We will show in Section 2 that it can be equiped with a G_∞-structure.

One can now define the generalization of Gerstenhaber morphisms:

Definition 2.10. A G_∞-morphism between two G_∞-algebras $(E_1, d_1 = d_1^1 + d_1^2 + \cdots)$ and $(E_2, d_2 = d_2^1 + d_2^2 + \cdots)$ is a morphism of differential coGerstenhaber coalgebras, of degree 0,

$$\varphi : \ (\Lambda^c \underline{T(E_1)}, d_1) \to (\Lambda^c \underline{T(E_2)}, d_2).$$

In particular $\varphi \circ d_1 = d_2 \circ \varphi$. As φ is a morphism of cofree coGerstenhaber coalgebras, φ is determined by its image on the cogenerators, i.e., by its components: $\varphi^{p_1,\dots,p_k}$: $\underline{{}^cT^{p_1}(E_1)}\Lambda\cdots\ \Lambda\underline{{}^cT^{p_k}(E_1)} \to E_2[1]$. As an example, for degree 0 elements α,β,γ in E_1, one has

$$\begin{aligned}\varphi((\alpha\otimes\beta)\Lambda\gamma) &= \varphi^{2,1}(\alpha\otimes\beta,\gamma)\\ &\quad+\varphi^1(\alpha)\otimes\varphi^{1,1}(\beta\Lambda\gamma)-\varphi^{1,1}(\gamma\Lambda\alpha)\otimes\varphi^1(\beta)\\ &\quad+\varphi^2(\alpha\otimes\beta)\Lambda\varphi^1(\gamma)\\ &\quad+(\varphi^1(\alpha)\otimes\varphi^1(\beta))\Lambda\varphi^1(\gamma).\end{aligned}$$

3. A G_∞-structure on the space of cochains

The objective of this section is to prove the following proposition ([22]).

Proposition 3.1. *There exists a G_∞-structure on D_{poly} given by a coderivation d_D such that if $d_D = \sum_{l\geq 1,\ p_1+\cdots p_n=l} d_D^{p_1,\dots,p_n}$, then $d_D\circ d_D=0$ and*

(1) *d_D^1 is the Hochschild differential b.*

(2) *$d_D^{1,1}$ is the Gerstenhaber bracket $[-,-]_G$.*

(3) *d_D^2 is the cup product $\cup$, up to a Hochschild coboundary.*

(4) *$d_D^{p_1,\dots,p_n}=0$ for $n>2$.*

3.1. Construction of the G_∞-structure

We first reformulate this problem: let $L_D = \oplus\ \underline{D_{\text{poly}}}^{\otimes n}$ be the cofree coLie coalgebra on D_{poly} (see Section 2 for the notation). Since L_D is a cofree coLie coalgebra, a differential Lie bialgebra structure on L_D is uniquely determined by the restriction to cogenerators of the Lie bracket and the differential (which are coderivations on L_D) and so by degree one maps l_D^n: $\underline{D_{\text{poly}}}^{\otimes n}\to D_{\text{poly}}$ (for the differential $L_D\to L_D$), and maps $l_D^{p_1,p_2}$: $\underline{D_{\text{poly}}}^{\otimes p_1}\Lambda\underline{D_{\text{poly}}}^{\otimes p_2}\to D_{\text{poly}}$ (for the Lie bracket $L_D\Lambda L_D\to L_D$). The following lemma is well known.

Lemma 3.2. *Suppose we have a differential Lie bialgebra structure on the coLie coalgebra L_D, with differential and Lie bracket respectively determined by maps l_D^n and $l_D^{p_1,p_2}$ as above. Then D_{poly} has a G_∞-structure* *given, for all $p, q, n \geq 1$, by*

$$d_D^n = l_D^n, \qquad d_D^{p,q} = l_D^{p,q} \quad \text{and} \quad d_D^{p_1,\dots,p_r} = 0 \text{ for } r \geq 3.$$

Proof. The map $d_D = \sum_{i\geq 0} l_D^i + \sum_{p_1,p_2\geq 0} l_D^{p_1,p_2} : \Lambda^{\cdot} L_D \to \Lambda^{\cdot} L_D$ is the Chevalley-Eilenberg differential on the differential Lie algebra L_D; it satisfies $d_D \circ d_D = 0$. □

Thus to obtain the desired G_∞-structure on D_{poly}, it is enough to define a differential Lie bialgebra structure on L_D given by maps l_D^n and $l_D^{p_1,p_2}$ with $l_D^1 = b$, $l_D^{1,1} = [-,-]_G$ and $l_D^2 = \cup$ "up to homotopy".

Let us now give an equivalent formulation of our problem, which is stated in terms of the associated operads in [22]:

Proposition 3.3. *Suppose we have a differential bialgebra structure on the cofree tensorial coalgebra $T_D = \oplus_{n\geq 0} {D_{\text{poly}}}^{\otimes n}$ with differential and multiplication given respectively by maps a_D^n: ${D_{\text{poly}}}^{\otimes n} \to D_{\text{poly}}$ and $a_D^{p_1,p_2}$: ${D_{\text{poly}}}^{\otimes p_1} \otimes {D_{\text{poly}}}^{\otimes p_2} \to D_{\text{poly}}$. Then we have a differential Lie bialgebra structure on the coLie coalgebra $L_D = \oplus_{n\geq 0} \underline{{D_{\text{poly}}}^{\otimes n}}$, with differential and Lie bracket respectively determined by maps l_D^n and $l_D^{p_1,p_2}$ where $l_D^1 = a_D^1$, $l_D^{1,1}$ is the anti-symmetrization of $a_D^{1,1}$ and $l_D^2 = a_D^2$ "up to homotopy".*

A differential bialgebra structure on the cofree tensorial coalgebra $\oplus V^{\otimes n}$ associated to a vector space V is often called a B_∞-structure on V, see [1].

Proof. The proof relies on the existence of a quantization/dequantization functor, that we will recall in the next subsection. Let V be a finite-dimensional vector space and V^* be the dual space. A differential bialgebra structure on the cofree coalgebra ${}^cTV = \oplus_{n\geq 0} V^{\otimes n}$ is defined on the cogenerators by maps a^n: $V^{\otimes n} \to V$ $(n \geq 2)$, corresponding to the differential $\sum_{n\geq 0} a^n$: ${}^cTV \to {}^cTV$, and maps a^{p_1,p_2}: $V^{\otimes p_1} \otimes V^{\otimes p_2} \to V$ $(p_1, p_2 \geq 0)$, corresponding to the product $\sum_{p_1,p_2\geq 0} a^{p_1,p_2}$: ${}^cTV \otimes {}^cTV \to {}^cTV$. We can define dual maps of those maps to get again a differential bialgebra with differential D: $\hat{T} \to \hat{T}$ and coproduct Δ: $\hat{T} \to \hat{T}\hat{\otimes}\hat{T}$, where $\hat{T}$ is the completion of the tensor algebra $\oplus_{n\geq 0} V^{*\otimes n}$. The differential and coproduct D and Δ are defined now on the generators of the free algebra $\hat{T}$ by maps a^{n*}: $V^* \to V^{*\otimes n}$ and a^{p_1,p_2*}: $V^* \to V^{*\otimes p_1} \otimes V^{*\otimes p_2}$. The tensor algebra $\oplus_{n\geq 0} V^{*\otimes n}$ is graded as follows: $|x| = p$ when $x \in V^{*\otimes p}$.

Similarly, if we consider a differential Lie bialgebra structure on the cofree coLie coalgebra $L = \oplus_{n\geq 0} \underline{V^{\otimes n}}$, the dual maps d and δ of the structure maps $\sum_{n\geq 0} l^n$ and $\sum_{p_1,p_2\geq 0} l^{p_1,p_2}$ induce a differential Lie bialgebra structure on $\hat{L}$, the completion of the free Lie algebra $\oplus_{n\geq 0} Lie(V^*)(n)$ on V^*, where $Lie(V^*)(n)$ is the subspace of element of degree n.

We now replace formally each element x of degree n in $\hat{T}$ (resp. $\hat{L}$) by $h^n x$, where h is a formal parameter. Letting $|h| = -1$, we easily see that it is equivalent to define

- a differential associative (respectively Lie) bialgebra structure on the associative (resp. Lie) algebras $(\oplus_{n\geq 0} V^{*\otimes n})[[h]]$ (resp. $(\oplus_{n\geq 0} Lie(V^*)(n))[[h]]$) with the product and coproduct being of degree zero
- or a differential associative (resp. Lie) bialgebra structure on the associative (resp. Lie) algebra $\hat{T}$ (resp. $\hat{L}$).

Note that those two bullets are dual. Thus we have a differential free coalgebra $(\hat{T}[[h]], D, \Delta)$.

We can apply now Etingof-Kazhdan's dequantization theorem for graded differential bialgebras ([7] and Appendix in [11]) to our particular case where we start from a differential bialgebra free as an algebra $(\hat{T}, \Delta, D)$: this proves that

Proposition 3.4. *There exists a Lie bialgebra $(\hat{L}, [-,-], \delta, d)$, generated as a Lie algebra by V^* and an injective map $I_{\mathrm{EK}}\colon \hat{L}[[h]] \to (\oplus_{n\geq 0} V^{*\otimes n})[[h]]$ such that*

1. *the restriction $I_{\mathrm{EK}}\colon V^* \to V^*$ is the identity,*
2. *the maps I_{EK}, δ and $[-,-]$ are given by universal formulas (i.e. depending only on Δ and the product of $\hat{T}$),*
3. *$I_{\mathrm{EK}}([a,b]) = I_{\mathrm{EK}}(a)I_{\mathrm{EK}}(b) - I_{\mathrm{EK}}(b)I_{\mathrm{EK}}(a) + O(h)$, for all $a, b \in \hat{L}[[h]]$,*
4. *$(\Delta - \Delta^{\mathrm{op}})\, I_{\mathrm{EK}} = hI_{\mathrm{EK}}\delta + O(h^2)$,*
5. *$I_{\mathrm{EK}} \circ d = D \circ I_{\mathrm{EK}}$*

(6) *if we apply Etingof-Kazhdan's quantization functor (see* [6]*) to the Lie bialgebra* $(\oplus_{n\geq 0} Lie(V^*)^n[[h]], \delta)$ *we get the bialgebra* $((\oplus_{n\geq 0} V^{*\otimes n})[[h]], \Delta)$ *back.*

The last condition implies that $\hat{L}$ is free as a Lie algebra because $\hat{T}$ is free as an algebra. Moreover the structure maps l_D^{p*} and $l_D^{p,q*}$ on $\hat{L}$ satisfy $l_D^{1*} = a_D^{1*}$, $l_D^{1,1*}$ is the anti-symmetrization of $a_D^{1,1*}$ ans $l_D^{2*} = a_D^{2*}$ "up to homotopy". Taking now dual maps, we get the result. □

Remark 3.5. Here one strongly used the quantization/dequantization theorem. Indeed, if one only takes the anti-symmetrization and the classical limit to get the wanted Lie algebra structure on L_D, one will lose the information on degree 2 maps and in particular the information on l_D^2. Recall that we wanted $l_D^2 = \cup$ "up to homotopy" and by taking the naive classical limit one would get $l_D^2 = 0$ which will then only give the Lie algebra structure on D_{poly} that we started with !

By Proposition 3.3, the problem of defining a differential Lie bialgebra structure on L_D given by maps l_D^n and $l_D^{p_1,p_2}$ with $l_D^1 = b$, $l_D^{1,1} = [-,-]_G$ and $l_D^2 = \cup$ "up to homotopy" is now equivalent to defining a differential bialgebra structure on T_D given by maps a_D^n: ${D_{\text{poly}}}^{\otimes n} \to D_{\text{poly}}$ and $a_D^{p_1,p_2}$: ${D_{\text{poly}}}^{\otimes p_1} \otimes {D_{\text{poly}}}^{\otimes p_2} \to D_{\text{poly}}$ where $a_D^1 = b$, $a_D^{1,1}$ is the product $\{-|-\}$ defined in Section 0 and $a_D^2 = \cup$ "up to homotopy". Indeed, the anti-symmetrization of $\{-|-\}$ is by definition $[-,-]_G$. The latter can be achieved using the braces operations (defined in [9]) acting on the Hochschild cochain complex $D_{\text{poly}} = C(A,A)$ for any algebra A. The braces operations are maps $a_D^{1,p}$: $D_{\text{poly}} \otimes {D_{\text{poly}}}^{\otimes p} \to D_{\text{poly}}$ $(p \geq 1)$ defined, for all homogeneous $D, E_1, \ldots, E_p \in {D_{\text{poly}}}^{\otimes p+1}$ and $x_1, \ldots, x_d \in A$ (with $d = |D| + |E_1| + \cdots + |E_p| + 1$), by

$$a_D^{1,p}(D \otimes (E_1 \otimes \cdots \otimes E_p))(x_1 \otimes \cdots \otimes x_d) = \sum (-1)^\tau D(x_1, \ldots, x_{i_1}, E_1(x_{i_1+1}, \ldots), \ldots, E_p(x_{i_p+1}, \ldots), \ldots)$$

where $\tau = \sum_{k=1}^{p} i_k(|E_k|+1)$. It is clear that $a_D^{1,1}$ corresponds to the map $\{-,-\}$. Now Theorem 3.1 in [23] asserts (see also [9] and [17]) that:

- The maps $a_D^{1,p} : D_{\text{poly}} \otimes {D_{\text{poly}}}^{\otimes p} \to D_{\text{poly}}$, $a_D^{q\geq 2,p} = 0$ and the degree 0 shuffle product determine a coderivation $\star = \sum a_D^{p,q}$ on the cofree tensorial coalgebra $T_D = \oplus_{n\geq 0} {D_{\mathrm{poly}}}^{\otimes n}$ which turns T_D into a bialgebra.

- Similarly taking a_D^1 to be the Hochschild coboundary b and a_D^2 to be the cup-product $\cup$, and $a_D^{q\geq 3} = 0$, the coderivation $d = \sum a_D^n$ defines a differential structure on the tensor coalgebra T_D.

- These maps yield a differential bialgebra structure $(T_D, \star, d)$ on the cofree coalgebra T_D.

Actually, one only need to prove the associativity condition as the differential is given by the commutator (with respect to the product $\star$) $[m, -]$ with the multiplication m on A. Let us prove the three points for the first orders with respect to the degree:

- Let us check that $a_D^1 + a_D^2$ is a differential. For A, B in D_{poly} one gets:

$$\begin{aligned}(a_D^1 + a_D^2) \circ (a_D^1 + a_D^2)(AB) &= (b + \cup)(bAB \pm AbB + A \cup B) \\ &= b(A \cup B) + bA \cup B \pm A \cup bB = 0.\end{aligned}$$

- Let us check the associativity of $\star = a_D^{1,1} + a_D^{1,2} + \cdots$ up to order 2. For A, B, C in D_{poly}, one gets (here we forget the signs):

$$\begin{aligned}(A \star B) \star C &= (AB + BA + \{A, B\}) \star C \\ &= \{A, C\}B + A\{B, C\} + \{B, C\}A + B\{A, C\} \\ &\quad + \{A, B\}C + C\{A, B\} + \{\{A, B\}, C\} \\ &\quad + ABC \text{ (and other permutations in } ABC) \\ A \star (B \star C) &= A \star (BC + CB + \{B, C\}) \\ &= \{A, B\}C + B\{A, C\} + \{A, C\}B + C\{A, B\} + a_D^{1,2}(A, BC) \\ &\quad + a_D^{1,2}(A, CB) + A\{B, C\} + \{B, C\}A + \{A, \{B, C\}\} \\ &\quad + ABC \text{ (and other permutations in } ABC),\end{aligned}$$

and the result follows from

$$\{\{A, B\}, C\} = a_D^{1,2}(A, BC) + a_D^{1,2}(A, CB) + \{A, \{B, C\}\}.$$

- Let us check the compatibility condition between $\star = a_D^{1,1} + a_D^{1,2} + \cdots$ and the differential $d = a_D^1 + a_D^2$ up to order 2. For A, B in D_{poly}, one gets (here again we forget the signs):

$$\begin{aligned} d(A \star B) &= (b + \cup)(AB + BA + \{A, B\}) \\ &= bAB + AbB + bBA + BbA + A \cup B + B \cup A + b\{A, B\}, \\ dA \star B + A \star dB &= bAB + BbA + AbB + bBA + \{bA, B\} + \{A, bB\}, \end{aligned}$$

and the result follows from

$$A \cup B + B \cup A = \{bA, B\} + \{A, bB\} - b\{A, B\}.$$

Using this result, we can successively apply Proposition 3.3 and Lemma 3.2 to obtain the desired G_∞-structure on D_{poly} given by maps $d_D^{p_1,\ldots,p_k}$ such that $d_D^1 = b$, $d_D^{1,1} = [-,-]_G$ and $d_D^2 = \cup$ "up to homotopy" (i.e., up to a coboundary). Moreover, one remembers that maps $d_D^{p_1,\ldots,p_k}$ are 0 for $k > 2$.

3.2. The quantization/dequantization fonctor

Let us recall the definition of a Drinfeld associator (*cf* [4]):

Let T_n be the algebra generated by elements t_{ij}, $1 \le i, j \le n$, $i \ne j$, with defining relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{lm}] = 0$ for i, j, l, m distincts and $[t_{ij}, t_{ik} + t_{jk}] = 0$ for i, j, k distincts. Let $P_1, \ldots, P_n$ be disjoint subsets of $\{1, \ldots, m\}$. There exists a unique homomorphism $\rho_{P_1,\ldots,P_n}\colon T_n \to T_m$ defined by

$$\rho_{P_1,\ldots,P_n}(t_{ij}) = \sum_{p \in P_i, q \in P_j} t_{pq}.$$

For any $X \in T_n$, we denote $\rho_{P_1,\ldots,P_n}(X)$ by $X^{P_1,\ldots,P_n}$. Let $\Phi \in T_3$. The relation

$$\Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3}$$

in $T_4[[\hbar]]$ is called the pentagon relation. Let $B = e^{\hbar t_{12}/2} \in T_2[[\hbar]]$. The relations

$$\begin{aligned} B^{12,3} &= \Phi^{3,1,2} B^{1,3} (\Phi^{1,3,2})^{-1} B^{2,3} \Phi^{1,2,3}, \\ B^{1,23} &= (\Phi^{2,3,1})^{-1} B^{1,3} \Phi^{2,1,3} B^{1,2} (\Phi^{1,2,3})^{-1} \end{aligned}$$

in $T_3[[\hbar]]$ are called the hexagon relations.

An element $\Phi \in T_3$ satisfying the pentagon and hexagon relations is called a Drinfeld associator. Such associators exist over $\mathbb{C}$ ([3]). They are obtained from the KZ equations. Drinfeld also prove that such associators exist over $\mathbb{Q}$.

In this subsection we recall the following theorem (see appendix in [11]) which gives, as a consequence, Proposition 3.4 (here is where an associator Φ is used):

Theorem 3.6. *There exists an equivalence of categories*

$$DQ_\Phi : \text{ DGQUE} \to \text{DGLBA}_\hbar$$

from the category of differential graded quantized universal enveloping graded algebras to that of differential graded Lie graded bialgebras such that if $U \in \mathrm{Ob}(\text{DGQUE})$ *and* $\mathfrak{a} = DQ_\Phi(U)$*, then* $U/\hbar U = \mathbb{U}(\mathfrak{a}/\hbar\mathfrak{a})$*, where* $\mathbb{U}$ *is the universal algebra functor, taking a differential graded Lie graded algebra to a differential graded graded Hopf algebra.*

This theorem is a consequence of the Etingof-Kazhdan quantization theorems. The key point is that the quantization theorem is "universal" and so will be valid for any symmetric category and so for complexes $(V^\cdot, d^\cdot)$. A right way to understand the "universality" is to use the language of operads and props. We will not recall the definitions in this paper.

Let us outline the construction of the quantization functor starting with an associator Φ. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Let $\mathfrak{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ be its associated Lie bialgebra double. Let $r \in \mathfrak{g} \otimes \mathfrak{g}^* \in \mathfrak{D}^{\otimes 2}$ be the canonical r-matrix (corresponding to the identity map) and $t = r + r^{2,1} \in S^2(\mathfrak{D})^{\mathfrak{D}}$. Let us consider the homomorphism $T_n \to U(\mathfrak{D})^{\otimes n}$ sending t_{ij} to $t^{i,j}$ (where components of t are put in the i-th and j-th place in the tensor product). We will still denote by Φ the image of Φ by this homomorphism.

We will use the standard notation for the coproduct-insertion maps: we say that an ordered set is a pair of a finite set S and a bijection $\{1, \dots, |S|\} \to S$. For $I_1, \dots, I_m$ disjoint ordered subsets of $\{1, \dots, n\}$, (U, Δ) a Hopf algebra and $a \in U^{\otimes m}$, we define

$$a^{I_1,\dots,I_n} = \sigma_{I_1,\dots,I_m} \circ (\Delta^{(|I_1|)} \otimes \dots \otimes \Delta^{(|I_n|)})(a),$$

with $\Delta^{(1)} = \mathrm{Id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\mathrm{Id}^{\otimes n-1} \otimes \Delta) \circ \Delta^{(n)}$, and $\sigma_{I_1,\dots,I_m} : U^{\otimes \sum_i |I_i|} \to U^{\otimes n}$ is the morphism corresponding to the map $\{1, \dots, \sum_i |I_i|\} \to \{1, \dots, n\}$ taking $(1, \dots, |I_1|)$ to I_1, $(|I_1| + 1, \dots, |I_1| +$

$|I_2|$) to I_2, etc. When U is cocommutative, this definition depends only on the sets underlying $I_1, \dots, I_m$.

We get that $(U(\mathfrak{D})[[\hbar]], m_0, \Delta_0, R_0 = e^{\hbar t/2}, \Phi)$ is a quasi-triangular quasi-Hopf algebra ([4]). Quasi-triangular means that

$$\Delta_0^{21}(a) = R\Delta_0(a)R^{-1}$$

for all $a \in U(\mathfrak{D})$ and quasi-Hopf means that the coproduct Δ_0 is quasi-coassociative, that is to say

$$(\mathrm{Id} \otimes \Delta_0)(\Delta_0(a)) = \Phi(\Delta_0 \otimes \mathrm{Id})\Delta_0(a)\Phi^{-1}$$

for all $a \in U(\mathfrak{D})$. To make this quasi-Hopf algebra into a Hopf algebra, one has to twist Φ into the identity, that is to say one has to construct $J \in U(\mathfrak{D})^{\otimes 2}$ such that

$$((1 \otimes J)((\mathrm{Id} \otimes \Delta_0)(J)))^{-1}(J \otimes 1)(\Delta_0 \otimes \mathrm{Id})(J) = (J^{2,3}J^{1,23})^{-1}J^{1,2}J^{12,3} = \Phi. \tag{3.1}$$

Then $(U(\mathfrak{D})[[\hbar]], m_0, \mathrm{Ad}(J) \circ \Delta_0, R = J^{2,1}e^{\hbar t/2}J^{-1})$ is a Hopf algebra. Suppose now we have constructed such a J (actually, we ask J to have also good "polarization" properties), set $H = \{(\xi \otimes \mathrm{Id})R, \xi \in U(\mathfrak{D})^*[[\hbar]]\}$. It is a Hopf subalgebra of $U(\mathfrak{D})[[\hbar]]$. Let $U_\hbar(\mathfrak{g})$ be $\hbar$-adic completion. More precisely, let I be the maximal ideal of H, $U_\hbar(\mathfrak{g})$ is the $\hbar$-adic completion of the subalgebra $\sum_{n \geq 0} \hbar^{-n} I^n$ in $H \otimes_{k[[\hbar]]} k((\hbar))$. It is clear that $U_\hbar(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[\hbar]]$ and so $(U_\hbar(\mathfrak{g}), \mathrm{Ad}(J) \circ \Delta_0)$ is then a quantization of $(\mathfrak{g}, \delta)$. Notice that the product in $U_\hbar(\mathfrak{g})$ is not the same as the one in $U_\hbar(\mathfrak{D})$ (and so the product in $U(\mathfrak{D})$) as the algebra isomorphism $U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})[[\hbar]]$ is not the identity (which itself is not an algebra morphism).

Let us end this subsection showing how one can construct the twist J. In [6], the construction was done using the "categorical yoga" and one gets a general formula:

$$J = (\phi^{-1} \otimes \phi^{-1})\big((\Phi^{1,2,34})^{-1}\phi^{2,3,4} s e^{\hbar t^{2,3}/2}(\phi^{2,3,4})^{-1}\Phi^{1,2,34}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-)\big),$$

where M_+ and M_- are respectively the Verma module $\mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{D}} 1$ and $\mathrm{Ind}_{\mathfrak{g}^*}^{\mathfrak{D}} 1$, 1_+, and 1_- are respectively the generators of those module over $U(\mathfrak{g}^*)$ and $U(\mathfrak{g})$ and ϕ is the isomorphism $U(\mathfrak{D}) \to M_+ \otimes M_-$ generated by the assignment $1 \to 1_+ \otimes 1_-$. Finally, s is the twist in the tensor product. As an exercise, let us calculate the first terms of J. Let $\{a_i\}$ be a basis of $\mathfrak{g}$

and $\{b^i\}$ its dual basis, a basis of $\mathfrak{g}^*$. So $r = \sum a_i \otimes b^i$. Let us write the structure constants:

$$[a_i, b^j] = c_{ij}^k a_k,\ \delta a_k = \sum f_k^{ij} a_i \Lambda a_j$$

and so $[b^i, b^j] = f_k^{ij} b^k$ and $[a_i, b^j] = f_i^{jk} a_k - c_{ik}^j b^k$. Starting from an associator $\Phi = 1 + \frac{\hbar^2}{24}[t_{12}, t_{23}] + O(\hbar^3)$, one gets

$$\begin{aligned} J = 1 &+ \frac{\hbar}{2} r + \hbar^2 \big(\frac{1}{4}(a_j a_i \otimes b^j b^i + f_i^{jk} c_{jl}^i a_k \otimes b^l) \\ &- \frac{c_{ik}^j}{12} b^i a_j \otimes b^k - \frac{c_{ik}^j}{24} b^i \otimes b^k a_j - \frac{f_i^{jk}}{12} a_k \otimes b^i a_j - \frac{f_i^{jk}}{24} b^i a_k \otimes a_j\big). \end{aligned}$$

To get universal formulas, one has then to reorder the terms in J.

In [5], Enriquez proposed a cohomological construction of the twist J. He looks for this element in a "universal" algebra U_{univ} made from the r-matrix. The definition is rather complicated and uses the language of props. We will retain that it is generated by the components of r, *i.e.* words in $\{a_i\}$ and $\{b^j\}$ (with as many a's as b's) and the relations (the r-matrix relations):

$$aba'b' = aa'bb' + aa'[b', b] + [a', a]bb'.$$

This allows to write all the a's on the left hand side and all the b's in the right hand side. In the same spirit one can define

$$U_{\text{univ}}^{\otimes n} = \oplus_{N \geq 0} \big(((\mathcal{FA}_N)^{\otimes n})_{\Sigma_i \delta_i} \otimes ((\mathcal{FA}_N)^{\otimes n})_{\Sigma_i \delta_i} \big)_{\sigma_N}$$

where $\mathcal{FA}_N$ is the free algebra with generators x_i, $i = 1, \ldots, N$, graded by $\oplus_i \mathbb{N}\delta_i$ (x_i has degree i). We view Φ as an element of $U_{\text{univ}}^{\otimes 3}$ and we will build $J = 1 + \hbar\frac{r}{2} + \cdots \in U_{\text{univ}}^{\otimes 2}$ such that equation (3.1) is fulfiled in $U_{\text{univ}}^{\otimes 3}$. The construction is made by induction. Suppose we have built $J = 1 + \cdots + \hbar^n J_n + \cdots$ up to order $n-1$. Equation (3.1) at order n is equivalent to

$$d_2^{\text{coHo}} J_n = \Phi_n + \langle J_1, \ldots, J_{n-1} \rangle$$

where Φ_n is the $\hbar^n$ component of Φ, $\langle J_1, \ldots, J_{n-1} \rangle$ is an expression involving only component J_k, $k \leq n-1$, and $d_n^{\text{coHo}}\colon U_{\text{univ}}^{\otimes n} \to U_{\text{univ}}^{\otimes n+1}$ is the coHochschild differential:

$$d_n^{\mathrm{coHo}}(j) = j^{12,3,\dots,n+1} - j^{1,23,\dots,n+1} + (-1)^{n+1} j^{1,2,\dots,nn+1} + j^{2,\dots,n,n+1} + (-1)^n j^{1,2,\dots,n}.$$

It is well known that $\ker d_n^{\mathrm{coHo}} = \mathrm{Im}\, d_{n-1}^{\mathrm{coHo}} \oplus \Lambda^n(\mathfrak{D}_{\mathrm{univ}})$ (this is true for any enveloping algebra). For any choice of J_k, $k \le n-1$, $\Phi_n + \langle J_1, \dots, J_{n-1}\rangle$ is in $\mathrm{Ker}\, d_2^{\mathrm{coHo}}$. Moreover, one can always replace J_{n-1} with $J_{n-1} + \lambda_{n-1}$ ($\lambda_{n-1} \in \Lambda^2(\mathfrak{D}_{\mathrm{univ}})$) so that we still have a solution up to order $n-1$. The equation we want to solve now is the following equation with unknown (J_n, λ_{n-1}):

$$d_2^{\mathrm{coHo}} J_n = \Phi_n + \langle J_1, \dots, J_{n-1} + \lambda_{n-1}\rangle = C_n + f(\lambda_{n-1}),$$

where $f\colon \Lambda^2(\mathfrak{D}_{\mathrm{univ}}) \to U_{\mathrm{univ}}^{\otimes 3}$, $\lambda_{n-1} \mapsto \langle J_1, \dots, J_{n-1}+\lambda_{n-1}\rangle - \langle J_1, \dots, J_{n-1}\rangle$, and $d_3^{\mathrm{coHo}} C_n = 0$, so $C_n = d_2^{\mathrm{coHo}} K_n + \mu_n$, with $\mu_n \in \Lambda^3(\mathfrak{D}_{\mathrm{univ}})$.
One has $d_3^{\mathrm{coHo}}(f(\lambda_{n-1})) = 0$ so $f(\lambda_{n-1}) = d_2^{\mathrm{coHo}}\ f'(\lambda_{n-1}) + \mathrm{Alt}(f(\lambda_{n-1}))$. We get after computation

$$\begin{aligned}\mathrm{Alt}(f(\lambda_{n-1})) = {} & \frac{1}{6}|[r, \lambda_{n-1}]| \\ & (= [r^{1,2}, \lambda_{n-1}^{1,3}] + [r^{1,2}, \lambda_{n-1}^{2,3}] + [r^{1,3}, \lambda_{n-1}^{2,3}] \\ & \quad + [\lambda_{n-1}^{1,2}, r^{1,3}] + [\lambda_{n-1}^{1,2}, r^{2,3}] + [\lambda_{n-1}^{1,3}, r^{2,3}]).\end{aligned}$$

So one wants to solve

$$d_2^{\mathrm{coHo}}(J_n - f'(\lambda_{n-1}) - K_n) = \frac{1}{6}|[r, \lambda_{n-1}]| + \mu_n.$$

Actually, we have a complex, making $|[r, -]|$ into a differential: when $0 \le k \le n-1$, let us define

$$(\mathrm{Id}^{\otimes k} \otimes \mathrm{Id}^{\otimes n-k-1})_{\mathrm{univ}} : (\mathfrak{D}^{\otimes n})_{\mathrm{univ}} \to (\mathfrak{D}^{\otimes k} \otimes \Lambda(\mathfrak{D}) \otimes \mathfrak{D}^{\otimes n-k-1})_{\mathrm{univ}}$$

by

$$a \mapsto [r^{k,k+1}, a^{1,\dots,k,k+2,\dots,n+1} + a^{1,\dots,k-1,k+1,\dots,n+1}].$$

Then we have a complex $((\Lambda^{\cdot}(\mathfrak{D}))_{\mathrm{univ}}, \partial^{\cdot})$ where

$$(\Lambda^k(\mathfrak{D}))_{\mathrm{univ}} \ni x \mapsto \partial^k(x) = \mathrm{Alt}((\partial \otimes \mathrm{Id}^{\otimes k-1})_{\mathrm{univ}}(x)) \in (\Lambda^{k+1}(\mathfrak{D}))_{\mathrm{univ}}.$$

It turns out that the 3-rd cohomology group of that complex is 0 if the "degree" in a's and b's is greater than 3 and is spanned by the class of

$[t^{1,2}, t^{2,3}]$ otherwise. Moreover, one checks that $\mathrm{Alt}((\partial\otimes\mathrm{Id}^{\otimes 2})_{\mathrm{univ}}(\mu_n)) = 0$ so there exists $\lambda_{n-1} \in \Lambda^2(\mathfrak{D}_{\mathrm{univ}})$ such that

$$\mu_n = -\frac{1}{6}\partial^2(\lambda_{n-1}) = -\frac{1}{6}|[r, \lambda_{n-1}]|$$

which gives the induction step and allows us to construct J.

Remark 3.7. Following Enriquez's proof, it seems that the term J_{n-1} in the $\hbar$-series of J is built from terms Φ_{n-1} **and** Φ_n in the $\hbar$-series of Φ, as we had to correct J_{n-1} by λ_{n-1} which seems to be dependent from Φ_n. On the other hand, it is clear, from the Etingof-Kazhdan's formula that their J_{n-1} do not depend from Φ_n. This is not surprising: in Enriquez's construction, the correcting term λ_{n-1} only depends on an anti-symmetrization of Φ_n which is unique (it is an easy check).

4. A G_∞-morphism between chains and tensor fields

4.1. A differential d'_T on $\Lambda^\cdot \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$ and G_∞-morphism $\psi : (\Lambda^\cdot \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}, d'_T) \to (\Lambda^\cdot \underline{D_{\mathrm{poly}}{}^{\otimes\cdot}}, d_D)$

The objective of this section is to prove the following proposition:

Proposition 4.1. *There exist a differential (and coderivation) d'_T on $\Lambda^\cdot \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$ and a morphism of differential coalgebras ψ: $(\Lambda^\cdot \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}, d'_T) \to (\Lambda^\cdot \underline{D_{\mathrm{poly}}{}^{\otimes\cdot}}, d_D)$ such that the induced map $\psi^1 : T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ is the Hochschild-Kostant-Rosenberg map φ^1 of Section 0.*

Proof. For $i = T$ or D and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1+\cdots+p_k=n} \underline{\mathfrak{g}_i^{\otimes p_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k\leq n} V_i^{[k]}$. Let $d_D^{p_1,\ldots,p_k}$: $\underline{D_{\mathrm{poly}}{}^{\otimes p_1}}\Lambda\cdots\Lambda\underline{D_{\mathrm{poly}}{}^{\otimes p_k}} \to D_{\mathrm{poly}}$ be the components of the differential d_D defining the G_∞-structure of D_{poly} (see Definition 2.6) and denote $d_D^{[n]}$ and $d_D^{[\leq n]}$ the sums

$$d_D^{[n]} = \sum_{p_1+\cdots+p_k=n} d_D^{p_1,\ldots,p_k} \qquad \text{and} \qquad d_D^{[\leq n]} = \sum_{p\leq n} d_D^{[p]}.$$

Clearly, $d_D = \sum_{n\geq 1} d_D^{[n]}$. In the same way, we define $d'^{[n]}_T$ and $d'^{[\leq n]}_T$. We know from Section 2 that a morphism ψ: $(\Lambda^\cdot \underline{T_{\text{poly}}{}^{\otimes\cdot}}, d'_T) \to (\Lambda^\cdot \underline{D_{\text{poly}}{}^{\otimes\cdot}}, d_D)$ is uniquely determined by its components

$$\psi^{p_1,\dots,p_k} : \underline{T_{\text{poly}}{}^{\otimes p_1}}\Lambda\cdots\Lambda\underline{T_{\text{poly}}{}^{\otimes p_k}} \to D_{\text{poly}}\,.$$

Similarly we set

$$\psi = \sum_{n\geq 1}\psi^{[n]} = \sum_{n\geq 1}\sum_{p_1+\cdots+p_k=n}\psi^{p_1,\dots,p_k} \qquad \text{and} \qquad \psi^{[\leq n]} = \sum_{1\leq k\leq n}\psi^{[k]}.$$

We have to build both the differential d'_T and ψ, the morphism of differential. In fact we will build the maps $d'^{[n]}_T$ and $\psi^{[n]}$ by induction. For the first terms, we set

$$d'^{[1]}_T = 0 \qquad \text{and} \qquad \psi^{[1]} = \varphi^1 \text{ (the H.-K.-R. map)}.$$

Suppose we have built maps $(d'^{[i]}_T)_{i\leq n-1}$ and $(\psi^{[i]})_{i\leq n-1}$ satisfying

$$\psi^{[\leq n-1]}\circ d'^{[\leq n-1]}_T = d_D^{[\leq n-1]}\circ\ \psi^{[\leq n-1]}$$

on $V_T^{[\leq n-1]}$ and $d'^{[\leq n-1]}_T \circ d'^{[\leq n-1]}_T = 0$ on $V_T^{[\leq n]}$. These conditions are enough to insure that d'_T is a differential and ψ a morphism of differential coalgebras. If we reformulate the identity $\psi\circ d'_T = d_D\circ\psi$ on $V_T^{[n]}$, we get

$$\psi^{[\leq n]}\circ d'^{[\leq n]}_T = d_D^{[\leq n]}\circ\psi^{[\leq n]}. \tag{4.1}$$

If we take now into account that $d'^{[1]}_T = 0$, and that on $V_T^{[n]}$ we have $\psi^{[k]}\circ d'^{[l]}_T = d_D^{[k]}\circ\psi^{[l]} = 0$ for $k+l > n+1$, the identity (4.1) becomes

$$\psi^{[1]}d'^{[n]}_T + B = d_D^{[1]}\psi^{[n]} + A \tag{4.2}$$

where $B = \sum_{k=2}^{n-1}\psi^{[\leq n-k+1]}d'^{[k]}_T$ and $A = d_D^{[1]}\psi^{[\leq n-1]} + \sum_{k=2}^{n} d_D^{[k]}\psi^{[\leq n-k+1]}$ (we now omit the composition sign $\circ$). The term $d_D^{[1]}$ in (4.2) is the Hochschild coboundary b. So thanks to the H.-K.-R. theorem identity (4.2) is equivalent to the cochain $B-A$ being a Hochschild cocycle *i.e.* that $d_D^{[1]}(B-A) = 0$ which is true by direct computation (see [11]). We also have to show that for any choice of those maps, we have

$$d'^{[\leq n]}_T d'^{[\leq n]}_T = 0 \text{ on } V_T^{[\leq n+1]}. \tag{4.3}$$

Again this is always true by direct computation (see again [11]). □

As an example, let us construct ${d'_T}^{[2]}$: for $n = 2$, we get $A = d_D^{[1]}\psi^{[1]} + d_D^{[2]}\psi^{[1]}$ and $B = 0$ so that

$$\psi^{[1]}{d'}_T^{[2]} = d_D^{[1]}(\psi^{[2]} + \psi^{[1]}) + d_D^{[2]}\psi^{[1]}.$$

Thus ${d'_T}^{[2]}$ is the image of $d_D^{[2]}$ through the projection on the cohomology of $D_{\rm poly}$ and as the Hochschild-Kostant-Rosenberg map $\psi^{[1]}$ is injective from $T_{\rm poly}{=}H(D_{\rm poly},\, b = d_D^{[1]})$ to $D_{\rm poly}$, we get

$${d'_T}^{[2]} = d_T^{[2]}.$$

Remark 4.2. The main tool we have used here is the existence of a quasi-isomorphism between the complexes $(T_{\rm poly}, 0)$ and $(D_{\rm poly}, b)$. Since we know explicit homotopy formulas for such a quasi-isomorphism (see [21], [13]), we can obtain explicit formulas for ${d'_T}^{[k]}$ and $\psi^{[k]}$.

4.2. A G_∞-morphism ψ': $(\Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}, d_T) \to (\Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}, d'_T)$

In this subsection, we will prove the following proposition.

Proposition 4.3. *If the complex* $\Big(\mathrm{Hom}(\Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}, \Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}), [d_T^{1,1}{+}d_T^2, -]\Big)$ *is acyclic, then there exists a* G_∞*-morphism* $\psi' \colon (\Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}, d_T) \to (\Lambda^\cdot \underline{{T_{\rm poly}}^{\otimes^\cdot}}, d'_T)$ *such that the induced map* ${\psi'}^{[1]} : T_{\rm poly} \to T_{\rm poly}$ *is the identity.*

We will use the same notations for $V_T^{[n]}$, $V_T^{[\le n]}$, ${d'}_T^{[n]}$ and ${d'}_T^{[\le n]}$ and we also denote $d_T = \sum_{n\ge 1} d_T^{[n]}$, $d_T^{[\le n]} = \sum_{1\le k\le n} d_T^{[k]}$, $\psi' = \sum_{n\ge 1}\psi^{[n]}$ and ${\psi'}^{[\le n]} = \sum_{1\le k\le n} {\psi'}^{[n]}$.

Proof. We will build the maps ${\psi'}^{[n]}$ by induction as before. For ${\psi'}^{[1]}$ we have to set:

$${\psi'}^{[1]} = \mathrm{Id} \ \ \text{(the identity map)}.$$

Suppose we have built maps $({\psi'}^{[i]})_{i\le n-1}$ satisfying ${\psi'}^{[\le n-1]} \circ d_T^{[\le n]} = {d'_T}^{[\le n]} \circ {\psi'}^{[\le n-1]}$ on $V_T^{[\le n]}$ (${d'_T}^{[\le n]}$ maps $V_T^{[\le l]}$ to $V_T^{[\le l-1]}$). Making explicit the equation $\psi' d_T = d'_T \psi'$ on $V_T^{[n+1]}$, we get

$${\psi'}^{[\le n]}\, d_T^{[\le n+1]} = {d'_T}^{[\le n+1]}\, {\psi'}^{[\le n]}. \tag{4.4}$$

If we now take into account that $d_T^{[i]} = 0$ for $i \neq 2$, ${d'_T}^{[1]} = 0$ and that on $V_T^{[n+1]}$ we have ${\psi'}^{[k]} d_T^{[l]} = {d'_T}^{[\leq k]} {\psi'}^{[l]} = 0$ for $k + l > n + 2$, the identity (4.4) becomes

$$ {\psi'}^{[\leq n]}\, d_T^{[2]} = \sum_{k=2}^{n+1} {d'_T}^{[k]}\, {\psi'}^{[\leq n-k+2]}. $$

We have seen in the previous section that ${d'_T}^{[2]} = {d_T}^{[2]}$. Thus (4.4) is equivalent to

$$ d_T^{[2]} {\psi'}^{[\leq n]} - {\psi'}^{[\leq n]} {d_T}^{[2]} = \left[{d_T}^{[2]}, {\psi'}^{[\leq n]} \right] = - \sum_{k=3}^{n+1} {d'_T}^{[k]} {\psi'}^{[\leq n-k+2]}. $$

Notice that $d_T^{[2]} = d_T^{1,1} + d_T^2$. By the acyclicity of the complex $(\mathrm{End}(\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}), [d_T^{[2]}, -])$, the construction of ${\psi'}^{[\leq n]}$ will be possible when $\sum_{k=3}^{n+1} {d'_T}^{[k]} {\psi'}^{[\leq n-k+2]}$ is a cocycle in this complex, which is true by direct computation (see [11]) $\square$

4.3. Acyclicity of the complex $\left(\mathrm{Hom}(\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}, \Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}), [d_T^{1,1} + d_T^2, -] \right)$

In this section the manifold M is supposed to be the Euclidian space $\mathbb{R}^d$ for $m \geq 1$. We prove the following proposition:

Proposition 4.4. *If $M = \mathbb{R}^d$, the cochain complex $(\mathrm{End}(\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}), [d_T^{1,1} + d_T^2, -])$ is acyclic.*

Proof. Since morphism of coalgebras $\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot} \to \Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}$ are in one to one correspondence with maps $\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot} \to T_{\mathrm{poly}}$, we are left to check that the cochain complex

$$ \left(\mathrm{Hom}(\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}, T_{\mathrm{poly}}), [d_T^{1,1} + d_T^2, -] \right) $$

is acyclic. Firstly, we introduce an "external" bigrading on the cochain complex induced by the following bigrading on $\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}$: if $x \in \underline{T_{\mathrm{poly}}}^{\otimes p_1} \Lambda \cdots \Lambda \underline{T_{\mathrm{poly}}}^{\otimes p_n}$, $|x|^{\mathrm{e}} = (p_1 - 1 + \cdots + p_n - 1, n - 1)$. This grading gives a bicomplex structure on the vectorial space $\left(\mathrm{Hom}(\Lambda^{\cdot} \underline{T_{\mathrm{poly}}}^{\otimes \cdot}, T_{\mathrm{poly}}), [d_T^{1,1} + d_T^2, -] \right)$ for which $d_T^{1,1} = [-,-]_S$ is of bidegree $(0,1)$ and $d_T^2 = \wedge$ is of bidegree $(1,0)$. We will first show that the complex

$$\left(\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}), [[-,-]_S + \wedge, -]\right)$$

is concentrated in bidegree $(0,0)$ if the complex

$$\left(\mathrm{Hom}_{T_{\mathrm{poly}}}(\Lambda^{\cdot}_{T_{\mathrm{poly}}}\Omega_{T_{\mathrm{poly}}}, T_{\mathrm{poly}}), d_{\mathrm{CE}}\right),$$

is concentrated in degree 0, where $d_{\mathrm{CE}} + d_H$ is the dual map of $[d_T^{1,1} + d_T^2, -] = [[-,-]_S + \wedge, -]$ (d_{CE} is the Chevalley-Eilenberg differential and d_{H} is the Harrison differential) and $\Omega_{T_{\mathrm{poly}}}$ is the module of 1-differential Kähler form of the algebra T_{poly}. We will then show that this complex is acyclic.

The exterior product d_T^2 makes T_{poly} into an associative algebra and so for any vector space V, the space $T_{\mathrm{poly}} \otimes V$ is a T_{poly}-module equipped with a T_{poly}-action by multiplication on the first factor. Observe that

$$\begin{aligned}
&\left(\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}, T_{\mathrm{poly}}), [d_T^{1,1} + d_T^2, -]\right)\\
&\quad\cong \left(\mathrm{Hom}_{T_{\mathrm{poly}}}(T_{\mathrm{poly}} \otimes \Lambda^{\cdot}\underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}, T_{\mathrm{poly}}), [d_T^{1,1} + d_T^2, -]\right),\\
&\quad\cong \left(\mathrm{Hom}_{T_{\mathrm{poly}}}(\Lambda^{\cdot}_{T_{\mathrm{poly}}}(T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}), T_{\mathrm{poly}}), [d_T^{1,1} + d_T^2, -]\right)
\end{aligned}$$

where T_{poly} acts (on the right and on the left) on itself by the multiplication d_T^2. The induce differential $[d_T^2, -]$ on this complex is the dual of a differential on $\Lambda^{\cdot}_{T_{\mathrm{poly}}} T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$ which is the Harrison differential d_{H} on each factor $T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$ (*i.e.* the image of the Hochschild differential d acting on $T_{\mathrm{poly}}{}^{\otimes\cdot+1}$ onto its quotient $T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$ by the shuffles). Indeed, for $\chi : \Lambda^{\cdot}_{T_{\mathrm{poly}}} T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}} \to T_{\mathrm{poly}}$ and $\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \in \Lambda^{\cdot}_{T_{\mathrm{poly}}} T_{\mathrm{poly}} \otimes \underline{T_{\mathrm{poly}}{}^{\otimes\cdot}}$, one has

$$\begin{aligned}
&[d_T^2, \chi](\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n)\\
&\qquad = \pm d_T^2(\gamma_1, \chi(\alpha \otimes \gamma_2 \cdots)) \pm d_T^2(\chi(\alpha \otimes \gamma_1 \cdots), \gamma_n)\\
&\qquad\quad + \sum \pm \chi(\alpha \otimes \gamma_1 \cdots d_T^2(\gamma_i, \gamma_{i+1}) \cdots)\\
&\qquad = \chi(d_T^2(\alpha, \gamma_1) \otimes \gamma_2 \cdots) + \sum \pm \chi(\alpha \otimes \cdots d_T^2(\gamma_i, \gamma_{i+1}) \cdots)\\
&\qquad = \chi(d_{\mathrm{H}}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n)).
\end{aligned}$$

We now use the fact that $(T_{\rm poly}, d_T^2) = (\Gamma(M, \Lambda TM), \wedge)$ is a polynomial algebra to show that

Proposition 4.5. *The cohomology of*

$$\Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\rm poly}}^{\otimes \cdot}, \Lambda^{\cdot}\underline{T_{\rm poly}}^{\otimes \cdot}), [[-,-]_S + \wedge, -]\Big)$$

is the cohomology of the complex $\Big(\mathrm{Hom}_{T_{\rm poly}}(\Lambda^{\cdot}_{T_{\rm poly}}\Omega_{T_{\rm poly}}, T_{\rm poly}), d_{\rm CE}\Big)$ *which sits in the complex*

$$(\mathrm{Hom}(\Lambda^{\cdot}\, T_{\rm poly}, T_{\rm poly}), d_{\rm CE}) \cong \Big(\mathrm{Hom}_{T_{\rm poly}}(T_{\rm poly} \otimes \Lambda^{\cdot}\, T_{\rm poly}, T_{\rm poly}), d_{\rm CE}\Big)\,.$$

In particular, the differential $d_{\rm CE}$ is induced by the usual exterior derivative (see [15]) on $\mathrm{Hom}_{T_{\rm poly}}(T_{\rm poly} \otimes \Lambda^{\cdot}\underline{T_{\rm poly}}^{\otimes \cdot}, T_{\rm poly})$. Proposition 4.5 can be proved using spectral sequences but can also be obtained directly.

Proof. We have explicit quasi-isomorphisms and homotopies between ${T_{\rm poly}}^{\otimes \cdot +1}$ and $\Lambda^{\cdot}\Omega_{T_{\rm poly}}$: $J : {T_{\rm poly}}^{\otimes \cdot +1} \to \Lambda^{\cdot}\Omega_{T_{\rm poly}}$ sending $\gamma_0 \otimes \cdots \otimes \gamma_n$ to $\gamma_0 d\gamma_1 \cdots d\gamma_n$, $I : \Lambda^{\cdot}\Omega_{T_{\rm poly}} \to {T_{\rm poly}}^{\otimes \cdot +1}$, the anti-symmetrization given by

$$J(\gamma_0 d\gamma_1 \cdots d\gamma_n) = \sum_{\varepsilon \in S_n} \frac{\mathrm{sgn}(\varepsilon)}{n!} \gamma_0 \otimes \gamma_{\varepsilon^{-1}(1)} \cdots \otimes \gamma_{\varepsilon^{-1}(n)},$$

and explicit homotopies $s : {T_{\rm poly}}^{\otimes \cdot +1} \to {T_{\rm poly}}^{\otimes \cdot +2}$ described in [13] such that $J \circ I = \mathrm{Id}$ and $I \circ J = \mathrm{Id} + d \circ s + s \circ d$. One can extend those maps to have quasi-isomorphisms and homotopies between $T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot}$ and $\Lambda^{\cdot}\Omega_{T_{\rm poly}}$. Finally, since $\Lambda^{\cdot}_{T_{\rm poly}}\, T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot}$ is a bicomplex with differential $d = d_{\rm CE} + d_{\rm H}$, it follows from [16], Section 3 that there exists a map $u : \Lambda^{\cdot}_{T_{\rm poly}}\Omega_{T_{\rm poly}} \to \Lambda^{\cdot}_{T_{\rm poly}}\, T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot}$ and a (degree one) map $H : \Lambda^{\cdot}_{T_{\rm poly}}\, T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot} \to \Lambda^{\cdot}_{T_{\rm poly}}\, T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot}[1]$ such that $pu = \mathrm{Id}$ and $up = \mathrm{Id} + d\,H + H\,d$ (p is the projection $\Lambda^{\cdot}_{T_{\rm poly}}\, T_{\rm poly} \otimes \underline{T_{\rm poly}}^{\otimes \cdot} \to \Lambda^{\cdot}_{T_{\rm poly}}\Omega_{T_{\rm poly}}$). □

To finish the proof of Proposition 4.4, we proceed as in [22] and [14]. Recall from the introduction that $A = C^\infty(\mathbb{R}^d)$ is the algebra of smooth functions on $\mathbb{R}^d$. Let $\mathrm{Der}(A) = \Omega^*_A$ be the space of smooth derivations on A. Since T_{poly} is a A-module, by transitivity of the space of Kähler differentials for smooth manifolds, one has

$$\Omega_{T_{\text{poly}}} \cong T_{\text{poly}} \otimes_A \Omega_A \oplus \Omega_{T_{\text{poly}}/A}.$$

Since $T_{\text{poly}} \cong \Lambda^*_A \text{Der}(A)$, we find that $\Omega_{T_{\text{poly}}/A} \cong T_{\text{poly}} \otimes \text{Der}(A)$ (with grading shifted by minus one on $\text{Der}(A)$). Hence (see [22].3.5) there is an isomorphism

$$\Big(\text{Hom}_{T_{\text{poly}}}(\Lambda^{\cdot}_{T_{\text{poly}}} \Omega_{T_{\text{poly}}}, T_{\text{poly}}), d_{\text{CE}}\Big) \cong \Big(\Lambda^{1+\cdot} \Omega_{T_{\text{poly}}}, d_{dR}\Big)$$

where d_{dR} is de Rham's differential (the degree on the left hand of the isomorphism is the one induced by the inner degree of T_{poly}). When $T_{\text{poly}} = \Gamma(\mathbb{R}^d, \Lambda\mathbb{R}^d)$ this complex is acyclic. □

Remark 4.6. At every step of this proof, it is possible to construct explicit homotopy formulas. So the coefficients ${\psi'}^{[n]}$ built in this section can be expressed in an explicit way from the G_∞-structure on D_{poly}.

Corollary 4.7. *If $T_{\text{poly}} = \Gamma(\mathbb{R}^d, \Lambda T\mathbb{R}^d)$, then there exists a G_∞-morphism ψ': $(\Lambda^{\cdot}\underline{{T_{\text{poly}}}^{\otimes \cdot}}, d_T) \to (\Lambda^{\cdot}\underline{{T_{\text{poly}}}^{\otimes \cdot}}, d'_T)$ such that the induced map ${\psi'}^{[1]}$: $T_{\text{poly}} \to T_{\text{poly}}$ is the identity.*

Proof. It is an immediate consequence of Propositions 4.3 and 4.4. □

Corollary 4.8. *The composition $\psi \circ \psi'$: $(\Lambda^{\cdot}\underline{{T_{\text{poly}}}^{\otimes \cdot}}, d_T) \to (\Lambda^{\cdot}\underline{{D_{\text{poly}}}^{\otimes \cdot}}, d_D)$ gives the wanted G_∞-morphism between T_{poly} and D_{poly}.*

5. Globalization of the formality maps

5.1. Globalization process

In this section, we recall the process of globalization of formality maps. Globalization was proven by Kontsevich in [18]. Here we will present Dolgushev's approach which uses Fedosov methods. This approach is actually very similar to the one of Kontsevich but maybe more explicit. The idea is to first write formality theorem locally on bundles that can be seen as bundles of the Taylor expansion (in the neighbourhood of the base points) of the considered objects. Let us define those bundles as done in [8] by Fedosov:

- $\mathcal{W} := \hat{S}(T^*M)$ is the bundle of formal fiberwise functions on TM. Local sections are given by formal power series
$$\sum_{l=0}^{\infty} s_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l}$$
where y^i are formal coordinates on the fibers of TM and $s_{i_1\dots i_l}$ are coefficients of a symmetric covariant tensor.

- $\mathcal{T}^{\cdot} := \mathcal{W}\otimes\Lambda^{\cdot+1}TM$ is the graded bundle of formal fiberwise polyvector fields. Local homogeneous sections of degree k are of the form
$$\sum_{l=0}^{\infty} v^{j_0\dots j_k}_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l}\frac{\partial}{\partial y^{j_0}}\Lambda\cdots\Lambda\frac{\partial}{\partial y^{j_k}}$$
where $v^{j_0\dots j_k}_{i_1\dots i_l}$ are coefficients of a tensor with symmetric covariant part (indices $i_1,\dots,i_l$) and antisymmetric contravariant part (indices $j_0, \dots, j_k$).

- $\mathcal{D}^{\cdot} := \mathcal{W}\otimes T^{\cdot+1}(SE)$ is the graded bundle of formal fiberwise polydifferential operators. Local homogeneous sections of degree k look like as follow
$$\sum_{l=0}^{\infty} P^{\alpha_0\dots\alpha_k}_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l}\frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}}\otimes\cdots\otimes\frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}}$$
where α_s are multi-indices, and $P^{\alpha_0\dots\alpha_k}_{i_1\dots i_l}$ are coefficients of a tensor with symmetric covariant part (indicies $i_1,\dots,i_l$) which is also symmetric in indices $\alpha^1_s,\dots,\alpha^d_s$ for any $s=0,\dots,k$.

From now on, and until the end of this section, $\mathcal{B}$ denotes any of these three bundles. For our purpose, we need to tensor $\mathcal{B}$ by the exterior algebra bundle ΛT^*M (in other words we consider differential forms with values in $\mathcal{B}$). These new bundles $\mathbf{B} := \Lambda T^*M\otimes\mathcal{B}$ carrie natural fiberwise algebraic structures; namely

- $\mathbf{W}$ is a bundle of graded commutative algebras with grading given by the exterior degree of forms, which is also filtered (as an algebra) by the polynomial degree in the fibers.

- **T** and **D** are endowed with fiberwise dgla-structures respectively induced by those of T_{poly} and D_{poly}. Grading is given by the sum of the exterior degree and the degree in $\mathcal{B}^{\cdot}$.

In what follows, and when it does not lead to any confusion, we denote the same operations on bundles **B** by the same letters. We also use dual local basis $(e_i)_i$ and $(\xi^i)_i$ of TM and T^*M in order to make explicit computations. Bundles **B** are viewed as graded $\mathcal{O}_M$-modules with grading given by the exterior degree of forms. The nilpotent differential $\delta := \xi^i \frac{\partial}{\partial y^i} : \mathbf{W}^* \to \mathbf{W}^{*+1}$ obviously extends to nilpotent differentials on **T** and **D**. Namely $\delta = [\xi^i \frac{\partial}{\partial y^i}, -]_S$ on **T** and $\delta = [\xi^i \frac{\partial}{\partial y^i}, -]_G$ on **D**. Before giving an explicit description of the cohomology of $(\mathbf{B}, \delta)$ let us remark that δ preserves the grading in $\mathcal{B}$ and decreases the polynomial degree in the fibers (*i.e.* degree in y's). Moreover δ is by definition a derivation of the graded Lie algebras **T** and **D**, and since the multiplication operator $m = 1 \otimes 1$ is δ-closed then δ (anti)commutes with the Hochschild coboundary $b = [m, -]_G$ in **D**. We summarize this by saying that δ is compatible with the dg-structures on **B**.

Proposition 5.1. *For all $n > 0$, $H^n(\mathbf{B}, \delta) = 0$. And $H^0(\mathbf{B}, \delta) = F^0\mathcal{B}$ is the sheaf of sections of $\mathcal{B}$ that are constant in the fibers.*

Proof. Let us introduce the operator $\delta^* = y^i \iota(e_i)$ of contraction with the Euler vector field $\Theta = y^i e_i$. Then we define the homotopy operator κ to be $\frac{1}{k+l}\delta^*$ on k-differential forms with value in $\mathcal{B}$ and l-polynomial in the fibers for $k + l > 0$, and 0 on sections of $\mathcal{B}$ constant in the fibers. Then by a direct computation one obtains

$$u = \delta\kappa u + \kappa\delta u + \mathcal{H}u \qquad (u \in \mathbf{B}) \tag{5.1}$$

where $\mathcal{H}u \in F^0\mathcal{B}$ is the *harmonic* part of u, that is to say its homogeneous part of zero exterior degree and constant in the fibers. □

Suppose now that we have a torsion free connection ∇. Such a connection, which always exists, defines a derivation of **W**, that we denote by the same symbol ∇. Namely, let $\Gamma_{ij}^k(x) :=< \xi^k, \nabla_{e_i} e_j >$ be Christoffel's symbols of ∇, then locally

$$\nabla = d - \xi^i \Gamma_{ij}^k y^j \frac{\partial}{\partial y^k}$$

It obviously extends to derivations of the graded Lie algebras **T** and **D**. Namely

$$\nabla = d - (\xi^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k})\cdot$$

where for any section V of $\mathbf{T} \hookrightarrow \mathbf{D}$, $V \cdot w$ means $V(w)$, $[V, w]_S$ or $[V, w]_G$ when w is a section of $\mathbf{W}, \mathbf{T}$ or $\mathbf{D}$, respectively. Moreover $dm = [\xi^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k}, m]_G = 0$, then $\nabla m = 0$ and thus ∇ (anti)commutes with b in **D**. Since the connection is torsion free one can also show by a direct computation that ∇ and δ (anti)commute.

The standard curvature tensor of ∇ induces an operator $\mathcal{R}$ on **B** which is given locally by

$$\mathcal{R} = -(\frac{1}{2} \xi^i \Lambda \xi^j \mathcal{R}^l_{ijk} y^k \frac{\partial}{\partial y^l})\cdot$$

Then we have $\nabla^2 = \mathcal{R}$ on **B**. Eventhough ∇ is not nilpotent in general, we use it to deform the differential δ on **B**. Namely

Theorem 5.2. *There exists a section A of $T^*M \otimes \mathcal{T}^0 \subset T^*M \otimes \mathcal{D}^0$ with a zero of order two in the fibers such that $\kappa A = 0$ and the derivation $\mathbb{D} := \nabla - \delta + A\cdot$ is nilpotent.*

Proof. Following Fedosov ([8]), one has to solve

$$A = \kappa A + \kappa(\nabla A + \frac{1}{2} A \cdot A).$$

This equation has a unique solution and using Bianchi's identity $\nabla\mathcal{R} = \delta\mathcal{R} = 0$, homotopy property (5.1), $\kappa A = \mathcal{H} A = 0$, and the fact that κ raises the polynomial degree in the fiber one can show that $\mathbb{D}^2 = 0$. □

In what follows we refer to the nilpotent differential $\mathbb{D}$ as the *Fedosov differential.*

The following theorem states that the δ-cohomology described in proposition 5.1 is equal to the cohomology given by Fedosov differential $\mathbb{D}$.

Theorem 5.3. *For all $n > 0$, $H^n(\mathbf{B}, \mathbb{D}) = 0$; and $H^0(\mathbf{B}, \mathbb{D}) = F^0\mathcal{B}$.*

Proof. This follows essentially from a spectral sequence argument. Namely, let us denote by $F^p\mathbf{B}$ the sheaf of homogeneous sections of polynomial degree p in the fibers; then remark that $\mathbb{D}(F^{\geq p+1}\mathbf{B}) \subset F^{\geq p}\mathbf{B}$ and that $\mathbb{D} = -\delta \bmod F^{\geq p+1}\mathbf{B}$. Thus there is a spectral sequence with $E_1^{p,q} \cong H^{p+q}(F^p\mathbf{B}, \delta)$ which converges to $H^*(\mathbf{B}, \mathbb{D})$; then we conclude using proposition 5.1. □

Following [2], one can define explicitly an isomorphism $\vartheta : F^0\mathcal{B} \to Z^0(\mathbf{B}, \mathbb{D})$: it is the linear map that assigns to any section u_0 of $F^0\mathcal{B}$ the unique section u of $\mathcal{B}$ satisfying the equation

$$u = u_0 + \kappa(\nabla u + A \cdot u) \tag{5.2}$$

It is proved in [2] (proof of theorem 3) that this defines a bijective linear map from $F^0\mathcal{B}$ to $Z^0(\mathbf{B}, \mathbb{D})$ with inverse $\mathcal{H}$ ($\mathcal{H} \circ \vartheta = \mathrm{id}$). When $\mathcal{B} = \mathcal{W}$ it is obvious that $\mathcal{H} : Z^0(\mathbf{W}, \mathbb{D}) \to F^0\mathcal{W} = \mathcal{O}_M$ is an isomorphism of commutative algebras. Moreover we get (see [2]):

Proposition 5.4. *$\mathcal{H}_T : Z^0(\mathbf{T}, \mathbb{D}) \to T_{\mathrm{poly}}{}^{\cdot}$ and $\mathcal{H}_D : Z^0(\mathbf{D}, \mathbb{D}) \to D_{\mathrm{poly}}{}^{\cdot}$ are dgla-morphisms.*

Taking the inverse maps, one gets L_∞-morphisms φ_T: $(T_{\mathrm{poly}}, d_T) \to (\mathbf{T}, d_T + \mathbb{D})$ and φ_D: $(D_{\mathrm{poly}}, d_D) \to (\mathbf{D}, d_D + \mathbb{D})$. We will now define a L_∞-morphism $\tilde{\varphi}$: $(\mathbf{T}, d_T + \mathbb{D}) \to (\mathbf{D}, d_D + \mathbb{D})$. We will suppose that the L_∞-morphism φ define in the previous sections satisfies the following conditions:

(1) The L_∞-morphism is local and it can be made equivariant with respect to linear transformations of the coordinates on $\mathbb{R}^d_0$.

(2) For any set of vector fields $(\alpha_i)_{1\leq i\leq 2} \in \Gamma(\mathbb{R}^d_0, T\mathbb{R}^d_0)$,

$$\varphi^{1,1}(\alpha_1 \Lambda \alpha_2) = 0. \tag{5.3}$$

(3) If $n \geq 2$ and $\alpha \in \Gamma(\mathbb{R}^d_0, T\mathbb{R}^d_0)$ is linear in the coordinates on $\mathbb{R}^d_0$, then for any set of multivector fields $\gamma_i \in \Gamma(\mathbb{R}^d_0, \Lambda T\mathbb{R}^d_0)$:

$$\varphi^{1,1,\ldots,1}(\alpha \Lambda \gamma_2 \Lambda \cdots \Lambda \gamma_m) = 0. \tag{5.4}$$

Thanks to the first conditions, it is obvious that such a morphism naturally extends to a morphsim $(\mathbf{T}, d_T) \to (\mathbf{D}, d_D)$. Moreover, it commutes with the differential d. Let us now write $\nabla = d + [B, -]$ and define $\tilde{\varphi}$, the twit of φ by B as follows:

$$\tilde{\varphi}(x_1 \Lambda \cdots \Lambda x_n) = \sum \varphi(x_1 \Lambda \cdots \Lambda x_n \Lambda B \Lambda \cdots \Lambda B).$$

It is a well known fact (see [11] for example) that $\tilde{\varphi}$ is a L_∞-isomorphism from $(\mathbf{T}, d_T + d + [B, -])$ to $(\mathbf{T}, d_T + d + [\sum \varphi(B \Lambda \cdots \Lambda B), -])$. Thanks to the second condition, we get $\sum \varphi(B \Lambda \cdots \Lambda B) = B$. Finally, one can prove (see [2] for example) that the term in B that depends on the choice of the local trivialization is linear in the fiber coordinates so $\tilde{\varphi}$ does not depend on a choice of local coordinate thanks to the third condition. Finally, we have the following diagram:

$$\begin{array}{ccc} (\mathbf{T}, d_T + \mathbb{D}) & \stackrel{\tilde{\varphi}}{\rightarrow} & (\mathbf{D}, d_D + \mathbb{D}) \\ \uparrow \varphi_T & & \downarrow \mathcal{H}_D \\ (T_{\text{poly}}, d_T) & & (D_{\text{poly}}, d_D), \end{array}$$

To end the proof, one has to show that the morphism $\tilde{\varphi} \circ \varphi_T$ can be deformed into a map $T_{\text{poly}} \to Z^0(\mathbf{D}, \mathbb{D}) \simeq D_{\text{poly}}{}^{\cdot}$. This can be done using general arguments on L_∞-isomorphisms or explicitally as in [2]

5.2. Existence of globalizable formality maps

In this part, we will show that one can construct a G_∞-morphism which, when reduced to a L_∞-morphism is globalizable that is to say satisfies the three conditions described in the previous subsection. Here is our main theorem:

Theorem 5.5. *Suppose $M = \mathbb{R}^d$ and we are given a G_∞-structure on D_{poly} given by a differential d_D as in Section 2. One can construct a G_∞-morphism φ: $T_{\text{poly}} \to D_{\text{poly}}$ satisfying the extra conditions:*

(1) *The G_∞-morphism is local (one can replace $\mathbb{R}^d$ by its formal completion $\mathbb{R}^d_0$ at the origin, or in other words, one can replace the functions with their Taylor expansion) and it can be made equivariant with respect to linear transformations of the coordinates on $\mathbb{R}^d_0$.*

(2) *For any set of vector fields $(\alpha_i)_{1 \leq i \leq 2} \in \Gamma(\mathbb{R}^d_0, T\mathbb{R}^d_0)$,*
$$\varphi^{1,1}(\alpha_1 \Lambda \alpha_2) = 0. \tag{5.5}$$

(3) *If $n \geq 2$ and $\alpha \in \Gamma(\mathbb{R}^d_0, T\mathbb{R}^d_0)$ is linear in the coordinates on $\mathbb{R}^d_0$, then for any set of tensor product of multivector fields $\gamma_i \in \underline{\Gamma(\mathbb{R}^d_0, \Lambda T\mathbb{R}^d_0)^{\otimes p_i}}$:*
$$\varphi^{1,p_2,\dots,p_n}(\alpha \Lambda \gamma_2 \Lambda \cdots \Lambda \gamma_m) = 0. \tag{5.6}$$

Corollary 5.6. *The restriction (that we still denote φ) of φ as a L_∞-morphism*
$$\varphi : (T_{\text{poly}}, [-,-]_S) \to (D_{\text{poly}}, [-,-]_G + b)$$

satisfies the conditions:

(1) *The L_∞-morphism is local and it can be made equivariant with respect to linear transformations of the coordinates on $\mathbb{R}_0^d$.*

(2) *For any set of vector fields $(\alpha_i)_{1\leq i\leq 2} \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$,*

$$\varphi^{1,1}(\alpha_1 \Lambda \alpha_2) = 0. \tag{5.7}$$

(3) *If $n \geq 2$ and $\alpha \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$ is linear in the coordinates on $\mathbb{R}_0^d$, then for any set of multivector fields $\gamma_i \in \Gamma(\mathbb{R}_0^d, \Lambda T\mathbb{R}_0^d)$:*

$$\varphi^{1,1,\ldots,1}(\alpha \Lambda \gamma_2 \Lambda \cdots \Lambda \gamma_m) = 0. \tag{5.8}$$

Those are exactly the conditions written in [19] *and* [2] *for globalization. So one can build a global L_∞-morphism using Tamarkin's methods.*

Proof. Let us first prove the following lemma:

Lemma 5.7. *The map $d_D^{1,p}$ satisfies $d_D^{1,p}(\alpha, \gamma_1 \cdots \gamma_p) = 0$ for $p > 1$ and any linear vector field α.*

Proof. By construction, the maps $d_D^{p,q}$ are invariant under the action of linear vector fields and even quadratic functions and constant 2-vector fields. In other words, those maps are invariant under the action of $\mathfrak{g}l_{d,d}$. Let us prove the lemma by induction on p. Supposoe the result is true for $p > 1$. Let us write $\gamma = \gamma_1 \cdots \gamma_{p+1}$. For $\alpha \in \mathfrak{g}l_{d,d}$, let us write $\alpha\cdot$ for the action of α. Then invariance under the action of $\mathfrak{g}l_{d,d}$ implies that, for any $\alpha, \beta \in \mathfrak{g}l_{d,d}$, one has

$$\beta \cdot d_D^{1,p+1}(\alpha, \gamma) = d_D^{1,p+1}(\beta \cdot \alpha, \gamma) + d_D^{1,p+1}(\alpha, \beta \cdot \gamma).$$

Let us now write the Jacoby identity for β, α and γ. Using the induction hypothesis, we get:

$$\begin{aligned} d_D^{1,1}(\beta, d_D^{1,p+1}(\alpha,\gamma)) + d_D^{1,p+1}(\beta, d_D^{1,1}(\alpha,\gamma)) &= d_D^{1,p+1}(d_D^{1,1}(\beta,\alpha),\gamma) \\ &+ d_D^{1,p+1}(\alpha, d_D^{1,1}(\beta,\gamma) + d_D^{1,1}(\alpha, d_D^{1,p+1}(\beta,\gamma)) \end{aligned}$$

As $d_D^{1,1}(\beta, -) = \beta\cdot$ for any vector fields β, we get

$$d_D^{1,p+1}(\beta, \alpha \cdot \gamma)) = \alpha \cdot d_D^{1,p+1}(\beta, \gamma)),$$

and so

$$d_D^{1,p+1}(\beta \cdot \alpha, \gamma) = 0$$

for any linear $\alpha, \beta \in \mathfrak{g}l_{d,d}$. Thus $d_D^{1,p+1}(\alpha, \gamma) = 0$ for any linear vector fields. $\square$

The theorem will now follow if we prove that points 2 and 3 of Tamarkin's construction are still true with ψ and ψ' satisfying the extra conditions of Theorem 5.5 and d'_T satisfying conditions (5.6) for $n \geq 3$ or $n = 2$ and $p_2 > 1$.

• We want first to construct the maps ${d'}_T^{[n]}$ and $\psi^{[n]}$ by induction with the initial condition

$$ {d'}_T^{[1]} = 0 \qquad \text{and} \qquad \psi^{[1]} = \varphi^1 \text{ (the H.-K.-R. map).} $$

Note that φ^1 satisfies the first conditions of Theorem 5.5.
Now suppose the construction is done for $n-1$ $(n \geq 2)$, i.e., we have built maps $({d'}_T^{[i]})_{i \leq n-1}$ and $(\psi^{[i]})_{i \leq n-1}$ satisfying the extra conditions of Theorem 5.5 and

$$ \psi^{[\leq n-1]} \circ {d'}_T^{[\leq n-1]} = d_D^{[\leq n-1]} \circ \ \psi^{[\leq n-1]} \text{ on } V_T^{[\leq n-1]} $$
$$ \text{and} \qquad {d'}_T^{[\leq n-1]} \circ {d'}_T^{[\leq n-1]} = 0 \text{ on } V_T^{[\leq n]}. \tag{5.9} $$

We have proved that for any such $({d'}_T^{[i]})_{i \leq n-1}$ and $(\psi^{[i]})_{i \leq n-1}$, one can construct ${d'}_T^{[n]}$ and $\psi^{[n]}$ such that condition (5.9) is true for n instead of $n-1$, as this last statement is equivalent to $\varphi^1 {d'_T}^{[n]} = b\psi^{[n]} + A$ where A is always a Hochschild cocycle.
- It is obvious (use homotopy formulas of [13]) that the first condition in Theorem 5.5 can then be satisfied for those maps ${d'_T}^{[n]}$ and $\psi^{[n]}$.
- Using Equation (5.9), condition (5.5) is equivalent to:

$$ \varphi^1([\alpha, \beta]_S) = [\varphi^1(\alpha), \varphi^1(\beta)]_G, $$

for any set of vector fields $\alpha, \beta \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$, which is true.
- Let us check conditions (5.6) for $d_T^{[n]}$ and $\psi^{[n]}$ when they are supposed to be true by induction for $k \leq n-1$. Using the induction hypothesis in Equation (5.9) and the fact that $d_D^{p_1,\ldots,p_n} = 0$ for $n > 2$ and $d_D^{1,p}(\alpha, \gamma_1 \cdots \gamma_p) = 0$ for $p > 1$ and any linear vector field α, one can see that those conditions are equivalent to

$$ [X, \psi^{[n-1]}(\cdots \Lambda \underline{x_i^1 \otimes \cdots \otimes x_i^{p_i}} \Lambda \cdots)]_G $$
$$ = \sum \pm \psi^{[n-1]}(\cdots \Lambda \underline{\cdots \otimes [X, x_i^{n_{ij}}]_S \otimes \cdots} \Lambda \cdots), \tag{5.10} $$

where X is a linear vector field and $x_i^{n_{ij}}$ are tensor fields, which is exactly the equivariance with respect to linear transformations of the coordinates on $\mathbb{R}_0^d$ and was already proved.
So one can construct $d_T^{[n]}$ and $\psi^{[n]}$ satisfying the conditions of Theorem 5.5.

• Let us now construct ψ' by induction. Suppose the construction is done for $n-1$, *i.e.* we have built maps $({\psi'}^{[i]})_{i\leq n-1}$ satisfying the extra conditions of Theorem 5.5 and

$$ {\psi'}^{[1]} = \mathrm{Id}, \qquad {\psi'}^{[\leq n-1]} d_T^{[\leq n]} = {d'_T}^{[\leq n]} {\psi'}^{[\leq n-1]} \tag{5.11} $$

on $V_T^{[\leq n-1]}$. Again, we proved that one can construct ${\psi'}^{[n]}$ such that condition (5.11) is true for n instead of $n-1$: this is equivalent to

$$ [d^{[2]}, {\psi'}^{[\leq n]}] = -\sum_{k=3}^{n+1} {d'_T}^{[k]} {\psi'}^{[n-k+2]} $$

where the complex $\Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}), [d^{[2]}, -]\Big)$ is acyclic and the right hand side is a cocycle in this complex. Let $\widetilde{\mathrm{Hom}}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}; \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot})$ be the subspace of $\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}; \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot})$ consisting of maps satisfying conditions of Theorem 5.5. It is clear from what we have done before that the right hand side of the previous equation is a cocycle in that complex.

Let us prove the acyclicity of $\Big(\widetilde{\mathrm{Hom}}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}), [d^{[2]}, -]\Big)$ (subcomplex of the acyclic complex

$$ \Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}), [d^{[2]}, -]\Big) : \mathrm{Hom}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot})) $$

can be seen as a subcomplex H of an extended complex $\widehat{H}$ where we do admit 1 on the left hand side. Both $\widehat{H}$ and H are acyclic (elements of H consist of all elements which are given by polydifferiential expressions and whose projection gives a polyvector field whose 0-ary component is a function vanishing at 0). Note now that H is a $\mathfrak{gl}_d[\epsilon]$-module, where $\mathfrak{gl}_d[\epsilon] = \mathfrak{gl}_d \oplus \mathfrak{gl}_d \cdot \epsilon$, $|\epsilon| = -1$, the differential is $\partial/\partial\epsilon$ and operations on H are given by maps L_X and i_X, respectively the natural action and the contraction by vector fields $X \in \mathfrak{gl}_d$.
The complex $\widetilde{\mathrm{Hom}}(\Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot}, \Lambda^{\cdot}\underline{T_{\mathrm{poly}}}^{\otimes\cdot})$ can be seen as a subcomplex $H' \subset H$ consisting of all $\mathfrak{gl}_d$-equivariant polyvector fields whose 0-ary component vanishes at 0 (and therefore vanishes itself), *i.e.* $U \in H$ is in

$H' \Leftrightarrow i_X U = L_X U = 0$. It suffices now to show H' is acyclic which is true because so is H and H' is quasi-isomorphic to the relative cochain complex $C^*(\mathfrak{gl}_d[\epsilon], \mathfrak{gl}_d; H)$.
To prove this quasi-isomorphism, split $\mathfrak{gl}_d$-equivariantly $T_{\mathrm{poly}} = \mathfrak{gl}_d \oplus h$; this induces an isomorphism of $\mathfrak{gl}_d[\epsilon]$-modules $H \cong \prod_i \hom(\Lambda^i \mathfrak{gl}_d, H')$. Let us discuss the differential on the right hand side of this formula corresponding to that on H under our identification. Let F be the filtration of H' given by $F^k H' = H' \cap F^k H$, where in turn, $F^k H$ consists of all elements which vanish on $\underline{T_{\mathrm{poly}}}^{\otimes p_1} \Lambda \cdots \Lambda \underline{T_{\mathrm{poly}}}^{\otimes p_i}$ as long as $p_1 + \cdots + p_i < k$. The differential is induced by that in $C^*(\mathfrak{gl}_d, H') \cong \prod_i \hom(\Lambda^i \mathfrak{gl}_d, H')$ modulo a term which increases F. An easy spectral sequence argument implies then the statement. $\square$

References

[1] J. H. Bauesi – The double bar and cobar constructions, *Compos. Math* **43** (1981), p. 331–341.

[2] V. Dolgushev – Covariant and equivariant formality theorems, *Adv. Math.* **191** (2005), p. 147–177.

[3] V. G. Drinfeld – Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990), p. 1419–1457.

[4] ______, Quantum groups, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, Amer. Math. Soc., Providence, RI, 1993, p. 798–820.

[5] B. Enriquez – A cohomological construction of quantization functors of Lie bialgebras, *Adv. Math.* **197** (2005), p. 430–479.

[6] P. Etingof & D. Kazhdan – Quantization of Lie bialgebras. I, *Selecta Math. (N.S.)* **2** (1996), p. 1–41.

[7] ______, Quantization of Lie bialgebras. II, III, *Selecta Math. (N.S.)* **4** (1998), p. 213–231, 233–269.

[8] B. Fedosov – A simple geometrical construction of deformation quantization, *J. Diff. Geom.* **40** (1994), p. 213–238.

[9] M. Gerstenhaber & A. Voronov – Homotopy G-algebras and moduli space operad, *Internat. Math. Res. Notices* **3** (1995), p. 141–153.

[10] G. Ginot – Homologie et modèle minimal des algèbres de Gerstenhaber, *Ann. Math. Blaise Pascal* **11** (2004), p. 95–127.

[11] G. Ginot & G. Halbout – A formality theorem for Poisson manifold, *Lett. Math. Phys.* **66** (2003), p. 37–64.

[12] V. Ginzburg & M. Kapranov – Koszul duality for operads, *Duke Math. J.* **76** (1994), p. 203–272.

[13] G. Halbout – Formule d'homotopie entre les complexes de Hochschild et de de Rham, *Compositio Math.* **126** (2001), p. 123–145.

[14] V. Hinich – Tamarkin's proof of Kontsevich's formality theorem, *Forum Math.* **15** (2003), p. 591–614.

[15] G. Hochschild, B. Kostant & A. Rosenberg – Differential forms on regular affine algebras, *Transactions AMS* **102** (1962), p. 383–408.

[16] C. Kassel – Homologie cyclique, caractère de Chern et lemme de perturbation, *J. Reine Angew. Math.* **408** (1990), p. 159–180.

[17] M. Khalkhali – Operations on cyclic homology, the X complex, and a conjecture of Deligne, *Comm. Math. Phys.* **202** (1999), p. 309–323.

[18] M. Kontsevich – Formality conjecture. deformation theory and symplectic geometry, *Math. Phys. Stud.* **20** (1996), p. 139–156.

[19] ______, Deformation quantization of Poisson manifolds, I, *Lett. Math. Phys.* **66** (2003), p. 157–216.

[20] M. Kontsevich & Y. Soibelman – Deformations of algebras over operads and the Deligne conjecture, (2000), p. 255–307.

[21] P. B. A. Lecomte & M. D. Wilde – A homotopy formula for the Hochschild cohomology, *Compositio Math.* **96** (1995), p. 99–109.

[22] D. Tamarkin – Another proof of M. Kontsevich's formality theorem, (1998), math.QA/9803025.

[23] A. Voronov – Homotopy Gerstenhaber algebras, in *Conférence Moshé Flato 1999, Vol. II (Dijon)* (K. A. Publ., éd.), Math. Phys. Stud., 22, 2000, p. 307–331.

Gilles Halbout
Institut de Recherche
Mathématique Avancée
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE
halbout@math.u-strasbg.fr