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Poincaré and log-Sobolev inequality for stationary Gaussian processes and moving average processes

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Abstract

For stationary Gaussian processes, we obtain the necessary and sufficient conditions for Poincaré inequality and log-Sobolev inequality of process-level and provide the sharp constants. The extension to moving average processes is also presented, as well as several concrete examples.

1 Introduction and Main Results

Let $X := (X_n)_{n\in\mathbb{Z}}$ be a real valued stationary Gaussian process with

$$\mathbb{E}X_0^2 = \sigma^2 > 0,\ \mathbb{E}X_0X_n = \sigma^2\rho(n),\ \forall n \in \mathbb{Z}. \tag{1.1}$$

It is a very important class of stochastic processes both in theory and applications. The limit theorems about stationary Gaussian processes are abundant, see Avram [1] for the central limit theorem, Donsker and Varadhan [3] and L. Wu [9] for the large deviations, H.Djellout and al. [2] for moderate deviations, and the references therein.

In this note we are mainly interested in Poincaré inequality and Logarithmic Sobolev inequality on the product space $\mathbb{R}^{\mathbb{Z}}$ for the law of the process X. As developed by Ledoux [5] and many other authors, the Poincaré or log-Sobolev yield sharp concentration inequalities, which are much more robust than the limit theorems quoted above.

We begin by describing those two inequalities on $E := \mathbb{R}^{\mathbb{Z}}$. Regarding

$$l^2(\mathbb{Z}) := \{h := (h_n)_{n\in\mathbb{Z}} | h_n \in \mathbb{R}, |h| := \sqrt{\sum_{n\in\mathbb{Z}} |h_n|^2} < +\infty\}$$

as a tangent space of $E = \mathbb{R}^{\mathbb{Z}}$, let ∇ be the corresponding gradient, i.e., for a function F on E, derivable at each coordinate x_i, i.e., $\partial_{x_i} F$ exists, let

$$\nabla F(x) = (\partial_{x_i} F(x))_{i \in \mathbb{Z}}.$$

If $F : E \to \mathbb{R}$ and $\nabla F : E \to l^2(\mathbb{Z})$ are continuous, we say that $F \in C^1(E)$.

Definition 1.1: We say that a $\mathbb{R}$-valued stochastic process $X = (X_n)_{n \in \mathbb{Z}}$ satisfies the Poincaré inequality, if there is some best constant $c_P(X) \in \mathbb{R}^+$ such that

$$Var_{\mathbb{P}}(F(X)) \le c_P(X) \mathbb{E} \sum_{i \in \mathbb{Z}} |\partial_{x_i} F|^2(X), \ \forall F \in C^1(E) \bigcap C_b(E), \tag{1.2}$$

and X satisfies the log-Sobolev inequality (LSI in short) if there is some best constant $c_{LS}(X) \in \mathbb{R}^+$ such that

$$Ent_{\mathbb{P}}(F^2(X)) \le 2c_{LS}(X) \mathbb{E} \sum_{i \in \mathbb{Z}} |\partial_{x_i} F|^2(X), \ \forall F \in C^1(E) \bigcap C_b(E). \tag{1.3}$$

Here $Ent_{\mathbb{P}}(F(X)) := \mathbb{E}^{\mathbb{P}} F(X) \log \frac{F(X)}{\mathbb{E}^{\mathbb{P}} F(X)}$ for $\mathbb{P}$-integrable nonnegative $F(X)$, is the Kullback entropy.

The same definition and notation apply also to the random vector in $\mathbb{R}^n$. It is well known that $c_P(X) \le c_{LS}(X)$ (cf. [5]).

For the stationary Gaussian process given in (1.1), if $\rho(m) = 0, \forall m \ne 0$, it becomes an i.i.d. sequence for which it is now well known (due to Gross [6], see [5])

$$c_P(X) = c_{LS}(X) = \sigma^2. \tag{1.4}$$

To state our main result, let us introduce the (nonnegative and bounded) spectral measure μ on the torus $\mathbb{T}$ identified as $[-\pi, \pi]$, which is determined by (Bochner's theorem)

$$\sigma^2 \rho(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} d\mu(t), \ \forall n \in \mathbb{Z} \tag{1.5}$$

The main result of this note is:

Theorem 1.2: *For the stationary Gaussian process in (1.1), we have*

$$c_P(X) = c_{LS}(X) = \begin{cases} \|f\|_\infty, & \textit{if } \ \mu \ll dt, f := d\mu/dt; \\ +\infty, & \textit{otherwise} \end{cases} \tag{1.6}$$

where $\|f\|_\infty = esssup_t f(t)$. *In particular, X satisfies the Poincaré or log-Sobolev inequality iff $\mu \ll dt$ and the density $f := d\mu/dt$ is bounded.*

When $\rho(\cdot) \in l^2(\mathbb{Z})$, the spectral density $f = d\mu/dt$ exists and belongs to $L^2([-\pi, \pi]) := L^2([-\pi, \pi], dt)$ and

$$f(t) = \sigma^2 \sum_{k \in \mathbb{Z}} \rho(k) e^{ikt} = \sigma^2 \left(1 + 2 \sum_{k \geq 1} \rho(k) \cos(kt) \right). \tag{1.7}$$

From the result above we have immediately

Corollary 1.3: *If $\rho(n) \geq 0$ for all n or if $(-1)^n \rho(n) \geq 0$ for all n, then*

$$c_P(X) = c_{LS}(X) = \sigma^2 \left(1 + 2 \sum_{n \geq 1} |\rho(n)| \right).$$

In particular, X satisfies the Poincaré or log-Sobolev inequality iff $\sum_{n \geq 1} |\rho(n)| < +\infty$.

In the literature, $\sum_{n \geq 1} |\rho(n)| < +\infty$ is often called *short range dependence*, cf. Taqqu [8].

Remark 1.4: For stationary Gaussian process $X = (X_k)_{k \in \mathbb{Z}}$ with law P (on $\mathbb{R}^{\mathbb{Z}}$), one can consider the abstract Wiener space $(\mathbb{R}^{\mathbb{Z}}, H, P)$, where $H \subset \mathbb{R}^{\mathbb{Z}}$ is the Cameron-Martin space associated with P. It is not difficult (but already more difficult than the proof of Theorem 1.2) to check that for any smooth $F : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ depending only on a finite number of variables,

$$\|\nabla_H F\|_H^2 = \langle \nabla F, \Gamma \nabla F \rangle_{l^2(\mathbb{Z})}$$

where ∇_H is the Malliavin gradient, and $\Gamma = (\sigma^2 \rho(k-l))_{k,l \in \mathbb{Z}}$ is the covariance matrix of X. By the Gross theorem,

$$Ent_{\mathbb{P}}(F^2(X)) = Ent_P(F^2) \leq 2 \int \|\nabla_H F\|_H^2 dP = 2\mathbb{E} \langle \nabla F, \Gamma \nabla F \rangle_{l^2(\mathbb{Z})}(X) \tag{1.8}$$

(an easy derivation of it is given in the proof of Lemma 2.1). This log-Sobolev inequality, though involving the covariance structure of X, is however less convenient (than Theorem 1.2) for the derivation of concentration inequalities. When $\Gamma : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is bounded, the right hand side (r.h.s. in short) of (1.8) is bounded by

$$\lambda_{\max}(\Gamma) \mathbb{E} |\nabla F|_{l^2(\mathbb{Z})}^2 (X).$$

By (2.2) below, $\lambda_{\max}(\Gamma) = \|f\|_\infty$.

This note is organized as follows. The next section is devoted to the proof of Theorem 1.2 and its counterpart for continuous time Gaussian processes. In §3, we present an extension to the moving average processes and provide several examples.

2 Proof of Theorem 1.2 and the continuous time counterpart

2.1 Proof of Theorem 1.2

It is based on the following (known) observation:

Lemma 2.1: *For a d-dimensional random vector X of law $N(0,\Gamma)$, where Γ is the covariance matrix, then*

$$c_P(X) = c_{LS}(X) = \lambda_{\max}(\Gamma)$$

where $\lambda_{\max}(\Gamma)$ denotes the maximal eigenvalue of Γ, i.e.,

$$\lambda_{\max}(\Gamma) = \sup_{x\in\mathbb{R}^d} \frac{< x, \Gamma x >}{< x, x >} .$$

We give its proof for the convenience of the reader and especially for its simplicity.

Proof: Let ξ be a random vector of law $\mathcal{N}(0, I)$ on $\mathbb{R}^d$. Then X and $\sqrt{\Gamma}\xi$ have the same law. Therefore by (1.4), we have for any bounded C^1 function F on $\mathbb{R}^d$,

$$\begin{aligned} Ent(F^2(X)) = Ent(F^2(\sqrt{\Gamma}\xi)) &\le 2\mathbb{E}|\nabla_\xi F(\sqrt{\Gamma}\xi)|^2 \\ &= 2\mathbb{E}|\sqrt{\Gamma}(\nabla F)(\sqrt{\Gamma}\xi)|^2 = 2\mathbb{E}\langle \Gamma(\nabla F)(X), (\nabla F)(X)\rangle \\ &\le 2\lambda_{\max}(\Gamma)\mathbb{E}|\nabla F(X)|^2 \end{aligned}$$

where it follows that $c_P(X) \le c_{LS}(X) \le \lambda_{\max}(\Gamma)$. Furthermore, letting x_0 be an unit eigenvector of Γ associated with $\lambda_{\max}(\Gamma)$ and $F(x) := \langle x, x_0\rangle$, we see that $|\nabla F(x)| = |x_0| = 1$ and $F(X)$ is centered, Gaussian with variance

$$Var(F(X)) = \langle x_0, \Gamma x_0\rangle = \lambda_{\max}(\Gamma) = \lambda_{\max}(\Gamma)\mathbb{E}|\nabla F(x)|^2$$

where it follows that $c_P(X) \geq \lambda_{\max}(\Gamma)$. The lemma is proved. $\square$

Proof: (Proof of Theorem 1.2) Considering (X_k/σ) if necessary, we may assume that $\sigma = 1$ without loss of generality. Let $X^{(n)} := (X_k)_{-n\leq k\leq n}$, which is centered, Gaussian with the covariance matrix given by the Toplitz matrix

$$\Gamma_n = (\rho(k-l))_{-n\leq k,l\leq n}.$$

In the definition 1.1, one can take only bounded C^1-function F depending on a finite number of variables (by approximation, the detail is left to the reader). In other words we always have

$$c_P(X) = \sup_n c_P(X^{(n)}), \quad c_{LS}(X) = \sup_n c_{LS}(X^{(n)}). \tag{2.1}$$

Then in the present situation we get by Lemma 2.1,

$$c_P(X) = \sup_n c_P(X^{(n)}) = \sup_n c_{LS}(X^{(n)}) = \sup_n \lambda_{\max}(\Gamma_n) = c_{LS}(X).$$

We divide the proof into two cases.

Case 1. $\rho(\cdot) \notin l^2(\mathbb{Z})$. In this case, we have

$$\lambda_{\max}(\Gamma_n) \geq \sup_{x\in\mathbb{R}^{2n+1};|x|=1} |(\Gamma_n x)_0| = \sup_{x\in\mathbb{R}^{2n+1};|x|=1} \sum_{k=-n}^{n} x_k\rho(k) = \sqrt{\sum_{k=-n}^{n}\rho(k)^2}$$

and thus $c_{LS}(X) = c_P(X) = +\infty$.

Case 2. $\rho(\cdot) \in l^2(\mathbb{Z})$. In this case $\mu \ll dt$ and the spectral density $f = d\mu/dt$ is in $L^2([-\pi,\pi])$. It remains to show that

$$\sup_n \lambda_{\max}(\Gamma_n) = \|f\|_\infty. \tag{2.2}$$

The following simple proof is due to the referee. By Rayleigh's principle, and noting that $\rho(k-l) = \frac{1}{2\pi}\int e^{-i(k-l)t}f(t)dt$, we have for any $n \geq 1$,

$$\lambda_{\max}(\Gamma_n) \leq \sup_{|x|\leq 1}\langle x, \Gamma_n x\rangle = \sup_{|x|\leq 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=-n}^{n} x_k e^{ikt}\right|^2 f(t)dt,$$

which is obviously bounded from above by $\|f\|_\infty$ by Parseval's equality. Conversely, using the equality above and the denseness of trigonometric polynomials in the Banach space $C_b\mathbb{T}$ of complex valued continuous and bounded functions on $\mathbb{T}$, we have furthermore

$$\sup_n \lambda_{\max}(\Gamma_n) = \sup_g \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^2 f(t)dt$$

where the supremum runs over all complex-valued $g \in C_b\mathbb{T}$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^2(t)dt \le 1.$$

This supremum equals to $\|f\|_\infty$.

$\square$

2.2 Continuous time stationary Gaussian processes

Let now $X = (X_t)_{t\in\mathbb{R}}$ be a real-valued stationary centered Gaussian process, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with continuous covariance function on $\mathbb{R}$,

$$\gamma(t) := \mathbb{E}X_0X_t, \ \forall t \in \mathbb{R}.$$

Let μ be the spectral measure of X on $\mathbb{R}$, determined by

$$\gamma(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-its}d\mu(s), \quad \forall t \in \mathbb{R}$$

(Bochner's theorem). It is nonnegative and bounded (indeed $\mu(\mathbb{R}) = \gamma(0)$). We can and will assume that for each $T > 0$, the sample paths of $X_{[-T,T]} = (X_t)_{t\in[-T,T]}$ are a.s. in $L^2[-T,T] := L^2([-T,T], dt)$ (such version exists for its covariance operator is of trace class, see the proof of Theorem 2.2).

Let $E = L^2_{loc}(\mathbb{R})$, the space of all real-valued locally ($dt-$) square integrable functions on $\mathbb{R}$, equipped with the project limit topology of $L^2[-T,T]$ as $T \to +\infty$. Regarding $L^2(\mathbb{R}) := L^2(\mathbb{R}, dt)$ as the tangent space of E, for any Gateaux-differentiable function F on E and $x \in E$ such that $|D_hF(x)| \le C_x\|h\|_2$ for all $h \in L^2(\mathbb{R})$, where $\|h\|_2 := \|h\|_{L^2(\mathbb{R},dt)}$ and

$$D_hF(x) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F(x+\varepsilon h) - F(x)),$$

we can define the gradient $\nabla F(x) = (\nabla_t F(x))_{t\in\mathbb{R}} \in L^2(\mathbb{R})$ by

$$\int_{\mathbb{R}} \nabla_t F(x)h(t)dt = D_hF(x), \ \forall h \in L^2(\mathbb{R}).$$

In the variational calculus, we have formally $\nabla_t F(x) = \frac{\delta}{\delta x(t)} F(x)$.

When $F : E \to \mathbb{R}$ and $\nabla F : E \to L^2(\mathbb{R})$ are continuous and bounded, we say that $F \in C_b^1(E)$. Similarly as in Definition 1.1, let $c_P(X) \in [0, +\infty]$ be the best constant for the following Poincaré inequality

$$Var_{\mathbb{P}}(F(X)) \le c_p(X) \mathbb{E} \int_{\mathbb{R}} |\nabla_t F|^2(X) dt, \quad \forall F \in C_b^1(E),$$

and $c_{LS}(X) \in [0, +\infty]$ be the best constant for the following log-Sobolev inequality

$$Ent_{\mathbb{P}}(F^2(X)) \le 2 c_{LS}(X) \mathbb{E} \int_{\mathbb{R}} |\nabla_t F|^2(X) dt, \quad \forall F \in C_b^1(E).$$

These functional inequalities with respect to the L^2-metric (instead of the Cameron-Martin metric) have been investigated by M. Gourcy and the fourth author [4] for diffusions. We have the following counterpart of Theorem 1.2.

Theorem 2.2: *We have*

$$c_P(X) = c_{LS}(X) = \begin{cases} \|f\|_\infty, & \text{if } \mu \ll dt, \ f := d\mu/dt; \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof: **1)** We begin with an extension of Lemma 2.1: let X be a centered Gaussian random variable valued in a separable Hilbert space H of law $\mathcal{N}(0, \Gamma)$, where the self-adjoint nonnegative definite covariance operator $\Gamma : H \to H$, determined by

$$\mathbb{E}\langle h_1, X\rangle\langle h_2, X\rangle = \langle h_1, \Gamma h_2\rangle.$$

It is well known that Γ is of trace class (and conversely if Γ is of trace class, then $\mathcal{N}(0, \Gamma)$ is a probability measure on H). Then using the usual pre-Dirichlet form $\mathbb{E}|\nabla F|_H^2(X)$ and letting $c_P(X), c_{LS}(X)$ be the best constants for the corresponding Poincaré and log-Sobolev inequalities, respectively, we have

$$c_P(X) = c_{LS}(X) = \lambda_{\max}(\Gamma). \tag{2.3}$$

Indeed, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H such that $\Gamma e_n = \lambda_n e_n$ where the sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ is ranged as non-increasing. As

$\{\langle X, e_n\rangle; n \in \mathbb{N}\}$ are independent with laws $\{\mathcal{N}(0, \lambda_n); n \in \mathbb{N}\}$, we have by the independent tensorization ([5]),

$$\begin{aligned}
&c_P(X) = \sup_n c_P(\langle X, e_n\rangle),\ c_{LS}(X) = \sup_n c_{LS}(\langle X, e_n\rangle),\\
&c_P(\langle X, e_n\rangle) = c_{LS}(\langle X, e_n\rangle) = \lambda_n
\end{aligned}$$

where (2.3) follows.

2) By approximation, it is easy to check that

$$c_P(X) = \sup_{T>0} c_P(X_{[-T,T]}),\ c_{LS}(X) = \sup_{T>0} c_{LS}(X_{[-T,T]})$$

where $c_P(X_{[-T,T]}), c_{LS}(X_{[-T,T]})$ are the best constants defined in Step 1) with $H = L^2[-T, T]$. The covariance operator of $X_{[-T,T]}$ are given by

$$(\Gamma_T h)(t) = \int_{[-T,T]} \gamma(t-s)h(s)ds,\ \forall h \in L^2[-T, T].$$

By Step 1), we have only to prove that

$$\sup_{T>0} \lambda_{\max}(\Gamma_T) = \begin{cases} \|f\|_\infty, & \text{if } \mu \ll dt,\ f := d\mu/dt;\\ +\infty, & \text{otherwise.}\end{cases}$$

Let $L^2_{\mathbb{C}}[-T, T]$, $L^2_{\mathbb{C}}(\mathbb{R})$ be the spaces of complex-valued L^2-integrable functions on $[-T, T]$ and $\mathbb{R}$, respectively. For any $h \in L^2_{\mathbb{C}}(\mathbb{R})$, let

$$\hat{h}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{its} h(s)ds$$

be the Fourier transform of h. It is well known that $h \to \hat{h}$ is unitary on $L^2_{\mathbb{C}}(\mathbb{R})$. Regarding $h \in L^2[-T, T]$ as an element of $L^2(\mathbb{R})$ by putting $h = 0$ out of $L^2[-T, T]$, we have

$$\begin{aligned}
\lambda_{\max}(\Gamma_T) &= \sup_{h \in L^2_{\mathbb{C}}[-T,T], \|h\|_2 \le 1} \langle h, \Gamma_T h\rangle\\
&= \sup_{h \in L^2_{\mathbb{C}}[-T,T], \|h\|_2 \le 1} \frac{1}{2\pi} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{its} h(s)ds \right|^2 d\mu(t)\\
&= \sup_{h \in L^2_{\mathbb{C}}[-T,T], \|h\|_2 \le 1} \int_{\mathbb{R}} |\hat{h}(t)|^2 d\mu(t).
\end{aligned}$$

Since any $h \in L^2_{\mathbb{C}}(\mathbb{R}) \bigcap L^1_{\mathbb{C}}(\mathbb{R})$, $\widehat{h1_{[-T,T]}} \to \hat{h}$ in $L^2_{\mathbb{C}}(\mathbb{R})$ and also uniformly on $\mathbb{R}$, we have

$$\sup_{T>0} \lambda_{\max}(\Gamma_T) = \sup_{h \in L^2_{\mathbb{C}}(\mathbb{R}) \bigcap L^1_{\mathbb{C}}(\mathbb{R}), \|h\|_2 \le 1} \int_{\mathbb{R}} |\hat{h}(t)|^2 d\mu(t).$$

Since the family $\mathcal{A}$ of all $\hat{h}$ with $h \in L^2_{\mathbb{C}}(\mathbb{R}) \bigcap L^1_{\mathbb{C}}(\mathbb{R})$ constitutes an algebra separating the points of $\mathbb{R}$, then by monotone class theorem, for any complex-valued measurable and bounded function g on $\mathbb{R}$, say $g \in b_{\mathbb{C}}\mathcal{B}(\mathbb{R})$, we can find a sequence ($g_n = \hat{h}_n \in \mathcal{A}$) such that $g_n \to g$ in $L^2(\mathbb{R}, dt + \mu)$. Thus

$$\sup_{T>0} \lambda_{\max}(\Gamma_T) = \sup_{g \in b_{\mathbb{C}}\mathcal{B}(\mathbb{R}), \|g\|_2 \le 1} \int_{\mathbb{R}} |g|^2(t) d\mu(t).$$

This implies easily the desired result. □

3 Extension and several examples

In this section we extend Theorem 1.2 to general moving average processes and provide some concrete examples.

3.1 Extension to moving average processes

Let $(\xi_k)_{k\in\mathbb{Z}}$ be a sequence of i.i.d. real-valued random variables such that $\mathbb{E}\xi_0 = 0$ and $\mathbb{E}\xi_0^2 = 1$, and $(a_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Consider the moving average process

$$X_n := \sum_{k=-\infty}^{+\infty} a_k \xi_{n+k}, \ n \in \mathbb{Z}. \tag{3.1}$$

It is a well defined stationary process with covariance function

$$\gamma(n) = \mathbb{E} X_0 X_n = \sum_{k=-\infty}^{+\infty} a_k a_{k-n}, \ \forall n \in \mathbb{Z}.$$

Its spectral density function is given by

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \gamma(n) e^{in\theta} = \left| \sum_{n=-\infty}^{+\infty} a_n e^{in\theta} \right|^2.$$

A stationary Gaussian process with spectral density $f \in L^2([-\pi,\pi])$ can be always written as a moving average process with driven noises (ξ_k) being i.i.d. with law $\mathcal{N}(0,1)$. So the following result partially generalizes Theorem 1.2.

Theorem 3.1: *Let $X = (X_n)$ be the moving average process given above with the spectral density f. Then*

$$c_P(X) \le \|f\|_\infty c_P(\xi_0),\ c_{LS}(X) \le \|f\|_\infty c_{LS}(\xi_0). \tag{3.2}$$

Remark 3.2: Bobkov and Götze [7] have found necessary and sufficient conditions for both characterizing $c_P(\xi_0) < +\infty$ and $c_{LS}(\xi_0) < +\infty$.

Proof: By (2.1), it is enough to prove that for any $n \ge 1$ and for any smooth and bounded function $F : \mathbb{R}^{I_n} \to \mathbb{R}$ where $I_n = [-n,n] \bigcap \mathbb{Z}$,

$$Var(F(X_{I_n})) \le \|f\|_\infty c_P(\xi_0)\mathbb{E}|\nabla F(X_{I_n})|^2;$$
$$Ent(F^2(X_{I_n})) \le \|f\|_\infty c_{LS}(\xi_0)\mathbb{E}|\nabla F(X_{I_n})|^2.$$

We prove here only the first Poincaré inequality (the proof of the LSI is completely similar). Let $A_n = (a_{k-l})_{k\in I_n, l\in\mathbb{Z}} : l^2(\mathbb{Z}) \to \mathbb{R}^{I_n}$ and $\xi = (\xi_l)_{l\in\mathbb{Z}}$ as a column vector. We have $X_{I_n} = A\xi$. Now for each $N \ge 1$, let $\bar{P}_N\xi = (1_{k\in I_N}\xi_k)_{k\in\mathbb{Z}}$ as before. Since $F(A_n\bar{P}_N\xi)$ is a bounded and smooth function of ξ_{I_N}, by using $c_P(\xi) = c_P(\xi_0)$ we have

$$\begin{aligned}
Var(F(A_n\bar{P}_N\xi)) &\le c_P(\xi_0)\mathbb{E}|\nabla_\xi F(A_n\bar{P}_N\xi)|^2 \\
&= c_P(\xi_0)\mathbb{E}|(A_n\bar{P}_N)^*(\nabla F)(A_n\bar{P}_N\xi)|^2 \\
&\le c_P(\xi_0)\lambda_{\max}((A_n\bar{P}_N)(A_n\bar{P}_N)^*)\mathbb{E}|(\nabla F)(A_n\bar{P}_N\xi)|^2
\end{aligned}$$

where A^* denotes the adjoint matrix or operator. Letting N go to infinity, as $A_n\bar{P}_N\xi \to A_n\xi = X_{I_n}, a.s.$, so we get by dominated convergence

$$Var(F(X_{I_n})) \le c_P(\xi_0)\sup_{N\ge1}\lambda_{\max}((A_n\bar{P}_N)(A_n\bar{P}_N)^*)\mathbb{E}|(\nabla F)(X_{I_n})|^2.$$

Furthermore, since

$$\langle (A_n\bar{P}_N)(A_n\bar{P}_N)^*x, x\rangle = |P_N^*A_n^*x|^2 \le |A_n^*x|^2 = \langle A_nA_n^*x, x\rangle,\ \forall x \in \mathbb{R}^{I_n},$$

we have $\lambda_{\max}((A_n\bar{P}_N)(A_n\bar{P}_N)^*) \le \lambda_{\max}(A_nA_n^*)$ for all N. But $A_nA_n^*$ coincides with the covariance matrix $\Gamma_n = (\rho(k-l))_{k,l\in I_n}$ of X_{I_n}, hence we obtain

$$Var(F(X_{I_n})) \le c_P(\xi_0)\lambda_{\max}(\Gamma_n)\mathbb{E}|(\nabla F)(X_{I_n})|^2.$$

Now the desired Poincaré inequality follows by (2.2). □

3.2 Several examples

Example 3.3: (Fractional Brownian Motion) Let $(B^H(t))_{t\geq 0}$ be the real fractional Brownian Motion with Hurst index $H \in (0,1)$, i.e., a centered Gaussian process with

$$\mathbb{E}B^H(s)B^H(t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \ \forall s,t \geq 0.$$

Let $X_n = B^H(n+1) - B^H(n), n \in \mathbb{N}$, which is stationary with covariance function

$$\rho(n) = Cov(X_0, X_n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right), \ n \geq 0$$

and $\rho(n) = \rho(-n)$, $n \leq 0$. Note that $\rho(n) \sim 2H(2H-1)n^{2(H-1)}$, as $n \to +\infty$.

1) If $H > 1/2$, then$\rho(n) > 0$ for all n and $\sum_{n\geq 0}\rho(n) = +\infty$. Thus by Corollary 1.3, X does not satisfy the Poincaré inequality, neither the log-Sobolev inequality.

2) If $H = 1/2$ (the trivial case), then $c_P(X) = c_{LS}(X) = 1$.

3) If $H < 1/2$, then $\rho(n) < 0$ for all n and $\sum_{n\geq 1}|\rho(n)| = 1/2$. Thus the spectral density f exists and it is continuous on the torus $\mathbb{T}$. Consequently $c_P(X) = c_{LS}(X) = \|f\|_\infty \leq 2$.

Example 3.4: (ARMA model) Consider the autoregressive process

$$X_{n+1} = \theta X_n + \sigma\xi_{n+1}, \ n \geq 0$$

where $\theta \in (-1,1)$, $(\xi_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. r.v. with $\mathbb{E}\xi_0 = 0$ and $\mathbb{E}\xi_0^2 = 1$, $\sigma^2 > 0$ is the strength of noise, and X_0 is independent of $(\xi_n)_{n\geq 1}$. Its unique invariant measure is the law of $\sum_{k=0}^{+\infty}\theta^k\xi_{-k}$. Below we take $X_0 = \sum_{k=0}^{+\infty}\theta^k\xi_{-k}$. In that case, $(X_n)_{n\geq 0}$ is a moving average process with $a_n = 1_{n\leq 0}\theta^{|n|}$ and with covariance function $\gamma(n) = \sigma^2(1-\theta^2)^{-1}\theta^{|n|}$. Its spectral density function is given by

$$f(t) = \frac{\sigma^2}{1-\theta^2}\sum_{n\in\mathbb{Z}}\theta^{|n|}e^{int} = \frac{\sigma^2}{1+\theta^2-2\theta\cos t}, \ \forall t \in \mathbb{T}.$$

Then $\|f\|_\infty = \sigma^2(1-|\theta|)^{-2}$. Consequently by Theorem 3.1,

$$c_P(X) \le \frac{\sigma^2}{(1-|\theta|)^2} c_P(\xi_0), \ c_{LS}(X) \le \frac{\sigma^2}{(1-|\theta|)^2} c_{LS}(\xi_0).$$

In practice the following noises are the most often used:

1) ξ_0 is Gaussian $\mathcal{N}(0,1)$. We have $c_P(\xi_0) = c_{LS}(\xi_0) = 1$ and then by Theorem 1.2, $c_P(X) = c_{LS}(X) = \sigma^2(1-|\theta|)^{-2}$.

2) ξ_0 is of law uniform on $[-a,a]$ where $a = (3/2)^{1/3}$ (so that $\mathbb{E}\xi_0^2 = 1$). In this case it is well known ([5]) that

$$c_P((\pi/a)\xi_0) = c_{LS}((\pi/a)\xi_0) = 1.$$

Then $c_P(\xi_0) = c_{LS}(\xi_0) = a^2\pi^{-2}$. Thus we get

$$c_P(X) \le c_{LS}(X) \le \left(\frac{3}{2\pi^3}\right)^{2/3} \frac{\sigma^2}{(1-|\theta|)^2}.$$

3) ξ_0 is of density $e^{-2|x|}dx$ (symmetric exponential law). Again it is well known that $c_P(2\xi_0) = 1$ but $c_{LS}(2\xi_0) = +\infty$ ([5]). Hence

$$c_P(X) \le \frac{\sigma^2}{4(1-|\theta|)^2}.$$

Since for any $\lambda > 0$, by Jensen's inequality, $\mathbb{E}e^{\lambda|X_0|^2} \ge \mathbb{E}e^{\lambda|\xi_0|^2} = +\infty$, we have $c_{LS}(X) = +\infty$ by [5].

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