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Abstract

We are concerned with some unbounded linear operators on the so-called p-adic Hilbert space $\mathbb{E}_\omega$. Both the Closedness and the self-adjointness of those unbounded linear operators are investigated. As applications, we shall consider the diagonal operator on $\mathbb{E}_\omega$, and the solvability of the equation $Au = v$ where A is a linear operator on $\mathbb{E}_\omega$.

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1 Introduction

The primary goal of this paper is to investigate upon some unbounded linear operators on the so-called p-adic Hilbert space $\mathbb{E}_\omega$. For that, we first introduce and give the required background on the p-adic Hilbert space $\mathbb{E}_\omega$. Next, we shall be dealing with natural issues such as the closedness and the self-adjointness of those unbounded linear operators. To do so, one equips the so-called direct sum $\mathbb{E}_\omega \times \mathbb{E}_\omega$ of $\mathbb{E}_\omega$ with itself, with both an ultrametric norm and a hilbertian structure. Afterwards, it goes back to introduce an unitary operator Γ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ which yields a remarkable description of A^*, the adjoint of a linear operator A defined on $\mathbb{E}_\omega$ which does have an adjoint, in terms of A.

Let us mention that the p-adic Hilbert space $\mathbb{E}_\omega$ will play a key role throughout the paper. Apart from their intrinsic interests, p-adic Hilbert spaces have found extensive applications in theoretical physics.

For more on these and related issues we refer the reader to([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], and [11]) and the references therein.

Let $\mathbb{K}$ be a complete ultrametric valued field. Classical examples of such a field include $\mathbb{Q}_p$ the field of p-adic numbers where $p \geq 2$ is a prime, $\mathbb{C}_p$ the field of p-adic complex numbers, and the field of formal Laurent series([4]).

An ultrametric Banach space $\mathbb{E}$ over $\mathbb{K}$ is said to be a *free Banach space* (see [2], [3], and [4]) if there exists a family $(e_i)_{i\in I}$ (I being an index set) of elements of $\mathbb{E}$ such that each element $x \in \mathbb{E}$ can be written in an unique fashion as:

$$x = \sum_{i\in I} x_i e_i, \quad \lim_{i\in I} x_i e_i = 0, \quad \text{and} \quad \|x\| = \sup_{i\in I} |x_i| \|e_i\|.$$

The family $(e_i)_{i\in I}$ is then called an *orthogonal basis* for $\mathbb{E}$, and if $\|e_i\| = 1$, for all $i \in I$, the family $(e_i)_{i\in I}$ is called an *orthonormal basis.* For a detailed description and properties of these spaces, we refer the reader to ([2], [3], [4], and [11]) and the references therein.

Up to now, we shall suppose that the index set I is $\mathbb{N}$, the set of all natural integers.

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$ denote its (topological) dual and $B(\mathbb{E})$ the Banach space of all bounded linear operators on $\mathbb{E}$ (see [2], [3], and [4]). Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are equipped with their respective natural norms.

For $(u, v) \in \mathbb{E} \times \mathbb{E}^*$ we define the linear operator $(v \otimes u)$ by setting:

$$\forall x \in \mathbb{E},\ (v \otimes u)(x) := v(x)\, u = \langle v, x\rangle\, u.$$

It follows that $(v \otimes u) \in B(\mathbb{E})$ and $\|v \otimes u\| = \|v\| \,.\, \|u\|$.

Let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal basis for $\mathbb{E}$. We then define $e'_i \in \mathbb{E}^*$ by

$$x = \sum_{i\in\mathbb{N}} x_i e_i, \quad e'_i(x) = x_i.$$

It turns out that $\|e'_i\| = \dfrac{1}{\|e_i\|}$. Furthermore, every $x' \in \mathbb{E}^*$ can be expressed as a pointwise convergent series: $x' = \sum_{i\in\mathbb{N}} \langle x', e_i\rangle\, e'_i$. In addition to that, we have that:

$$\|x'\| := \sup_{i\in\mathbb{N}} \left[\frac{|\langle x', e_i\rangle|}{\|e_i\|}\right].$$

Now let us recall that every bounded linear operator A on $\mathbb{E}$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{ij} a_{ij}(e'_j \otimes e_i), \text{ and for any } \ j \in \mathbb{N}, \ \lim_{i\to\infty} |a_{ij}| \, \|e_i\| = 0. \tag{1.1}$$

Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i\in\mathbb{N}} a_{ij} \, e_i$ and its norm is defined by:

$$\|A\| := \sup_{i,j} \left[\frac{|a_{ij}| \, \|e_i\|}{\|e_j\|}\right].$$

In this paper we shall make extensive use of the p-adic Hilbert space $\mathbb{E}_\omega$ whose definition is given below. Again, for details, we refer the reader to ([2], [3], and [4]) and the references therein.

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of non-zero elements in a complete non-Archimedean field $\mathbb{K}$. We define the space $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega := \left\{ u = (u_i)_{i\in\mathbb{N}} \mid \forall i, \ \ u_i \in \mathbb{K} \text{ and } \lim_{i\to\infty} |u_i| \, |\omega_i|^{1/2} = 0 \right\}.$$

Clearly, $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ if and only if $\lim_{i\to\infty} u_i^2 \omega_i = 0$. Actually $\mathbb{E}_\omega$ is an ultrametric Banach space over $\mathbb{K}$ with the norm given by

$$u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega, \ \ \|u\| = \sup_{i\in\mathbb{N}} |u_i| \, |\omega_i|^{1/2}.$$

Let us also notice that $\mathbb{E}_\omega$ is a free Banach space (see [3]) and it has a *canonical orthogonal basis*. Namely, $(e_i)_{i\in\mathbb{N}}$, where e_i is the sequence all of whose terms are 0 except the i-th term which is 1, in other words, $e_i = (\delta_{ij})_{j\in\mathbb{N}}$ where δ_{ij} is the usual Kronecker symbol. We shall make extensive use of such a canonical orthogonal basis throughout the paper. It should be mentioned that for each i, $\|e_i\| = |\omega_i|^{1/2}$. Now if $|\omega_i| = 1$ we shall refer to $(e_i)_{i\in\mathbb{N}}$ as the *canonical orthonormal basis.*

Let $\langle , \rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ be the $\mathbb{K}$-bilinear form defined by

$$\forall u, v \in \mathbb{E}_\omega, \ u = (u_i)_{i\in\mathbb{N}}, \ v = (v_i)_{i\in\mathbb{N}}, \ \ \langle u, v\rangle := \sum_{i\in\mathbb{N}} \omega_i \, u_i v_i.$$

Then, $\langle , \rangle$ is a symmetric, non-degenerate form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with value in $\mathbb{K}$, and it satisfies the Cauchy-Schwarz inequality:

$$|\langle u, v\rangle| \leq \|u\| \, . \, \|v\|, \ \ \forall u, v \in \mathbb{E}_\omega.$$

Let us also mention that elements $(e_i)_{i\in\mathbb{N}}$ of the canonical orthogonal basis for $\mathbb{E}_\omega$ satisfy

$$\langle e_i, e_j\rangle = \omega_i\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ \\ \omega_i & \text{if} \quad i = j. \end{cases}$$

The space $\mathbb{E}_\omega$ endowed with the bilinear form $\langle , \rangle$ defined above is called a p-adic Hilbert space.

It should also be observed that for every bounded linear operator A on $\mathbb{E}_\omega$, the domain $D(A)$ ($D(A) := \{u = (u_i) \in \mathbb{E}_\omega : \lim_i |u_i| \|Ae_i\| = 0\}$) of A is the whole of $\mathbb{E}_\omega$. This is actually due to the fact that for every $u = \sum_{i\in\mathbb{N}} u_i e_i$ in $\mathbb{E}_\omega$: $|u_i|\ \|Ae_i\| \ \leq\ \|A\|\ |u_i|\ |\omega_i|^{1/2}$.

It is well-known ([2] and [3]) that one can find bounded linear operators on $\mathbb{E}_\omega$ which do not have adjoints. Similarly, we shall see that the latter is still true for unbounded linear operators we shall deal with. Consequently, we denote by $B_0(\mathbb{E}_\omega)$ the space of all bounded linear operators which do have adjoints with respect to the non-degenerate form $\langle , \rangle$ defined above. It is known ([2], [3] and [4]) that a bounded linear operator $A = \sum_{ij} a_{ij}(e'_j \otimes e_i)$ is in $B_0(\mathbb{E}_\omega)$ if and only if: $\forall i, \ \lim_{j\to\infty} |a_{ij}|\, |\omega_j|^{-1/2} = 0$.

Let us recall that $B_0(\mathbb{E}_\omega)$ is stable under the operation of taking an adjoint and for any $A \in B_0(\mathbb{E}_\omega) : (A^*)^* = A$ and $\|A\| = \|A^*\|$.

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero elements in a (complete) non-Archimedean field $\mathbb{K}$. Consider the corresponding p-adic Hilbert space given by $(\mathbb{E}_\omega, \langle , \rangle)$.

In this paper, we initiate a theory for some unbounded linear operators within the p-adic framework, that is, on the p-adic Hilbert spaces $\mathbb{E}_\omega$. As for bounded linear operators on $\mathbb{E}_\omega$, some of the results go along the classical line and others deviate from it. For the most part, the statements of the results are inspired by their classical settings. However their proofs may depend heavily on the ultrametric nature of $\mathbb{E}_\omega$ and the ground field $\mathbb{K}$. We especially emphasis on both the closedness and the self-adjointness of those unbounded linear operators on $\mathbb{E}_\omega$ (Theorems 3.3 and 3.4). To deal with those issues, we equip the direct sum $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with both a complete ultrametric norm as well as a hilbertian structure (it will be shown that this can be even done for a finite direct sum $\mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times ... \times \mathbb{E}_{\omega_n}$ of $\mathbb{E}_{\omega_1}$, $\mathbb{E}_{\omega_2}$, ..., and $\mathbb{E}_{\omega_n}$ where the sequences $\omega_k = (\omega_i^k)_{i\in\mathbb{N}}$ for $k = 1, 2, ..., n$ are respectively nonzero elements in $\mathbb{K}$). Next we consider an unitary operator Γ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ which enables to describe the adjoint A^* of a given operator A in terms of A.

To illustrate our abstract results, a few examples are discussed. In particular, a special attention is paid to the diagonal operator on $\mathbb{E}_\omega$ as well as the solvability of the equation $Au = v$ where A is (possibly unbounded) a linear operator on $\mathbb{E}_\omega$ (Theorems 5.1 and 5.4).

2 Unbounded linear operators on $\mathbb{E}_\omega$

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$, $\varpi = (\varpi_i)_{i\in\mathbb{N}}$ be sequences of non-zero elements in a complete non-Archimedean field $\mathbb{K}$, and let $\mathbb{E}_\omega$, $\mathbb{E}_\varpi$ be their corresponding p-adic Hilbert spaces. We suppose that $(e_i)_{i\in\mathbb{N}}$ and $(h_j)_{j\in\mathbb{N}}$ are respectively the canonical orthogonal bases associated to the p-adic Hilbert spaces $\mathbb{E}_\omega$ and $\mathbb{E}_\varpi$.

Let $D \subset \mathbb{E}_\omega$ be a subspace and let $A : D \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\varpi$ be a linear transformation. As for bounded linear operators one can decompose A as a pointwise convergent series defined by:

$$A = \sum_{i,j} a_{i,j}\ e'_j \otimes h_i \quad \text{and,} \quad \forall j \in \mathbb{N}, \quad \lim_{i\to\infty} |a_{i,j}|\ \|h_i\| = 0.$$

Definition 2.1: An unbounded linear operator A from $\mathbb{E}_\omega$ into $\mathbb{E}_\varpi$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}_\omega$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\varpi$.

Throughout the paper, we consider unbounded linear operators A whose domain $D(A)$ consists of all $u \in \mathbb{E}_\omega$ such that $Au \in \mathbb{E}_\varpi$, that is,

$$\begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \quad \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\}, \\ \\ A = \sum\limits_{i,j\in\mathbb{N}} a_{i,j}\ e'_j \otimes h_i, \quad \forall j \in \mathbb{N},\ \lim\limits_{i\to\infty} |a_{i,j}|\ \|h_i\| = 0. \end{cases} \tag{2.1}$$

We denote the collection of those unbounded linear operators by $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$. Clearly, $B(\mathbb{E}_\omega, \mathbb{E}_\varpi) \subset U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$.

We have seen that if $A \in B(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ then its domain is the whole of $\mathbb{E}_\omega$. We shall see that one can find elements of $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ whose domains differ from $\mathbb{E}_\omega$ (see Remark 4.2). As for bounded operators, there are elements of $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ which do not have adjoints (see Example 2.3 below). Actually in the next definition we state conditions which do guarantee the existence of the adjoint. Without lost of generality we shall suppose that $\mathbb{E}_\omega = \mathbb{E}_\varpi$. As usual we denote $U(\mathbb{E}_\omega, \mathbb{E}_\omega)$ by $U(\mathbb{E}_\omega)$.

In what follows, $(\mathbb{K}, |.|)$ denotes a complete non-Archimedean field.

Definition 2.2: Let $\omega = (\omega_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero terms and let $(\mathbb{E}_\omega, \langle , \rangle)$ be the corresponding p-adic Hilbert space. An operator $A = \sum_{i,j} a_{i,j}\ e'_j \otimes e_i$ in $U(\mathbb{E}_\omega)$ is said to have an adjoint $A^* \in U(\mathbb{E}_\omega)$ if and only if

$$\lim_{j\to\infty} \left[\frac{|a_{i,j}|}{|\omega_j|^{1/2}}\right] = 0, \quad \forall i \in \mathbb{N}. \tag{2.2}$$

Furthermore, the adjoint A^* of A is uniquely expressed by

$$\begin{cases} D(A^*) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|A^* e_i\| = 0\}, \\ \\ A^* = \sum\limits_{i,j\in\mathbb{N}} a^*_{i,j}\ e'_j \otimes e_i, \quad \forall j \in \mathbb{N},\ \lim\limits_{i\to\infty} |a^*_{i,j}|\ |\omega_i|^{\frac{1}{2}} = 0, \end{cases}$$

where $a^*_{i,j} = w_i^{-1} w_j a_{j,i}$.

We denote by $U_0(\mathbb{E}_\omega)$ the collection of operators in $U(\mathbb{E}_\omega)$ which do have adjoint operators. Clearly, $B_0(\mathbb{E}_\omega) \subset U_0(\mathbb{E}_\omega)$.

Example 2.3:(Unbounded operator with no adjoint). Set $\mathbb{K} = \mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic norm $|.|_p$ and let $\omega_i = p^{3i}$ (in the appropriate $\mathbb{Q}_p$) so that $|\omega_i|^{\frac{1}{2}} = p^{-\frac{3}{2}i}$. Now let $A = \sum_{i,j} a_{i,j}\ (e'_j \otimes e_i)$ be the linear operator defined by its coefficients:

$$a_{ij} = \begin{cases} p^{-j} & \text{if } \ i < j \\ 1 & \text{if } \ i = j \\ p^{-i} & \text{if } \ i > j. \end{cases} \quad \text{and} \quad |a_{ij}| = \begin{cases} p^{j} & \text{if } \ i < j \\ 1 & \text{if } \ i = j \\ p^{i} & \text{if } \ i > j. \end{cases}$$

We have

Proposition 2.4: *The linear operator* $A = \sum_{i,j} a_{i,j}\ (e'_j \otimes e_i)$ *defined above is in* $U(\mathbb{E}_\omega)$ *and it does not have an adjoint.*

Proof: Since $\forall j$, $\lim_i |a_{ij}||\omega_i|^{\frac{1}{2}} = \lim_{i>j} p^i p^{-\frac{3}{2}i} = 0$ it is clear that A is well-defined. Furthermore, $\forall i, j \in \mathbb{N}$,

$$\lambda_{i,j} := \frac{|a_{ij}||\omega_i|^{\frac{1}{2}}}{|\omega_j|^{\frac{1}{2}}} = \begin{cases} p^{\frac{3}{2}(j-i)+j} & \text{if } \ i<j \\ 1 & \text{if } \ i=j \\ p^{i+\frac{3}{2}j-\frac{3}{2}i} & \text{if } \ i>j. \end{cases}$$

$$\text{Consequently,} \quad \forall i,j, \quad \lambda_{i,j} := \frac{|a_{ij}||\omega_i|^{1/2}}{|\omega_j|^{1/2}} \geq \begin{cases} p^j & \text{if } \ i<j \\ 1 & \text{if } \ i=j \\ p^{-\frac{3}{2}i} & \text{if } \ i>j. \end{cases}$$

Clearly, $\|A\| := \sup_{i,j} \lambda_{i,j} = \infty$, hence $A \in U(\mathbb{E}_\omega)$. To complete the proof we have to show that $\forall i$, $\lim_j \left[\frac{|a_{ij}|}{|\omega_j|^{\frac{1}{2}}}\right] \neq 0$. Indeed, $\forall i$, $\lim_j \left[\frac{|a_{ij}|}{|\omega_j|^{\frac{1}{2}}}\right] = \lim_{j>i} p^j p^{\frac{3}{2}j}$ $= \infty$, hence the adjoint of A does not exist. □

3 Closed linear operators on $\mathbb{E}_\omega$

Let $A \in U(\mathbb{E}_\omega)$. As in the classical setting we define the graph of the linear operator A by

$$\mathcal{G}(A) := \{(x, Ax) \in \mathbb{E}_\omega \times \mathbb{E}_\omega : \ x \in D(A)\}.$$

Definition 3.1: An operator $A \in U(\mathbb{E}_\omega)$ is said to be closed if its graph is a closed subspace in $\mathbb{E}_\omega \times \mathbb{E}_\omega$. The operator A is said to be closable if it has a closed extension.

As in the classical setting we characterize the closedness of an operator $A \in U(\mathbb{E}_\omega)$ as follows: $\forall u_n \in D(A)$ such that $\|u - u_n\| \mapsto 0$ and $\|Au_n - v\| \mapsto 0$ $(v \in \mathbb{E}_\omega)$ as $n \mapsto +\infty$, then $u \in D(A)$ and $Au = v$.

It is now clear that if $A \in B(\mathbb{E}_\omega)$, it is closed. Indeed since A is bounded, $D(A) = \mathbb{E}_\omega$. Moreover if $x_n \in \mathbb{E}_\omega$ such that $x_n \mapsto x$ on $\mathbb{E}_\omega$, then by the boundedness of A it follows that $Ax_n \mapsto Ax$, that is, $(x_n, Ax_n) \mapsto (x, Ax)$ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, hence $\mathcal{G}(A)$ is closed.

We denote the collection of closed linear operators $A \in U(\mathbb{E}_\omega)$ by $C(\mathbb{E}_\omega)$. In view of the above, $B(\mathbb{E}_\omega) \subset C(\mathbb{E}_\omega)$.

Let us notice that in the classical context, the collection of closed operators is not a vector space. However if A is an unbounded linear operator acting in a Banach space $\mathbb{X}$ and if $B \in B(\mathbb{X})$ then $A + B$ their algebraic sum is closed. In the p-adic setting, there is no reason that the situation differs from the classical one. Similarly, if $A \in C(\mathbb{E}_\omega)$ and if $B \in B(\mathbb{E}_\omega)$ then $A + B \in C(\mathbb{E}_\omega)$.

Definition 3.2: An operator $A \in U_0(\mathbb{E}_\omega)$ is said to be self-adjoint if $D(A) = D(A^*)$ and $Au = A^*u$ for each $u \in D(A)$.

We shall prove the following important theorem:

Theorem 3.3: *Let $A \in U_0(\mathbb{E}_\omega)$, then its adjoint A^* is a closed linear operator. In particular if A is self-adjoint, then it is closed.*

As previously mentioned, to deal with the closedness of an operator $A \in U(\mathbb{E}_\omega)$ we first equip the direct sum $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with both an ultrametric norm and a p-adic hilbertian structure. Actually, we shall show that this can be even done for a finite direct sum of Hilbert spaces $\mathbb{E}_\omega$.

Throughout the rest of this section, we suppose that $char(\mathbb{K})$ the characteristic of the field $\mathbb{K}$ is zero. Notice that examples of such fields include $\mathbb{Q}_p$ ($p \geq 2$ being prime) the field of p-adic numbers.

Let $\omega_k = (\omega_i^k)_{i\in\mathbb{N}}$ with $k = 1, 2, ..., n$ be (n being fixed) n sequences of nonzero terms in $\mathbb{K}$. For each k ($k = 1, 2, ...n$) we consider the corresponding p-adic Hilbert space $\mathbb{E}_{\omega_k}$. Now let us consider the so-called finite direct sum $\mathbb{E}_n := \mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times ... \times \mathbb{E}_{\omega_n}$ of $\mathbb{E}_{\omega_1}$, $\mathbb{E}_{\omega_2}$, ..., and $\mathbb{E}_{\omega_n}$, respectively. Let us mention that $\mathbb{E}_n$ is also denoted by: $\mathbb{E}_{\omega_1} \oplus \mathbb{E}_{\omega_2} \oplus ... \oplus \mathbb{E}_{\omega_n}$.

We now equip $\mathbb{E}_n$ with the ultrametric norm defined by

$$||(u_1, u_2, ..., u_n)||_n := \max\left(\|u_1\|, \|u_2\|, ..., \|u_n\|\right), \tag{3.1}$$

$\forall (u_1, u_2, ..., u_n) \in \mathbb{E}_n$, where $\|.\|$ is the norm on each $\mathbb{E}_{\omega_k}$ ($k = 1, 2, ...n$).

Now arguing that each $(\mathbb{E}_{\omega_k}, \|.\|)$ for $k = 1, 2, ...n$ is an ultrametric Banach space it follows that $(\mathbb{E}_n, ||.||_n)$ is also an ultrametric Banach space. It also clear that $\mathbb{E}_n$ (n fixed) is a free Banach space with orthogonal basis as $\{(L_i^k)_{i\in\mathbb{N}},\ \ k = 1, 2, ..., n\}$ where $L_i^k = (0, 0, ...e_i^k, ..0, 0, 0)$ whose terms are 0 except the i-th term which is e_i^k ($(e_i^k)_{i\in\mathbb{N}}$ being the canonical orthogonal basis for $\mathbb{E}_{\omega_k}$ with $k = 1, 2, .., n$). Actually, $\|L_i^k\|_n = \|e_i^k\| = |\omega_i^k|^{1/2}$ for all $i \in \mathbb{N}$ and for each $k = 1, 2, ..., n$. We confer to $\{(L_i^k)_{i\in\mathbb{N}},\ \ k = 1, 2, ..., n\}$ as the canonical orthogonal basis for $\mathbb{E}_n$.

Let $\langle, \rangle_n : \mathbb{E}_n \times \mathbb{E}_n \mapsto \mathbb{K}$ be the $\mathbb{K}$-bilinear form defined by

$$\langle (u_1, u_2, ..., u_n), (v_1, v_2, ..., v_n)\rangle_n := \sum_{k=1}^{n} \langle u_k, v_k\rangle, \tag{3.2}$$

for all $(u_1, u_2, ..., u_n), (v_1, v_2, ..., v_n) \in \mathbb{E}_n$ where $\langle , \rangle$ is the corresponding inner product of each $\mathbb{E}_{\omega_k}$ for $k = 1, 2, ..., n$.

Then, $\langle , \rangle_n$ is a symmetric, non-degenerate form on $\mathbb{E}_n \times \mathbb{E}_n$.

The space $\mathbb{E}_n := \mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times ... \times \mathbb{E}_{\omega_n}$ endowed with the $\mathbb{K}$-bilinear form $\langle , \rangle_n$ defined in (3.2) will be called a p-adic Hilbert space.

One can easily see that if $n = 1$ we retrieve the p-adic Hilbert space $(\mathbb{E}_\omega, \langle , \rangle)$ that we had introduced in Section 1.

Clearly, the norm (3.1) is not deduced from the $\mathbb{K}$-bilinear form (3.2). However we have the Cauchy-Schwarz inequality:

$$|\langle (u_1, u_2, ..., u_n), (v_1, v_2, ..., v_n)\rangle_n| \leq ||(u_1, u_2, ..., u_n)||_n \ . \ ||(v_1, v_2, ..., v_n)||_n,$$

for all $(u_1, u_2, ..., u_n), (v_1, v_2, ..., v_n) \in \mathbb{E}_n$.

We shall prove the Cauchy-Schwarz inequality above only for $n = 2$ since the proof of the general case follows along the same lines.

Proof: For all $(u, v),\ (x, y) \in \mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2}$,

$$\begin{aligned} |\langle (u, v), (x, y)\rangle_2| &= |\langle u, x\rangle + \langle v, y\rangle| \\ &\leq \max(|\langle u, x\rangle|, |\langle v, y\rangle|) \\ &\leq \max(\|u\| \ \|x\|, \|v\| \ \|y\|) \\ &\leq \max(||(u, v)||_2 \ \|x\|, ||(u, v)||_2 \ \|y\|) \\ &\leq \max(||(u, v)||_2 \ ||(x, y)||_2, ||(u, v)||_2 \ ||(x, y)||_2) \\ &= ||(u, v)||_2 \ ||(x, y)||_2. \end{aligned}$$

□

In view of the above, the direct sum $\mathbb{E}_\omega \times \mathbb{E}_\omega$ is a p-adic Hilbert space.

Let us mention that if $M \subset \mathbb{E}_\omega \times \mathbb{E}_\omega$ is a subspace, then its orthogonal complement $M^\perp$ with respect the $\mathbb{K}$-bilinear form $\langle , \rangle_2$ is defined by

$$M^\perp := \{(u, v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega : \ \langle (u, v), (x, y)\rangle_2 = 0, \ \forall (x, y) \in M\}.$$

One can easily check that $M^\perp$ is closed. This is actually a consequence of the continuity of the bilinear form defined by $\Phi_{(x,y)} : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ and

$$\Phi_{(x,y)}(u, v) = \langle (u, v), (x, y)\rangle_2, \ \ \forall (u, v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega,$$

where $(x, y) \in \mathbb{E}_\omega \times \mathbb{E}_\omega$ is fixed.

Let Γ be the bounded linear operator which goes from $\mathbb{E}_\omega \times \mathbb{E}_\omega$ into itself defined by

$$\Gamma(u, v) := (-v, u), \ \forall (u, v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega.$$

Clearly, the operator Γ defined above satisfies:

(i) $\Gamma^2(u, v) = \Gamma(-v, u) = (-u, -v) = -(u, v), \ \forall (u, v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega$;

(ii) $\Gamma^2(M) = M$ for any subspace M of $\mathbb{E}_\omega \times \mathbb{E}_\omega$.

From (i) it follows that $\Gamma^2 = -I$ (Γ is an unitary operator) where I is the identity operator of $\mathbb{E}_\omega \times \mathbb{E}_\omega$.

As we mentioned in Section 1, the unitary operator Γ enables us to describe yield A^* (if $A \in U_0(\mathbb{E}_\omega)$) in terms of A.

We have

Theorem 3.4: *If $A \in U_0(\mathbb{E}_\omega)$, then $\mathcal{G}(A^*) = [\Gamma \mathcal{G}(A)]^\perp$ where $[\Gamma \mathcal{G}(A)]^\perp$ is the orthogonal complement of* $[\Gamma \mathcal{G}(A)]$.

Proof: (Theorem 3.4). Since the adjoint A^* of A does exist one defines its graph by:

$$\mathcal{G}(A^*) := \{(x, A^*x) \in \mathbb{E}_\omega \times \mathbb{E}_\omega : \ x \in D(A^*)\}.$$

Thus for $(x, y) \in D(A^*) \times D(A)$ one has

$$\begin{aligned} \langle (x, A^*x), \Gamma(y, Ay) \rangle_2 &= \langle (x, A^*x), (-Ay, y) \rangle_2 \\ &= -\langle x, Ay \rangle + \langle A^*x, y \rangle \\ &= 0, \end{aligned}$$

hence $\mathcal{G}(A^*) \subset [\Gamma \mathcal{G}(A)]^\perp$.

Conversely, if $(x, y) \in [\Gamma \mathcal{G}(A)]^\perp$, $\forall z \in D(A)$, then,

$$\begin{aligned} 0 &= \langle (x, y), (-Az, z) \rangle_2 \\ &= -\langle x, Az \rangle + \langle y, z \rangle. \end{aligned}$$

It follows that $\langle Az, x \rangle = \langle z, y \rangle$. Now by uniqueness of the adjoint, we obtain that: $x \in D(A^*)$ and $A^*x = y$, hence $(x, y) \in \mathcal{G}(A^*)$. □

Proof: (Theorem 3.3). Since the adjoint of A^* of A does exist and that $\Gamma \mathcal{G}(A)$ is a subspace of $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then using Theorem 3.4 it follows that $\mathcal{G}(A^*) = [\Gamma \mathcal{G}(A)]^\perp$ is closed, hence A^* is closed. □

Let $A \in U(\mathbb{E}_\omega)$. We define the resolvent set $\rho(A)$ of A as the set of all $\lambda \in \mathbb{K}$ such that the operator $A_\lambda := \lambda I - A$ is one-to-one and that $A_\lambda^{-1} \in B(\mathbb{E}_\omega)$. In this event, the spectrum $\sigma(A)$ of A is defined as the complement of $\rho(A)$ in $\mathbb{K}$.

4 The diagonal operator on $\mathbb{E}_\omega$

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero terms in $\mathbb{K}$ and let $(\mathbb{E}_\omega, \langle , \rangle)$ be the corresponding p-adic Hilbert space. Define the diagonal operator $A \in U(\mathbb{E}_\omega)$ by its coefficients: $a_{i,j} = \lambda_i \delta_{i,j}$ where $\delta_{i,j}$ is the usual Kronecker symbols and $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero elements satisfying

$$\lim_{i\to\infty} |\lambda_i| = \infty. \tag{4.1}$$

Observe that: $D(A) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K} : \lim_i |\lambda_i||x_i|\|e_i\| = 0\}$.

Proposition 4.1: *Under previous assumptions, suppose that (4.1) holds. Then A is self-adjoint. Furthermore, $\rho(A) = \mathbb{K} - \{\lambda_i\}_{i\in\mathbb{N}}$.*

Proof: First of all, let us make sure that the operator A is well-defined. For that note that $|a_{i,i}| = |\lambda_i|$ and that $|a_{i,j}| = 0$ if $i \neq j$. We have, $\lim_i |a_{i,j}|\ |\omega_i|^{1/2} = \lim_{i>j} |a_{i,j}|\ |\omega_i|^{1/2} = 0$, hence A is well-defined. Next, it is clear that $\dfrac{|a_{i,j}|\ |\omega_i|^{1/2}}{|\omega_j|} = |\lambda_i|$ if $i = j$ and 0 if $i \neq j$. It follows that $\|A\| := \sup_{i,j} \left[\dfrac{|a_{i,j}|\ |\omega_i|^{1/2}}{|\omega_j|}\right] = \sup_i |\lambda_i| = \infty$, hence $A \in U(\mathbb{E}_\omega)$.

Let us show that the adjoint A^* of A does exist. This is actually obvious since $\forall i \in \mathbb{N},\ \ \lim_j |a_{i,j}|\ |\omega_j|^{-1/2} = \lim_{j>i} |a_{i,j}|\ |\omega_j|^{-1/2} = 0$. Now the adjoint A^* is defined by $A^* = \displaystyle\sum_{i,j} b_{i,j}\ e'_j \otimes e_i$, where $b_{i,j} = w_i^{-1} w_j a_{j,i} = a_{i,j}$ for all $i,j \in \mathbb{N}$. The latter yields $A = A^*$.

To complete the proof one needs to compute $\rho(A)$. For that, we have to solve the equation $(A - \lambda I)x = y$, where $x = \displaystyle\sum_i x_i e_i \in D(A)$ and $y = \displaystyle\sum_i y_i e_i \in \mathbb{E}_\omega$. Considering the previous equation on $(e_i)_{i\in\mathbb{N}}$ and using the fact A is self-adjoint it follows that: $\forall i \in \mathbb{N},\ (\lambda_i - \lambda).\langle e_i, x\rangle = \langle e_i, y\rangle$. Equivalently,

$\forall i \in \mathbb{N},\ (\lambda_i - \lambda).\omega_i x_i = \omega_i y_i$. Now if $\forall i \in \mathbb{N},\ \lambda_i \neq \lambda$ the previous equation has a unique solution x. Moreover, $x = (A-\lambda)^{-1}y = \sum_i \frac{y_i}{\lambda_i - \lambda} e_i$.

Let us show that $(A-\lambda)^{-1}y$ given above is well-defined. For that it is sufficient to prove that $\lim_i \left[\frac{|y_i|}{|\lambda_i - \lambda|} \ \|e_i\| \right] = 0$. According to (4.1) the sequence $\left(\frac{1}{|\lambda_i - \lambda|} \right)_{i \in \mathbb{N}}$ is bounded, hence $\lim_i \left[\frac{|y_i|}{|\lambda_i - \lambda|} \ \|e_i\| \right] = 0$.

It remains to find conditions on λ so that x defined above belongs to $D(A)$. For that it is sufficient to find conditions so that: $\lim_i \frac{|y_i|}{|\lambda_i - \lambda|} \|Ae_i\| = \lim_i \frac{|\lambda_i|}{|\lambda_i - \lambda|} \ |y_i| \ \|e_i\| = 0$. Actually, since $\lim_i |y_i| \ \|e_i\| = 0$ one has:

$$0 \leq \lim_i \frac{|y_i|}{|\lambda_i - \lambda|} \|Ae_i\| \leq \lim_i \frac{|\lambda_i|}{|\ |\lambda_i| - |\lambda|\ |} \ . \ \lim_i |y_i| \ \|e_i\| = 0.$$

From (4.1) it follows that $(A-\lambda)^{-1} \in B(\mathbb{E}_\omega)$ for all $\lambda \neq \lambda_i,\ i \in \mathbb{N}$. Actually, $\|(A-\lambda)^{-1}\| \leq \sup_{i \in \mathbb{N}} \left[\frac{1}{|\lambda_i - \lambda|} \right] < \infty$, hence $\rho(A) = \mathbb{K} - \{\lambda_i\}_{i \in \mathbb{N}}$. $\square$

Remark 4.2: Let us notice that the domain $D(A)$ of the diagonal operator A may not be equal to the whole of $\mathbb{E}_\omega$. To see it, suppose that the ground field $\mathbb{K}$ *contains a square of each of its elements* and choose $\tilde{x} = (\tilde{x}_i)_{i \in \mathbb{N}}$ where $\tilde{x}_i$ is given by: $\tilde{x}_i^2 = \frac{1}{\lambda_i^2 \ \omega_i}$ for all $i \in \mathbb{N}$. According to the assumption on the field $\mathbb{K}$ it is clear that for all $i \in \mathbb{N}$, $\tilde{x}_i$ lies in $\mathbb{K}$. Now, $\tilde{x} \in \mathbb{E}_\omega$ since $\lim_{i \to \infty} |\tilde{x}_i| \|e_i\| = \lim_{i \to \infty} \frac{1}{|\lambda_i|} = 0$, by (4.1). Meanwhile, one can easily see that $\tilde{x} \notin D(A)$ since $\lim_i \ |\tilde{x}_i| \ |\lambda_i| \|e_i\| = 1 \neq 0$.

5 The equation $Ax = y$ on $\mathbb{E}_\omega$

5.1 The bounded case

Let $A \in B(\mathbb{E}_\omega)$ be a bounded linear operator on $\mathbb{E}_\omega$. Consider the existence and uniqueness of a solution $x = \sum_{i \in \mathbb{N}} x_i e_i \in \mathbb{E}_\omega$ to the equation:

$$Ax = y, \tag{5.1}$$

where $y = \sum_{i\in\mathbb{N}} y_i e_i \in \mathbb{E}_\omega$ is an arbitrary element.

We give sufficient conditions ensuring the existence and uniqueness of a solution to (5.1). Next this will be applied to the perturbations of bases. In the last part of this section, we consider the existence and uniqueness of solutions to(5.1) when A is a (particular) unbounded linear operator.

Theorem 5.1: *Let $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be a bounded linear operator on $\mathbb{E}_\omega$. Suppose that $\|I - A\| < 1$, then (5.1) has a unique solution given by $x = \sum_{n\in N} (I - A)^n y$. Moreover, the solution x satisfies: $\|x\| = \|y\|$.*

Proof: The first part of the proof follows along the same lines as in the classical setting. Indeed, since A belongs to $B(\mathbb{E}_\omega)$ which is Banach algebra with unit and since $\|I - A\| < 1$ it follows that A is invertible. Thus, setting $B = I - A$ it follows that the only solution to (5.1) is given by $x = (I - B)^{-1}y$, that is, $x = \sum_{n\in\mathbb{N}} B^n y = \sum_{n\in\mathbb{N}} (I - A)^n y$. Now from the assumption $\|I - A\| < 1$ it is clear that $\|A\| = 1$, hence $\|x\| = \|Ax\| = \|y\|$. □

In view of the above it clear that $0 \in \rho(A)$. In addition to that, Theorem 5.1 is still valid in any arbitrary non-Archimedean Banach space $\mathbb{E}$.

Example 5.2: Let us illustrate Theorem 5.1 by an example. Consider the p-adic space $(\mathbb{E}_\omega, \langle,\rangle)$ over $\mathbb{Q}_p$ corresponding to $\omega_k = p^k$ for all $k \in \{1, 2, 3, ...\}$.

Now define the operator A on $\mathbb{E}_\omega$ by its coefficients a_{ij}:

$$a_{ij} = \begin{cases} p^{i-j} & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{if } i < j. \end{cases} \quad \text{and} \quad |a_{ij}| = \begin{cases} p^{-(i-j)} & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{if } i < j. \end{cases}$$

First of all, let us make sure that $A = \sum_{i,j\geq 1} a_{ij}\, e'_j \otimes e_i$ is well-defined. Indeed, one has: $\lim_i |a_{ij}|\, \|e_i\| = \lim_{i>j} p^{-(i-j)} \,.\, p^{-i/2} = 0$. Next, one shows that A is bounded. For that, we note that

$$\frac{|a_{ij}|\, \|e_i\|}{\|e_j\|} = \begin{cases} p^{-\frac{3}{2}\,(i-j)} & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{if } i < j, \end{cases}$$

and $\dfrac{|a_{ij}|\ \|e_i\|}{\|e_j\|} \leq 1$ for all $i, j = 1, 2, 3, ...$ Thus, $\|A\| := \sup\limits_{i,j} \dfrac{|a_{ij}|\ \|e_i\|}{\|e_j\|} \leq 1$, hence A is bounded. Actually, one can easily show that the adjoint A^* does exist. Nevertheless, A is not self-adjoint.

Consider $B = I - A$, where I is the identity operator of $\mathbb{E}_\omega$. It is then clear that $B = \sum\limits_{i,j \geq 1} b_{ij}\ e'_j \otimes e_i$, where

$$b_{ij} = \begin{cases} -p^{i-j} & \text{if} \quad i > j \\ 0 & \text{if} \quad i \leq j. \end{cases} \quad \text{and} \quad |b_{ij}| = \begin{cases} p^{-(i-j)} & \text{if} \quad i > j \\ 0 & \text{if} \quad i \leq j. \end{cases}$$

Similarly,

$$\frac{|b_{ij}|\ \|e_i\|}{\|e_j\|} = \begin{cases} p^{-\frac{3}{2}\ (i-j)} & \text{if} \quad i > j \\ 0 & \text{if} \quad i \leq j. \end{cases}$$

Thus it is clear that $\dfrac{|b_{ij}|\ \|e_i\|}{\|e_j\|} \leq p^{-\frac{3}{2}}$, that is, $\|B\| \leq p^{-\frac{3}{2}} < 1$, hence for each $y \in \mathbb{E}_\omega$, (5.1) has a unique solution x given by $x = \sum\limits_{n \in \mathbb{N}} B^n y$. Moreover, $\|x\| = \|y\|$, by Theorem 5.1.

Another application to Theorem 5.1 concerns the perturbation of bases for $\mathbb{E}_\omega$. This is discussed in the next subsection.

5.2 Application to the perturbation of bases on $\mathbb{E}_\omega$

We now apply Theorem 5.1 to the perturbation of bases. Let $(e_i)_{i \in \mathbb{N}}$ be the canonical basis for $\mathbb{E}_\omega$ and let $(f_i)_{i \in \mathbb{N}} \subset \mathbb{E}_\omega$ be a family of vectors.

We require the following assumption:

$$\sup_{n \in \mathbb{N}} \left[\frac{\|e_n - f_n\|}{\|e_n\|} \right] < 1. \tag{5.2}$$

Corollary 5.3: *Let $(e_i)_{i \in \mathbb{N}}$ be the canonical basis for $\mathbb{E}_\omega$ and let $(f_i)_{i \in \mathbb{N}} \subset \mathbb{E}_\omega$ be a family of vectors. Suppose that (5.2) holds true. Then $(f_i)_{i \in \mathbb{N}}$ is also a basis for $\mathbb{E}_\omega$.*

Proof: One defines a linear operator $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ by: $\forall x = \sum\limits_{k \in \mathbb{N}} x_k e_k$, $A(\sum\limits_{k \in \mathbb{N}} x_k e_k) = \sum\limits_{k \in \mathbb{N}} x_k f_k$. In particular, $Ae_k = f_k,\ \forall k \in \mathbb{N}$. Now from (5.2) one has $\|e_n - f_n\| < \|e_n\|,\ \forall n \in \mathbb{N}$, that is $\|e_n\| = \|f_n\|,\ \forall n \in \mathbb{N}$.

In view of the above, we have:

(i) The decomposition $Ax = \sum_{k\in\mathbb{N}} x_k f_k$ is well-defined;

(ii) $\|A\| := \sup_{n\in\mathbb{N}} \frac{\|Ae_n\|}{\|e_n\|} = \sup_{n\in\mathbb{N}} \frac{\|f_n\|}{\|f_n\|} = 1$, in particular A is a bounded linear operator on $\mathbb{E}_\omega$. Furthermore, $\|I - A\| := \sup_{n\in\mathbb{N}} \frac{\|e_n - Ae_n\|}{\|e_n\|} < 1$.

According to Theorem 5.1 the operator A is a bijective and isometric. Since in the p-adic context, an isometric linear bijection transforms an orthogonal basis into an orthogonal basis, then $(f_n)_{n\in\mathbb{N}}$ is an orthogonal basis for $\mathbb{E}_\omega$. □

5.3 The unbounded case

Let $(\varpi_n) \subset \mathbb{K}$ be sequence of nonzero terms and let $(f_i)_{i\in\mathbb{N}}$ be another orthogonal basis for $\mathbb{E}_\omega$. We shall suppose that there exists a nontrivial isometric linear bijection T (see Subsection 5.2) such that $Te_i = f_i$ for all $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical orthogonal basis for $\mathbb{E}_\omega$. As a consequence, $|\omega_i| = |\varpi_i|$ for all $i \in \mathbb{N}$. However, let us notice that the fact $|\omega_i| = |\varpi_i|$ for each $i \in \mathbb{N}$, the sequences $(\omega_i)_{i\in\mathbb{N}}$ and $(\varpi_i)_{i\in\mathbb{N}}$ need not be equal.

Since $(f_i)_{i\in\mathbb{N}}$ is an orthogonal basis for $\mathbb{E}_\omega$, for each $x \in \mathbb{E}_\omega$, $x = \sum_{n\in\mathbb{N}} x_n f_n$ with $\lim_{n\to\infty} |x_n|\|f_n\| = 0$, where $\|f_i\| =: |\varpi_i|^{1/2} = |\omega_i|^{1/2}$, $\forall i \in \mathbb{N}$, and $\langle f_i, f_j\rangle = \varpi_i \delta_{i,j}$ ($\delta_{i,j}$ being the usual Kronecker symbol).

We shall study the existence and uniqueness of solutions to (5.1) in the particular case where A is the linear operator defined by

$$D(A) = \{x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty} |x_i|\ |\mu_i|\ \|f_i\| = 0\},$$

and

$$Ax := \sum_{i\in\mathbb{N}} \mu_i\ x_i\ f_i, \quad \text{for each} \quad x = (x_i)_{i\in\mathbb{N}} \in D(A),$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ is a given sequence of nonzero terms.

It is not hard to see that A is well-defined on $D(A)$ and that it is an element of $U(\mathbb{E}_\omega)$. Now since $f_m \in \mathbb{E}_\omega$ it can be written as: $\forall\ m \in \mathbb{N}$,

$$f_m = \sum_i a_{im} e_i \quad \text{with} \quad \lim_i |a_{im}|\ \|e_i\| = 0. \tag{5.3}$$

We require the following assumption:

$$\lim_{m\to\infty}\left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right]=0. \tag{5.4}$$

Theorem 5.4: *Under assumptions (5.3)-(5.4), the equation (5.1) has a unique solution* $x=\sum_{m\in\mathbb{N}} x_m e_m \in D(A)$ *where*

$$x_m=\frac{1}{\mu_m\ \varpi_m}\ \left[\sum_{n\in\mathbb{N}}\ \omega_n\ y_n a_{nm}\right],\quad \forall m\in\mathbb{N}.$$

Proof: Let $y=\sum_{n\in\mathbb{N}} y_n e_n \in \mathbb{E}_\omega$ $(\lim_{n\to\infty}|y_n|\ |\omega_n|^{1/2}=0)$. Now from (5.3) it is clear that $\langle Ax, f_m\rangle=\langle\sum_{n\in\mathbb{N}} y_n e_n, f_m\rangle$, hence $\mu_m\varpi_m x_m=\sum_{n\in\mathbb{N}}\omega_n\ y_n\ a_{nm}$. It follows that a solution x to (5.1) is given by its coefficients:

$$x_m=\frac{1}{\mu_m\ \varpi_m}\ \left[\sum_{n\in\mathbb{N}}\ \omega_n\ y_n\ a_{nm}\right],\quad \forall m\in\mathbb{N}.$$

Since $(f_s)_{s\in\mathbb{N}}$ is an orthogonal basis for $\mathbb{E}_\omega$ it is then clear that $N(A)=\{0\}$ ($N(A)$ being the kernel of A), hence the above solution is unique. Now from (5.3) it is clear that x_m is well-defined. The remaining task now is to prove that the solution $x\in D(A)$. Indeed from the expression of x_m one has:

$$\begin{aligned}|x_m|\ |\mu_m|\ \|f_m\| &= |x_m|\ |\mu_m|\ |\varpi_m|^{1/2}\\ &= \left[\frac{|\sum_{n\in\mathbb{N}}\omega_n\ y_n\ a_{nm}|}{|\varpi_m|^{1/2}}\right]\\ &\le \left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right]\ .\ \|y\|.\end{aligned}$$

The previous inequality and (5.4) yield: $\lim_{m\to\infty} |x_m||\mu_m|\|f_m\| = 0$, hence $x \in D(A)$. $\square$

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