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Abstract

In this paper, we establish an one-to-one mapping between complex-valued functions defined on $R^+ \cup \{0\}$ and complex-valued functions defined on p-adic number field Q_p, and introduce the definition and method of Weyl-Heisenberg frame on hormonic analysis to p-adic anylysis .

1 Introduction

Wavelet transform was introduced to the field of p-adic numbers in [1]. In [4],[3] some theory of wavelet analysis and affine frame introduced to the field of p-adic numbers,respectively,on the basis of a mapping $P : R^+ \cup \{0\} \to Q_p(field\ of\ \text{p}-adic\ numbers)$. This paper considers on the basis of the mapping P, gives the Weyl-Heisenberg frame in field of p-adic numbers.

The field Q_p of the p-adic numbers is defined as the completion of field Q of rational with respect to the p-adic metric induced by the p-adic norm $|.|_p$,(see [5]). A p-adic number $x \neq 0$ is uniquely represented in the canonical form

$$x = p^{-r} \sum_{k=0}^{\infty} x_k p^k, \quad |x|_p = p^r \tag{1.1}$$

where p is prime and $r \in Z$ (Z is integer set) , $0 \leq x_k \leq$p-1, $x_0 \neq 0$. For $x, y \in Q_p$, we define $x < y$,either when $|x|_p < |y|_p$ or when $|x|_p = |y|_p$, but there exists an integer j such that $x_0 = y_0, \cdots, x_{j-1} = y_{j-1}, x_j < y_j$ from viepoint of (1.1).By interval $[a, b]$, we mean the set defined by $\{x \in Q_p | a \leq x \leq b\}$.

It is known that if $x = p^r \sum_{k=0}^{n} x_k p^{-k} \in R^+ \cup \{0\} and\ x_0 \neq 0,\ 0 \leq x_k \leq p-1, k = 1, 2, \cdots$, then there is another expression;

$$x = p^r(\sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1)p^{-n} + (p-1) \sum_{k=n+1}^{\infty} p^{-k}) \tag{1.2}$$

But we won't use that expression (1.2) in this paper.

A mapping $P : R^+ \cup \{0\} \rightarrow \mathbf{Q_p}$ was introduced in [4],[3],[2], as for $x = p^r \sum_{x=0}^{\infty} x_k p^{-k} \in R,\ x_0 \neq 0,\ 0 \leq x_k \leq p-1,\ k = 1, 2, \cdots$

$$P(x) = p^{-r} \sum_{k=0}^{\infty} x_k p^k$$

Let $M_p = P(M_{_R})$,

$$M_{_R} = \{x_{_R} | x_{_R} = p^r \sum_{k=0}^{n-1} x_k p^{-k} + (n-1)p^{-n} + (p-1) \sum_{k=0}^{\infty} p^{-k}, n \in Z^+ \cup \{0\}\}$$

In the following to distinguish between real number field R and p-adic number field Q_p, number in R denotes by the subscript R, and number with the subscript p belongs to Q_p. For example $x_{_R}, a_{_R}, b_{_R}$ in R; $x_{_p}, a_{_p}$ in Q_p.

In [2] a measure is constructed using the mapping P from $R^+ \cup \{0\}$ into $Q_p \backslash M_p$ and Lebesgue measure on $R^+ \cup \{0\}$, the symbol $\sum$ is the set of all compact subsets of Q_p, and S is the $\sigma - ring$ generated by $\sum$.

Definition 1.1:[2] Let $E \in S$, and put $E_P = E \backslash M_P$, and $E_R = P^{-1}(E_P)$.If E_R is a measurable set on $R^+ \cup \{0\}$, the we call E is a measurable set on Q_p, and define a set function $\mu_p(E)$ on S

$$\mu_p(E) = \frac{1}{p} \mu(E_R)$$

where $\mu(E_R)$ is the Lebesgue measure on E_R.This $\mu_p(E)$ is called the measure on E.

By the Definition 1.1, some examples can given immediately:
(1) Let $a_p, b_p \in Q_P$, then $\mu_p\{[a_p, b_p]\} = (b_R - a_R)/p$
(2) Let $B_r(0) = \{x_p | |x_p|_p \leq p^r, x_p \in Q_p\}$, then $\mu_p\{B_r(0)\} = p^r$
(3) Let $S_r(0) = \{x_p | |x_p|_p = p^r, x_p \in Q_p\}$, then $\mu_p\{S_r(0)\} = p^r(1 - \frac{1}{P})$
(4) $\mu_p\{M_p\} = 0$

According to the above definition 1.1 of measure, we can define integration over measurable sets E in Q_p

$$\int_E f(x_p) d\mu_p(x_p) \quad or \quad \int_E f(x_p) dx_p$$

By the definition 1.1 of measure we have following theorem.

Theorem 1.2:_(see [2]) Suppose $f(x_p)$ is a Complex-Valued function on Q_p, the $f(x_p)$ is integrable over the interval $[a_p, b_p](a_p, b_p \in Q_p)$, if and if the real function $f_R(x_R)$ defined on $R^+ \cup \{0\}$ is Lebesgue integrable over the interval $[a_R, b_R]$, and_

$$\int_{[a_p,b_p]} f(x_p)dx_p = \frac{1}{p}\int_{a_R}^{b_R} f(x_R)dx_R \tag{1.3}$$

where $f_R(x_R)$ is defined

$$f(x_p) = f(P \circ P^{-1}(x_p)) = (f \circ P)(x_R) \stackrel{def}{=} f_R(x_R), x_p = P(x_R) \in Q_p \backslash M_p$$

2 Weyl-Heisenberg frame on p-adic number field

In real analysis, if there exists constants A and B, $A, B > 0$, such that

$$A\|f\|^2_{L^2(R)} \leq \sum_{m,n} |(f, g_{m,n})_{L^2(R)}|^2 \leq B\|f\|^2_{L^2(R)}$$

holds for $\forall f \in L^2(R)$, then $g_{mn}(x)$ is called the Weyl-Heisenberg frame, where $g \in L^2(R)$,$p_0, q_0 \in R$,$g_{mn}(x) = g(x - nq_0)e^{2\pi imp_0x}, m, n \in Z$ and $(f, g_{mn})_{L^2(R)}$ is inner product in $L^2(R)$,

$$(f, g_{mn})_{L^2(R)} = \int_R f(x)\overline{g_{mn}}(x)dx.$$

In this section we give the definition of a Weyl-Heisenberg frame in Q_p by

$$g_{mn}(x_p) = g(\alpha_{mn}(x_p) - x_p)\exp(2\pi imp_0\rho(x_p)) \tag{2.1}$$

where

$$\alpha_{mn}(x_p) = P(|x_R + nq_0|) + x_p \tag{2.2}$$

and $m, n, p_0, q_0 \in Z$, $x_R = P^{-1}(x_p)$,$x_p \in Q_p\backslash M_p$. If $g_{mn}(x_p)$ satisfies the frame condition :

$$A\|f\|^2_{L^2} \leq \sum_{m,n} |(f, g_{m,n})_{L^2}|^2 \leq B\|f\|^2_{L^2}, A, B > 0, \forall f \in L^2(Q_p)$$

then we have

$$f(x_p) = \sum_{m,n} (f, g^*_{m,n})_{L^2} g_{m,n}(x_p) = \sum_{m,n} (f, g_{m,n}) g^*_{m,n}(x_p), \; x_p \in Q_p$$

where $\{g^*_{m,n}\}$ is the dual frame of $\{g_{m,n}\}$:

$$g^*_{m,n} = S^{-1} g_{m,n}$$

and S is the frame operator:

$$Sf = \sum_{m,n} (f, g_{m,n})_{L^2} g_{m,n}$$

and

$$(f, g_{mn})_{L^2} = \int_{Q_p} f(x_p) g_{mn}(x) dx$$

Theorem 2.1: *Suppose $f, g \in L^2(Q_p)$ are complex-valued functions defined on Q_p , $p_0 = p^{r_p}, q_0 = p^{r_q}, r_p, r_q \in Z$, if support $\widehat{g_R}(|w|) \subset [\frac{-1}{2q_0}, \frac{1}{2q_0}]$, and $\exists A, B > 0$, such that*

$$A \leq \sum_{m \in Z} |\widehat{g_R}(|w - mp_0|)|^2 \leq B, w \in R \;, \forall w \neq 0,$$

then functions $g_{mn}(x_p)$ defined by (2.1) construct a frame of $L^2(Q_p)$, where $g_R(|t|) = (g \circ P)(|t|) = g(x_p), (P(|t|) = x_p, g_R = g \circ P), t \in R$, and $\widehat{g_R}(|w|)$ is the Fourier transform of $g_R(|t|)$ in real analysis

$$\widehat{g_R}(|w|) = \int_R g_R(|t|) \exp(-2\pi i w t) dt$$

Proof. From formula (2.2) or $g_(\alpha_{mn}(x_p) - x_p) = (g \circ P)(|x_R + nq_0|) = g_R(|x_R + nq_0|)$, for $\forall f \in L^2(Q_p)$, we have

$$\sum_{m,n \in Z} |(f, g_{m,n})_{L^2}|^2 = \sum_{m,n \in Z} | \int_{Q_p} f(x_p) \overline{g_{m,n}}(x_p) \mathrm{d}x_p|^2$$

$$= \sum_{m,n \in Z} | \int_{Q_p} f(x_p) \overline{g}(\alpha^{(n)}_m(x_p) - x_p) \exp(-2\pi i m p_0 \rho(x_p))|^2$$

$$= \frac{1}{p} \sum_{m,n \in Z} | \int_{R^+} f_R(x_R) \overline{g_R}(|x_R + nq_0|) \exp(-2\pi i m p_0 x_R) \mathrm{d}x_R|^2 \tag{2.3}$$

where we used (1.3) and $f_R(x_R) = (f \circ P^{-1})(x_p)$ in section 1
Let

$$f_R^+(x) = \begin{cases} f_R(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

From (2.3), we obtain

$$\sum_{m,n \in Z} |(f, g_{m,n})|^2$$

$$= \sum_{m,n \in Z} | \int_R f_R^+(x)(\overline{g_R}(|x + nq_0|) \ \exp(-2\pi i m p_0 x) dx \ |^2$$

$$= \sum_{m,n \in Z} | \int_R \widehat{f_R^+}(w) \overline{[g_R(|\cdot + nq_0|) \exp(2\pi i m p_0 \cdot)]^\wedge}(w) \mathrm{d}w|^2 \tag{2.4}$$

where sign "$\cdot$" is the argument on the function, for Fourier transform. But

$$[g_R(|\cdot + nq_0|) \exp(2\pi i m p_0 \cdot)]^\wedge(w)$$

$$= \widehat{g_R}(|w - mp_0|) \exp(2\pi i n q_0 (w - mp_0))$$

Hence from the support $\hat{g} \subset [-\frac{1}{2q_0}, \frac{1}{2q_0}]$ in condition of the theorem and (2.4) we have

$$\sum_{m,n \in Z} |(f, g_{mn})|^2$$

$$= \sum_{m,n \in Z} | \int_R \widehat{f_R^+}(w + mp_0) \widehat{g_R}(|w|) \exp(-2\pi i n q_0 w) \mathrm{d}w|^2$$

$$= \sum_{m,n \in Z} | \int_{-\frac{1}{2q_0}}^{\frac{1}{2q_0}} \widehat{f_R^+}(w + mp_0) \widehat{g_R}(|w|) \exp(-2\pi i n q_0 w) \overline{dw}|^2 \tag{2.5}$$

We know that

$$c_n = q_0 \int_{\frac{-1}{2q_0}}^{\frac{1}{2q_0}} \widehat{f_R^+}(w+mp_0)\widehat{g_R}(|w|)\exp(-2\pi i n q_0 w)\mathrm{d}w, n \in Z$$

are Fourier coefficient of $\widehat{f_R}(w+mp_0)\widehat{g_R}(|w|)$ on $[\frac{-1}{2q_0}, \frac{1}{2q_0}]$. Hence by virtue of Parseval equality we have

$$\sum_{n\in Z} |c_n|^2$$

$$= \sum_{n=-\infty}^{\infty} |q_0 \int_{\frac{-1}{2q_0}}^{\frac{1}{2q_0}} \widehat{f_R^+}(w-mp_0)\widehat{g_R}(|w|)\exp(-2\pi i n q_0 w)\mathrm{d}w|^2$$

$$= q_0 \int_{-\frac{1}{2q_0}}^{\frac{1}{2q_0}} |\widehat{f_R^+}(w+mp_0)\widehat{g_R}(|w|)|^2 \mathrm{d}w$$

$$= q_0 \int_R |\widehat{f_R^+}(w+mp_0)\widehat{g_R}(|w|)|^2 dw$$

$$= q_0 \int_R |\widehat{f_R^+}(w)\widehat{g_R}(|w-mp_0|)|^2 \mathrm{d}w \tag{2.6}$$

Comparing (2.5) and (2.6), we have

$$\sum_{m,n\in Z} |\ (f_p, g_{m,n})_{L^2}\ |^2 = \frac{1}{q_0}\int_R |\widehat{f_R^+}(w)|^2 G(w)\mathrm{d}w \tag{2.7}$$

where

$$G(w) = \sum_{m\in Z} |\widehat{g_R}(|w-mp_0|)|^2$$

Finally, by virtue of the conditions of the theorem, we have

$$\sum_{m\in Z, n\in Z} |(f, g_{m,n})_{L^2}|^2 = \begin{cases} \geq \frac{A}{q_0}\int_R |\widehat{f_R^+}(w)|^2\mathrm{d}w \\ \leq \frac{B}{q_0}\int_R |\widehat{f_R^+}(w)|^2\mathrm{d}w \end{cases}$$

But

$$\int_R |\widehat{f_R^+}(w)|^2\mathrm{d}w = \int_R |f_R^+(x)|^2\mathrm{d}x = \int_{R^+} |f_R(x_R)|^2\mathrm{d}x_R = \int_{Q_p} |f(x_p)|^2\mathrm{d}x_p = \|f\|_{L^2}.$$

Hence we completed our proof.

3 Dual frame

In the section, we will give a formula to calculate the dual frame. By (2.7), we have

$$(Sf,f)_{L^2} = \sum_{m,n\in Z} |(f,g_{m,n})_{L^2}|^2 = \frac{1}{q_0}\int_R \widehat{f_R^+}(w)G(w)\overline{\widehat{f_R^+}}(w)dw$$

$$= \frac{1}{q_0}\int_R (\widehat{f_R^+}(\cdot)G(\cdot))^\vee(x)\overline{f_R^+}(x)\mathrm{d}x$$

$$= \frac{1}{q_0}\int_{R^+} (\widehat{f_R^+}(\cdot)G(\cdot))^\vee(x_R)\overline{f_R(x_R)}\mathrm{d}x_R$$

where sign " v " is the inverse Fourier transform. Therefore

$$(Sf,f)_{L^2} = \frac{1}{q_0}\int_{Q_p} (\widehat{f_R^+}(\cdot)G(\cdot))^\vee(P^{-1}(x_p))\overline{f(x_p)}\mathrm{d}x_p$$

$$= \frac{1}{q_0}((\widehat{f_R^+}(\cdot)G(\cdot))^\vee(P^{-1}(x_p)), f(x_p))_{L^2}$$

Since f is an arbitrary function in $L^2(Q_p)$, we conclude that

$$(Sf)(x_p) = \frac{1}{q_0}(\widehat{f_k^+}(\cdot)G(\cdot))^\vee(P^{-1}(x_p))$$

or for $x \in R^+\bigcup\{0\}$ we conclude that

$$(Sf)_R(x_R) = \frac{1}{q_0}(\widehat{f_R^+}(\cdot)G(\cdot))^\vee(x_R), x_R \geq 0 \tag{3.1}$$

where $(Sf)_R = (Sf)P^{-1}$

Bases on (3.1), we will extend the domain of $(Sf)_R(x_R)$ from $R^+\bigcup\{0\}$ onto R such that $(Sf)_R(t), t \in R$ is an even function on R. Taking Fourier transform on both sides of (3.1), we have

$$((Sf)_R)^\wedge(w) = \frac{1}{q_0}\widehat{f_R^+}(w)G(w) \tag{3.2}$$

After replace f with $S^{-1}f$ in formula (3.2), we have

$$\widehat{f_R}(w) = \frac{1}{q_0}\{(S^{-1}f)_R^+\}^\wedge(w)G(w)$$

Which leads to

$$\{(S^{-1}f)_R^+\}^\wedge(w) = \frac{q_0\widehat{f_R}(w)}{G(w)} \tag{3.3}$$

Then we take Fourier inverse transformation on both sides of (3.3), we have

$$(S^{-1}f)_R^+(x) = \{\frac{q_0\widehat{f_R}(\cdot)}{G(\cdot)}\}^\vee(x)$$

So, for $x \geq 0$,

$$(S^{-1}f)_R(x_R) = \{\frac{q_0\widehat{f_R(\cdot)}}{G(\cdot)}\}^\vee(x_R)$$

is valied or

$$(S^{-1})f(x_p) = \{\frac{q_0\widehat{f_R(\cdot)}}{G(\cdot)}\}^\vee(P^{-1}(x_p)) \tag{3.4}$$

Finally, let $f(x_p) = g_{mn}(x_p)$ in formula (3.4) , we obtain

$$g_{m,n}^*(x_p) = \{\frac{q_0\widehat{f_R(\cdot)}}{G(\cdot)}\}^\vee(P^{-1}(x_p))$$

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