ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 1, n° 1 (1994), p. 21-26
<http://www.numdam.org/item?id=AMBP_1994__1_1_21_0>

Ann. Math. Blaise Pascal, Vol. 1 , N° 1, 1994, pp. 21-26

GENERATING FUNCTION AND ORTHOGONALITY PROPERTY OF A CLASS OF POLYNOMIALS OCCURRING IN QUANTUM MECHANICS

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ABSTRACT : In this paper, we present a generating function and an orthogonality property of a class of polynomials occuring in quantum mechanics.

Key words : Generating function, Orthogonality property, Hermite Polynomials, Quantum mechanics.

AMS (MOS) : Subject classification : 33C25 , 81

INTRODUCTION : The object of this paper is to present a generating function and an orthogonality property of the polynomials ${}_1F_1(-n; b + 3/2; x^2)$, which occurs in the radical wave function of isotropic harmonic oscillator [4, p. 36, (6.60)].

The generating function for the polynomials ${}_1F_1(-n; b+3/2; x^2)$ has been obtained as a particular case of the generating function of B-polynomials, which has recently been defined by the author [2]. We obtain the orthogonality property of the polynomials ${}_1F_1(-n; b+3/2; x^2)$ as a bonus in our attempts to establish an orthogonality property of B-polynomials. We shall use the symbol $H_n^b(x)$ to denote the polynomials ${}_1F_1(-n; b+3/2; x^2)$.

It is interesting to note that the polynomials $H_n^b(x)$ appear to lead to the generalization of the Hermite polynomilas $H_n(x)$ [5, p. 380, (25)].

We visualize at least three orthogonality properties of the B-polynomials for different weight functions on different intervals. Howerver, we have not been successful to establish any of them. The proofs are difficult in view of the general nature of B-polynomials.

In what follows for sake of brevity, the symbol a_p is used to denote $a_1,...,a_p$, the symbol $1-a_p-m$ is used to denote $1-a_1-m,...,1-a_p-m$ and the notation $\prod_{j=1}^{p}(a_j)_m$ stands for the product $(a_1)_m...(a_p)_m$. Further, the expression

$$ {}_pF_q\left[\begin{matrix} a_p; z \\ b_q \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n z^n}{(b_1)_n...(b_q)_n n!} \tag{1.0} $$

is known as the generalized hypergeometric series or generalized hypergeometric function. Here p and q are positive integers or zero, and we assume that the variable z, the numerator parameters $a_1,...,a_p$ and the denominator parameters $b_1,...,b_q$ take on complex values, provided that no $b_j (j = 1,...,q)$ is zero or a negative integer.

Recently [2], we have defined the B-polynomials :

$$ B_m(x) = \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} r+q+1^F s+p\left[\begin{matrix} c_r, 1-b_q-m, -m; \frac{\beta}{\alpha}x(-1)^{p-q-1} \\ d_s, 1-a_p-m \end{matrix}\right](\alpha)^m, \tag{1.1} $$

by means of the generating function :

$$ {}_pF_q\left[\begin{matrix} a_p; \alpha t \\ b_q \end{matrix}\right] {}_rF_s\left[\begin{matrix} c_r; \beta x t \\ d_s \end{matrix}\right] = \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} $$

$$ \cdot r+q+1^F s+p\left[\begin{matrix} c_r, 1-b_q-m, -m; \frac{\beta}{\alpha}x(-1)^{p-q-1} \\ d_s, 1-a_p-m \end{matrix}\right] \tag{1.2} $$

<u>The generating function of the polynomials $H_n^b(x)$</u> :

In (1.2), putting $\alpha = \beta = 1, p = q = r = 0, s = 1, d_1 = b + 3/2$, and setting t^2 for t and $-x^2$ for x, we obtain the generating function for ${}_1F_1(-m; b+3/2; x^2)$:

$$ {}_0F_0(-;-;t^2)\,{}_0F_1(-;b+3/2;-t^2x^2) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!}\,{}_1F_1(-m;b+3/2;x^2) \tag{1.3} $$

In (1.3), setting ${}_0F_0(-;-;t^2) = e^{t^2}, {}_0F_1(-;b+3/2;-t^2x^2) =$ $(tx)^{b/2+1/4}\Gamma(b+3/2)J_{b+1/2}(2\sqrt{t}x)$ and ${}_1F_1(-m,b+3/2;x^2) = H_m^b(x)$, we have

$$ e^{t^2}(tx)^{b/2+1/4}\Gamma(b+3/2)J_{b+1/2}(2\sqrt{t}x) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} H_m^b(x) \tag{1.4} $$

The following formulae are required in the proofs :

The integral :

$$ \int_{-\infty}^{\infty} x^{2u} e^{-x^2}{}_pF_q\left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix}\right] dx $$

$$ = \Gamma(u+1/2)\,{}_{p+1}F_q\left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix}\right], \tag{1.5} $$

where $p < q+1$ (or $p = q+1$ and $|z| < 1), u = 0,1,2,.....$

The integral (1.5) can easily be established by expressing the hypergeometric function in the integrand as [1 , p. 322, (10.1)] and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and summation involved in the process, and evaluating the inner-integral with the help of the following integral :

$$ \int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \Gamma(n+1/2), \ n = 0,1,2,\ldots \tag{1.6} $$

The integral :

$$ \int_{-\infty}^{\infty} x^{2u} e^{-x^2}{}_pF_q\left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix}\right] {}_rF_s\left[\begin{matrix} c_r; yx^2 \\ d_s \end{matrix}\right] dx $$

$$ = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{r}(c_j)_m}{\prod_{j=1}^{s}(d_j)_m} \frac{y^m}{m!} \Gamma(m+u+1/2)\,{}_{p+1}F_q\left[\begin{matrix} a_p, m+u+1/2; x \\ b_q \end{matrix}\right], \tag{1.7} $$

where in addition to the conditions of (1.5), $r < s+1$ (or $r = s+1$ and $|y| < 1$).

To derive (1.7), we use the series representation of ${}_rF_s$ interchange the order of integration and summation and evaluate the resulting integral with the help of (1.5).

The Vandermonde's theorem [3 , p. 110, (4.1.2)] :

$$ {}_2F_1\left[\begin{matrix}-n, b; 1\\ c\end{matrix}\right] = \frac{(c-b)_n}{(c)_n}, n = 0,1,2,...; \tag{1.8}$$

The modified form of the relation [1 , p. 308, (9.37)] :

$$H_{2n}(x) = (-1)^n(2)^{2n}(1/2)_n F_1\left[\begin{matrix}-n; x^2\\ 1/2\end{matrix},\right] \tag{1.9}$$

The modified from of the relation [1 , p. 312, (6)] :

$$H_{2n+1}(x) = (-1)^n 2^{2n+1}(3/2)_n\ {}_1F_1\left[\begin{matrix}-n; x^2\\ 3/2\end{matrix}\right] \tag{1.10}$$

The Legendre duplication formula [1, p. 58, (2.24)]

$$2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\Gamma(2x) \tag{1.11}$$

The following well known relations [1, pp. 275, 323] :

$$ {}_0F_0(-; -; x) = e^x \tag{1.12}$$

$$ {}_0F_1\left[\begin{matrix}-; -\frac{x^2}{4}\\ 1/2\end{matrix}\right] = cosx \tag{1.13}$$

$$x\,{}_0F_1\left[\begin{matrix}-; -\frac{x^2}{4}\\ 3/2\end{matrix}\right] = sinx \tag{1.14}$$

$$(-k)_n = \begin{cases} 0, n > k \\ \qquad (k = 1,2,3,...) \\ (-1)^n n!, k = n \end{cases} \tag{1.15}$$

2. ORTOGONALITY PROPERTY OF THE POLYNOMIALS $H_n^b(x)$.

The polynomials $H_n^b(x)$ are orthogonal with weight $x^{2(b+1)}e^{-x^2}$ on the interval $(-\infty, \infty)$, i.e.

$$\int_{-\infty}^{\infty} x^{2(b+1)}\ e^{-x^2} H_n^b(x)H_k^b(x)dx = \begin{cases} 0, k \neq n \\ \frac{\Gamma(b+3/2)n!}{(b+3/2)_n}, k = n\end{cases} \tag{2.1}$$

where $b = -1, 0, 1, 2, ...$

PROOF . In (1.7), setting $y = z = 1, u = b+1, p = q = r = s = 1, a_1 = -n, b_1 = b + 3/2; c_1 = -k, d_1 = b + 3/2$, we have

$$\int_{-\infty}^{\infty} x^{2(b+1)}e^{-x^2}\ {}_1F_1(-n; b+3/2; x^2)\,{}_1F_1(-k; b+3/2; x^2)dx$$

$$= \sum_{m=0}^{\infty} \frac{(-k)_m}{(b+3/2)_m}\ \frac{1}{m!}\Gamma(m+b+3/2)\,{}_2F_1\left[\begin{matrix}-n, m+b+3/2; 1\\ b+3/2\end{matrix}\right] \tag{2.2}$$

Now, using the notation $H_n^b(x)$ for ${}_1F_1(-n; b+3/2; x^2)$ and Vandermonde's theorem (1.8) , (2.2) reduces to the form :

$$\int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} H_n^b(x) H_k^b(x) dx$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(b+3/2)(-k)_m(-m)_n}{m!(b+3/2)_n} \tag{2.3}$$

From (1.15), it is evident that all terms of the series (2.3) are zero for $m > k \neq n$ and $m < n \neq k$.

If $k = n = m$, we have

$$\int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} \left\{ H_n^b(x) \right\}^2 dx = \frac{\Gamma(b+3/2)n!}{(b+3/2)_n} \tag{2.4}$$

This proves (2.1)

3. THE POLYNOMIALS $H_n^b(x)$ AND THE HERMITE POLYNOMIALS $H_n(x)$.

(a) Generating functions

(i) In (1.3), putting $b = -1$, and applying (1.9), (1.11) , (1.12) and (1.13), it reduces to the generating function [1, p. 174, 2(a)] for the Hermite polynomials.

(ii) In (1.3), setting $b = 0$, and using (1.10) , (1.11) , (1.12) and (1.14), it yields the generating function [1, p. 174, 2(b)] for the Hermite polynomials.

(b) Orthogonality properties

(i) In (2.1), putting $b = -1$, and applying (1.9), (1.11), (1.12) and (1.13), we obtain the following orthogonality property of the Hermite polynomials :

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2n}(x) H_{2k}(x) dx = \begin{cases} 0, k \neq n \\ 2^{2n}(2n)!\sqrt{\pi}, k = n \end{cases} \tag{3.1}$$

(ii) In (2.1), setting $b = 0$, and using (1.10), (1.11) , (1.12) and (1.14), it yields the following orthogonality proerty of the Hermite polynomials :

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2n+1}(x) H_{2k+1}(x) dx = \begin{cases} 0, k \neq n \\ 2^{2n+1}(2n+1)!\sqrt{\pi}, k = n \end{cases} \tag{3.2}$$

From (3.1) and (3.2), the orthogonality property of the Hermite polynomials [1, pp. 170-171, (5.17) - (5.22)] follows.

ACKNOWLEDGEMENT

I wish to express my sincere thanks to the referee for his useful suggestions for the revision of this paper.

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manuscrit reçu le 26 Octobre 1992