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Generalized Besov type spaces on the Laguerre hypergroup

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Abstract

In this paper we study generalized Besov type spaces on the Laguerre hypergroup and we give some characterizations using different equivalent norms which allows to reach results of completeness, continuous embeddings and density of some subspaces. A generalized Calderón-Zygmund formula adapted to the harmonic analysis on the Laguerre Hypergroup is obtained inducing two more equivalent norms.

1 Introduction

Schwartz's theory of Fourier transform and the Lebesgue spaces has been exploited by many authors in the study of Besov spaces on $\mathbb{R}^n$ ([3], [24], [6]). This theory has been generalized to different spaces, and was applied further to investigate spaces analogous to the classical Besov spaces ([4], [2]).

In the present work, we study Besov type spaces on the Laguerre hypergroup, so we fix $\alpha \geq 0$ and $\mathbb{K} = [0, +\infty[\times\mathbb{R}$ and we define Besov type spaces using the harmonic analysis on the Laguerre hypergroup which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see [1]).

We consider the following system of partial differential operators:

$$\begin{cases} D_1 &= \dfrac{\partial}{\partial t}, \\ D_2 &= \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2}; \qquad (x,t) \in]0,\infty[\times\mathbb{R}. \end{cases}$$

For $\alpha = n-1$; $n \in \mathbb{N}\backslash\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. We denote by $\varphi_{\lambda,m}$, $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, the unique solution of the following system:

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda|(m + \frac{\alpha+1}{2})u; \\ u(0,0) = 1, \ \frac{\partial u}{\partial x}(0,t) = 0 \qquad \textit{for all } t \in \mathbb{R}. \end{cases}$$

One knows that $\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^\alpha(|\lambda|x^2)$, where $\mathcal{L}_m^\alpha$ is the Laguerre functions defined on $\mathbb{R}_+$ by $\mathcal{L}_m^\alpha(x) = e^{-\frac{x}{2}}\frac{L_m^\alpha(x)}{L_m^\alpha(0)}$ and L_m^α is the Laguerre polynomial of degree m and order α (see [17], [11], [13], [16]).

We recall that for $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$ and for a suitable function $f:\mathbb{K}\longrightarrow\mathbb{C}$ the Fourier-Laguerre transform $\mathcal{F}(f)(\lambda,m)$ of f at (λ,m) is defined by ([19], [22, 23], [12]):

$$\mathcal{F}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)d\mu_\alpha(x,t), \tag{1.1}$$

where $d\mu_\alpha(x,t) = \dfrac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}$.

It has been proved in [19, Theorem II.1] that the Fourier-Laguerre transform is a topological isomorphism from $S_*(\mathbb{K})$ onto $S(\mathbb{R}\times\mathbb{N})$ where

- $S_*(\mathbb{K})$ is the Schwartz space of functions $\psi:\mathbb{R}^2\longrightarrow\mathbb{C}$ even with respect to the first variable, $\mathcal{C}^\infty$ on $\mathbb{R}^2$ and rapidly decreasing together with all their derivatives; i.e. for all $k,p,q\in\mathbb{N}$ we have

$$\widetilde{\mathcal{N}}_{k,p,q}(\psi) = \sup_{(x,t)\in\mathbb{K}}\Big\{(1+x^2+t^2)^k\Big|\frac{\partial^{\,p+q}}{\partial x^p\partial t^q}\psi(x,t)\Big|\Big\} < \infty. \tag{1.2}$$

- $\mathcal{S}(\mathbb{R}\times\mathbb{N})$ the space of functions $\Psi:\mathbb{R}\times\mathbb{N}\longrightarrow\mathbb{C}$ satisfying :
  i) For all $m,p,q,r,s\in\mathbb{N}$, the function

$$\lambda \longmapsto \lambda^p\Big(|\lambda|(m+\frac{\alpha+1}{2})\Big)^q\Lambda_1^r\Big(\Lambda_2+\frac{\partial}{\partial\lambda}\Big)^s\Psi(\lambda,m)$$

  is bounded and continuous on $\mathbb{R}$, $\mathcal{C}^\infty$ on $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and such that the left and the right derivatives at zero exist.
  ii) For all $k,p,q\in\mathbb{N}$, we have

$$\mathcal{V}_{k,p,q}(\Psi) = \sup_{(\lambda,m)\in\mathbb{R}^*\times\mathbb{N}}\Big\{\Big(1+\lambda^2(1+m^2)\Big)^k\Big|\Lambda_1^p\Big(\Lambda_2+\frac{\partial}{\partial\lambda}\Big)^q\Psi(\lambda,m)\Big|\Big\} < \infty.$$

where

- $\Lambda_1\Psi(\lambda,m) = \frac{1}{|\lambda|}\Big(m\Delta_+\Delta_-\Psi(\lambda,m) + (\alpha+1)\Delta_+\Psi(\lambda,m)\Big)$.

- $\Lambda_2\Psi(\lambda,m) = \frac{-1}{2\lambda}\Big((\alpha+m+1)\Delta_+\Psi(\lambda,m) + m\Delta_-\Psi(\lambda,m)\Big)$.

- $\Delta_+\Psi(\lambda,m) = \Psi(\lambda,m+1) - \Psi(\lambda,m)$.

- $\Delta_-\Psi(\lambda,m) = \Psi(\lambda,m) - \Psi(\lambda,m-1)$, if $m \geq 1$ and $\Delta_-\Psi(\lambda,0) = \Psi(\lambda,0)$.

We note that $S_*(\mathbb{K})$ (resp. $S(\mathbb{R}\times\mathbb{N})$) equipped with the semi-norms $\widetilde{\mathcal{N}}_{k,p,q}$ (resp. $\mathcal{V}_{k,p,q}$), $k,p,q \in \mathbb{N}$, is a Fréchet space ([19]).

This paper deals with generalized Besov-Laguerre type spaces defined on $\mathbb{K}$ and it is organized as follows: in the second section, we collect some harmonic analysis properties of the Laguerre hypergroup which are given in [19] and [18]. Next, we state a version of Schur lemma which will be useful for our purpose. In the third section we introduce the homogeneous Besov-Laguerre type spaces $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ $(1 \leq p,q \leq \infty,\ \gamma \in \mathbb{R})$. The definition of the so called spaces is given in terms of convolution $f{\#}\psi_r$ with different kinds of smooth functions ψ. Next we characterize these spaces using discrete norms replacing the group $\mathbb{R}^*_+ =]0,+\infty[$ by the 2-powers group $\mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\}$ and we introduce some results and embeddings properties of these spaces with respect to their parameters p,q and γ. In the fourth section we establish some new harmonic analysis results on usual spaces on $\mathbb{K}$, essentially we give a Delsarte type development and a Calderón-Zygmund type formula. Finally we study the non homogeneous Besov-Laguerre type spaces $\Lambda^{\gamma}_{p,q}(\mathbb{K})$ $(1 \leq p,q \leq \infty,\ 0 < \gamma < 2)$ introduced as intersection of the homogenous ones with L^p-spaces and we give some characterizations with equivalent norms using the differences $\Delta_{(x,t)}f = T^{(\alpha)}_{(x,t)}f - f$. In proving these results, the main tool used is the harmonic analysis on the Laguerre hypergroup.

Finally, we mention that, C will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

2 Preliminaries

Throughout this paper we fix $\alpha \geq 0$ and we denote by

- $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{R}_+^* =]0, +\infty[$
- $\mathbb{K} = [0, \infty[\times\mathbb{R}$.
- $\mathcal{C}_*(\mathbb{K})$ the space of continuous functions on $\mathbb{R}^2$ even with respect to the first variable.
- $\mathcal{C}_{*,c}(\mathbb{K})$ the subspace of $\mathcal{C}_*(\mathbb{K})$ consisting of functions with compact support.
- $\mathcal{C}_*^\infty(\mathbb{K})$ the space of functions $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, even with respect to the first variable and $\mathcal{C}^\infty$ on $\mathbb{R}^2$.
- $S_{*,0}(\mathbb{K})$ the subset of functions ψ in $S_*(\mathbb{K})$ such that $\mathcal{F}\psi \in \mathcal{D}(\mathbb{R}^* \times \mathbb{N})$.
- $S_{*,0}^1(\mathbb{K})$ the subset of functions ψ in $S_{*,0}(\mathbb{K})$ such that

$$\int_0^\infty \Big(\mathcal{F}\psi(r^2\lambda, m)\Big)^2 \frac{dr}{r} = 1, \qquad for\ (\lambda, m) \in \mathbb{R}^* \times \mathbb{N}.$$

  These functions are known as generalized wavelets on $\mathbb{K}$ ([19]).

- $\mathcal{D}(\mathbb{R} \times \mathbb{N})$ the subspace of $S(\mathbb{R} \times \mathbb{N})$ of functions ψ satisfying the following:
  - i) There exists $m_0 \in \mathbb{N}$ satisfying $\psi(\lambda, m) = 0$, for all $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ such that $m > m_0$.
  - ii) For all $m \leq m_0$, the function $\lambda \longmapsto \psi(\lambda, m)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$, with compact support and vanishes in a neighborhood of zero.
- $L^p(\mathbb{K}) = L^p(\mathbb{K}, d\mu_\alpha)$, $1 \leq p \leq \infty$, the space of Borel measurable functions on $\mathbb{K}$ such that $\|f\|_p < \infty$, where

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{K}} |f(x,t)|^p d\mu_\alpha(x,t)\right)^{\frac{1}{p}}, \qquad if \quad p \in [1, \infty[, \\ \|f\|_\infty &= \operatorname*{esssup}_{(x,t)\in\mathbb{K}} |f(x,t)|, \end{aligned}$$

  $d\mu_\alpha$ being the positive measure defined on $\mathbb{K}$ given in the introduction. Each of these spaces is equipped with its usual topology.

Definition 2.1:

- The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function f by:

$$T^{(\alpha)}_{(x,t)}f(y,s)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f((x^2+y^2+2xy\cos\theta)^{\frac{1}{2}},t+s+xy\sin\theta)d\theta, \\ \hfill if\ \alpha=0, \\ \dfrac{\alpha}{\pi}\displaystyle\int_0^1\int_0^{2\pi} f((x^2+y^2+2xy\rho\cos\theta)^{\frac{1}{2}},t+s+xy\rho\sin\theta)\times \\ \hfill \rho(1-\rho^2)^{\alpha-1}d\theta d\rho, \quad if\ \alpha>0.\end{cases}$$

- The generalized convolution product on the Laguerre hypergroup is defined for a pair of functions f and g in $\mathcal{C}_{*,c}(\mathbb{K})$ by:

$$f\#g(x,t)=\int_{\mathbb{K}} T^{(\alpha)}_{(x,t)}f(y,s)g(y,-s)d\mu_\alpha(y,s) \qquad for\ all\ (x,t)\in\mathbb{K}.$$

We recall that $(\mathbb{K},*,i)$ is an hypergroup in the sense of Jewett ([15], [5]) where i denotes the involution defined on $\mathbb{K}$ by $i(x,t)=(x,-t)$. This hypergroup is the Laguerre hypergroup which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see [1]).

Notation 2.2: Let $r>0$. We will denote by

- $(x,t)_r=(\frac{x}{r},\frac{t}{r^2})$ the dilated of $(x,t)\in\mathbb{K}$.
- $f_r(x,t)=r^{-(2\alpha+4)}f((x,t)_r)$ the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure $d\mu_\alpha$, in the sense that

$$\int_{\mathbb{K}} f_r(x,t)d\mu_\alpha(x,t)=\int_{\mathbb{K}} f(x,t)d\mu_\alpha(x,t), \quad \forall\ r>0\ and\ f\in L^1(\mathbb{K}). \tag{2.1}$$

- $\Delta_{(x,t)}f=T^{(\alpha)}_{(x,t)}f-f$, for all $(x,t)\in\mathbb{K}$.

Proposition 2.3: *The following properties hold*

1) *For all $f \in \mathcal{C}_{*,c}(\mathbb{K})$, we have (see [19])*

$$\text{(i)}\ T^{(\alpha)}_{(0,0)} f(y,s) = f(y,s), \qquad \forall (y,s) \in \mathbb{K}.$$

$$\text{(ii)}\ T^{(\alpha)}_{(0,t)} f(y,s) = f(y,s+t), \qquad \forall (y,s) \in \mathbb{K},\ t \in \mathbb{R}.$$

$$\text{(iii)}\ T^{(\alpha)}_{(x,t)} f(y,s) = T^{(\alpha)}_{(y,s)} f(x,t) \qquad \forall (x,t),(y,s) \in \mathbb{K}.$$

2) (i) *For all $f \in \mathcal{C}_{*}(\mathbb{K})$ and $(x,t),(y,s) \in \mathbb{K}$, we have (see [18])*

$$T^{(\alpha)}_{(x,t)} f(y,s) = \int_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v)) f(z,v) z^{2\alpha+1} dz dv$$

where $W_\alpha((x,t),(y,s),(z,v))$ is given by

$$\frac{\alpha}{\pi (xyz)^{2\alpha}} \Big[x^2y^2 - \Big(\frac{z^2-(x^2+y^2)}{2} \Big)^2 - (v-(s+t))^2 \Big]^{\alpha-1},$$

if $(z,v) \in S_\alpha((x,t),(y,s))$ and $W_\alpha((x,t),(y,s),(z,v))$ equals 0 otherwise. $S_\alpha((x,t),(y,s))$ is given, for $\alpha \neq 0$, by

$$S_\alpha((x,t),(y,s)) = \Big\{ (z,v) \in \mathbb{K};\ \Big(\frac{z^2-(x^2+y^2)}{2} \Big)^2 + (v-(s+t))^2 \leq x^2y^2 \Big\}$$

and

$$S_0((x,t),(y,s)) = \Big\{ (z,v) \in \mathbb{K};\ \Big(\frac{z^2-(x^2+y^2)}{2} \Big)^2 + (v-(s+t))^2 = x^2y^2 \Big\}.$$

(ii) *Let f be in $L^p(\mathbb{K})$, $1 \leq p \leq \infty$. Then for all $(x,t) \in \mathbb{K}$, the function $T^{(\alpha)}_{(x,t)} f$ belongs to $L^p(\mathbb{K})$ and we have*

$$\|T^{(\alpha)}_{(x,t)} f\|_p \leq \|f\|_p.$$

3) *For f in $L^p(\mathbb{K})$ and g in $L^q(\mathbb{K})$, $1 \leq p,q \leq \infty$, the function $f_{\#}g$ belongs to $L^r(\mathbb{K})$; $\frac{1}{p}+\frac{1}{q} = 1 + \frac{1}{r}$, and we have*

$$\|f_{\#}g\|_r \leq \|f\|_p \|g\|_q.$$

4) *(i)* *Let f be in $L^1(\mathbb{K})$. Then the function $\mathcal{F}(f)$ is bounded on $\mathbb{R}\times\mathbb{N}$ and we have*

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}\times\mathbb{N})} \le \|f\|_1$$

where $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}\times\mathbb{N})} = \operatorname*{ess\,sup}_{(\lambda,m)\in\mathbb{R}\times\mathbb{N}} |\mathcal{F}(f)(\lambda,m)|.$

(ii) *Let f and g in $L^1(\mathbb{K})$, then we have*

$$\mathcal{F}(f_{\#}g) = \mathcal{F}(f)\mathcal{F}(g).$$

(iii) *Let f be in $L^1(\mathbb{K})$. Then for all (x,t) in $\mathbb{K}$ and (λ,m) in $\mathbb{R}\times\mathbb{N}$, we have*

$$\mathcal{F}(T^{(\alpha)}_{(x,t)}f)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}(f)(\lambda,m).$$

Proposition 2.4: *(See [1]) Let ψ in $S_*(\mathbb{K})$ and $(x,t)\in\mathbb{K}$. Then $T^{(\alpha)}_{(x,t)}\psi$ belongs to $S_*(\mathbb{K})$ and we have for all $p,q\in\mathbb{N}$*

$$D_1^p D_2^q \left(T^{(\alpha)}_{(x,t)}\psi\right) = T^{(\alpha)}_{(x,t)}\left(D_1^p D_2^q\psi\right).$$

Proposition 2.5: *Let ψ in $\mathcal{C}_*(\mathbb{K})$. Then ψ belongs to $S_*(\mathbb{K})$ if and only if*

(i) *For all $p,q\in\mathbb{N}$ the function $(x,t)\longmapsto D_1^p D_2^q\psi(x,t)$ is of class $\mathcal{C}^2$ on $\mathbb{R}^2$.*

(ii) *For all $k,p,q\in\mathbb{N}$ we have*

$$\mathcal{N}_{k,p,q}(\psi) = \sup_{(x,t)\in\mathbb{K}}\left\{(1+N^2(x,t))^k\left|D_1^p D_2^q\psi(x,t)\right|\right\} < \infty,$$

where $N(x,t)=(x^2+|t|)^{1/2}$ is the norm of $(x,t)\in\mathbb{K}$.

Proof: We obtain the desired result by using Proposition II.7 in [19] and the fact that

$$(1+x^2+t^2)^k \le (1+N^2(x,t))^{2k} \le 2^{2k}(1+x^2+t^2)^{2k}, \qquad \forall\,(x,t)\in\mathbb{K} \text{ and } k\in\mathbb{N}.$$

□

In the sequel we equip $S_*(\mathbb{K})$ with the semi-norms $\mathcal{N}_{k,p,q}$ which define the standard topology on $S_*(\mathbb{K})$.

We finish this preliminary section by giving a version of Schur lemma that will be useful for our purposes.

Lemma A. *(Schur lemma).* *Let $1 < q < \infty$ and q' its conjugate exponent. Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be a pair of σ-finite measure spaces and let $F : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}_+$ be a measurable function. Define $T_F f$ for all measurable positive function f on Ω_1 by*

$$T_F f(\omega_2) = \int_{\Omega_1} F(\omega_1, \omega_2) f(\omega_1) d\mu_1(\omega_1), \qquad \text{for all } \omega_2 \in \Omega_2.$$

If there exist $C > 0$ and measurable functions $h_i : \Omega_i \longrightarrow]0, +\infty[$ $(i = 1, 2)$ such that

$$\int_{\Omega_1} F(\omega_1, \omega_2) h_1^{q'}(\omega_1) d\mu_1(\omega_1) \leq C h_2^{q'}(\omega_2) \qquad \mu_2 - a.e.$$

$$\int_{\Omega_2} F(\omega_1, \omega_2) h_2^{q}(\omega_2) d\mu_2(\omega_2) \leq C h_1^{q}(\omega_1) \qquad \mu_1 - a.e.$$

Then T_F can be extended as a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.

3 Generalized homogeneous Besov-Laguerre type spaces

In what follows we equip the spaces $\mathbb{R}_+^*$ and $\mathbb{D}_2$ by the invariant measure $\frac{dr}{r}$ and the counting measure respectively.

Definition 3.1: Let $1 \leq p, q \leq \infty$, $\gamma \in \mathbb{R}$ and $\psi \in S^1_{*,0}(\mathbb{K})$. We define the generalized homogeneous Besov-Laguerre type spaces $\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$ as the set of tempered distributions f such that

$$f = \int_0^\infty f_{\#} \psi_{r\#} \psi_r \frac{dr}{r} \tag{3.1}$$

and $\|f\|_{\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})} < \infty$, where

$$\|f\|_{\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})} = \begin{cases} \left(\displaystyle\int_0^\infty \Big(\frac{\|f\#\psi_r\|_p}{r^\gamma}\Big)^q \frac{dr}{r}\right)^{\frac{1}{q}}, & \textit{if} \quad 1 \leq q < \infty, \\ \operatorname*{esssup}_{r>0} \Big(\dfrac{\|f\#\psi_r\|_p}{r^\gamma}\Big), & \textit{if} \quad q = \infty. \end{cases}$$

Remarks 3.2: 1) We begin by mentioning that the definition of the generalized homogeneous Besov-Laguerre type spaces given here is the same than that introduced by Chemin in the classical case (see [8]) and generalized by Bahouri, Gérard and Xu on the Heisemberg group (see [2]). We do not choose the classical definition introduced by Peetre (see [20]) in which $\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$ is defined as a set of distributions modulo polynomials. In fact in the case $\gamma < \frac{2\alpha+4}{p}$, the condition $\|f\|_{\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})} < \infty$ implies the convergence of the integral

$$\int_0^\infty f\#\psi_r\#\psi_r \frac{dr}{r}$$

in the sense of distribution and not only in the sense of distribution modulo polynomials, thus the two points of view are equivalent. We note finally that, similarly to the classical case, for $\gamma \geq \frac{2\alpha+4}{p}$, the space $\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$, as we define, is not a Banach space.

2) We note here that the expression (3.1) is independent, in $S'_*(\mathbb{K})$, of the choice of ψ in $S^1_{*,0}$ and it corresponds to the analogous one given in [2, Definition 3.1, p.12] replacing the diadic decomposition by the continuous decomposition.

3) If f belongs to $L^2(\mathbb{K})$, then (3.1) holds in $L^2(\mathbb{K})$. Which is a consequence of Plancherel's formula (see [18]). Hence one can write

$$\left\|f - \int_{1/\varepsilon}^{\varepsilon} f\#\psi_r\#\psi_r \frac{dr}{r}\right\|_2^2 = \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{+\infty} L_m^\alpha(0) \Big|\mathcal{F}f(\lambda,m)\Big|^2 \times \right.$$
$$\left. \left|1 - \int_{1/\varepsilon}^{\varepsilon} \Big(\mathcal{F}\psi_r(\lambda,m)\Big)^2 \frac{dr}{r}\right|^2 \right\} |\lambda|^{\alpha+1} d\lambda.$$

And, using Lebesgue theorem, the right hand side of the above equality tends to zero as ε tends to $+\infty$. Indeed

$$\left|1 - \int_{1/\varepsilon}^{\varepsilon} \Big(\mathcal{F}\psi_r(\lambda,m)\Big)^2 \frac{dr}{r}\right|^2 \longrightarrow 0 \qquad \text{as } \varepsilon \longrightarrow +\infty,$$

$$\sum_{m=0}^{+\infty} L_m^\alpha(0)\Big|\mathcal{F}f(\lambda,m)\Big|^2\Big|1-\int_{1/\varepsilon}^{\varepsilon}\Big(\mathcal{F}\psi_r(\lambda,m)\Big)^2\frac{dr}{r}\Big|^2 \leq \sum_{m=0}^{+\infty} L_m^\alpha(0)\Big|\mathcal{F}f(\lambda,m)\Big|^2.$$

and the right hand side of the above inequality is in $L^1(\mathbb{R},|\lambda|^{\alpha+1}d\lambda)$.

4) The expression (3.1) is not true in $S'_*(\mathbb{K})$ if f is a polynomial function on $\mathbb{K}$. Indeed in this case, for all $r>0$, we have $f{\#}\psi_r=0$.

Proposition 3.3: *Let $1\leq p,q\leq\infty$ and $\gamma\in\mathbb{R}$. Then the space $\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$ is independent of the choice of the function ψ in $S^1_{*,0}(\mathbb{K})$.*

Proof: Assume ψ and ϕ be a pair of functions belonging to $S^1_{*,0}(\mathbb{K})$. To get the desired result it suffices to prove that $\|f\|_{\dot{\Lambda}^{\gamma,\phi}_{p,q}(\mathbb{K})}\leq C\|f\|_{\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})}$, for all $f\in\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$. Since $\mathcal{F}\psi$ and $\mathcal{F}\phi$ belong to $\mathcal{D}(\mathbb{R}^*\times\mathbb{N})$, then there exist $\alpha,\beta>0$ such that $(Supp\mathcal{F}\psi_r)\cap(Supp\mathcal{F}\phi_\rho)=\emptyset$, for all $(r/\rho)\notin[\alpha,\beta]$. This implies that $\psi_r{\#}\phi_\rho=0$, for all $(r/\rho)\notin[\alpha,\beta]$. And so, using (3.1), one can write

$$f{\#}\phi_\rho=\int_0^\infty f{\#}\phi_\rho{\#}\psi_r{\#}\psi_r\frac{dr}{r}=\int_\alpha^\beta f{\#}\psi_{r\rho}{\#}\phi_\rho{\#}\psi_{r\rho}\frac{dr}{r}.$$

And by Minkowski's inequality

$$\begin{aligned}\frac{\|f{\#}\phi_\rho\|_p}{\rho^\gamma} &\leq C\int_\alpha^\beta\frac{\|f{\#}\psi_{r\rho}\|_p}{\rho^\gamma}\frac{dr}{r}\\ &= C\int_0^\infty r^\gamma \mathbb{1}_{[\alpha,\beta]}(r)\frac{\|f{\#}\psi_{r\rho}\|_p}{(r\rho)^\gamma}\frac{dr}{r}\\ &= C(H\star G)(\rho)\end{aligned}$$

with $H(s)=s^\gamma\mathbb{1}_{[\alpha,\beta]}(s)$, $G(s)=\dfrac{\|f{\#}\psi_s\|_p}{s^\gamma}$ and $H\star G$ is the convolution of H and G on the group $(\mathbb{R}^*_+,\frac{dr}{r})$. Now, by Young's inequality, it holds

$$\begin{aligned}\|f\|_{\dot{\Lambda}^{\gamma,\phi}_{p,q}(\mathbb{K})} &= \Big(\int_0^\infty\Big(\frac{\|f{\#}\phi_\rho\|_p}{\rho^\gamma}\Big)^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}\\ &\leq C\Big\|H\star G\Big\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})}\\ &\leq C\|H\|_{L^1(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})}\|G\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})}\\ &= C\|f\|_{\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})}.\end{aligned}$$

This completes the proof of the proposition. □

Remark 3.4: In view of their independence with respect to ψ the spaces $\dot{\Lambda}^{\gamma,\psi}_{p,q}(\mathbb{K})$ $(1 \leq p, q \leq \infty$ and $\gamma \in \mathbb{R})$ will be denoted indifferently with or without ψ, which will be chosen adequately in $S^1_{*,0}(\mathbb{K})$.

In what follows we give some properties of the generalized Besov-Laguerre type spaces.

Proposition 3.5: *Let $1 \leq p, q \leq \infty$, $\gamma \in \mathbb{R}$. The Besov-Laguerre type space $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ is homogeneous of degree $d(p,\gamma) = \frac{2\alpha+4}{p} - \gamma$ in the sense that, for all $f \in \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$*

$$\|d_r f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} = r^{\frac{2\alpha+4}{p}-\gamma}\|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})}, \qquad \textit{for all } r > 0$$

where $d_r f(x,t) = f((x,t)_r)$, for all $(x,t) \in \mathbb{K}$.

Proof: Assume f in $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$, then for all $1 \leq p, q \leq \infty$ and $\gamma \in \mathbb{R}$ we have

$$\begin{aligned}
\|d_r f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} &= \left\| \frac{\|(d_r f)_{\#}\psi_\rho\|_p}{\rho^\gamma} \right\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})} \\
&= r^{\frac{2\alpha+4}{p}} \left\| \frac{\|f_{\#}\psi_{\frac{\rho}{r}}\|_p}{\rho^\gamma} \right\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})} \\
&= r^{\frac{2\alpha+4}{p}-\gamma} \left\| \frac{\|f_{\#}\psi_\rho\|_p}{\rho^\gamma} \right\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{d\rho}{\rho})}.
\end{aligned}$$

The proposition is proved. □

Proposition 3.6: *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $\gamma \in \mathbb{R}$. The subspace $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K}) \cap \mathcal{C}^\infty_*(\mathbb{K})$ is dense in $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$.*

Proof: Let $\phi \in S^1_{*,0}(\mathbb{K})$ and $f \in \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$. Then for $\varepsilon > 1$, the function

$$f_\varepsilon = \int_{1/\varepsilon}^{\varepsilon} f_{\#}\phi_{r\#}\phi_r \frac{dr}{r}$$

is obviously $\mathcal{C}^\infty$ and belongs to $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$. Moreover the same reasoning given in Proposition 3.3 leads to

$$\|f_\varepsilon - f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} \leq C \left\| \frac{\|f_{\#}\phi_r\|_p}{r^\gamma} \; 1\!\!1_\varepsilon(r) \right\|_{L^q(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\frac{dr}{r})}$$

where $1\!\!1_\varepsilon$ is the characteristic function of the set $\mathbb{R} \setminus [1/\varepsilon, \varepsilon]$. And the right hand side of the above inequality tends to zero as ε tends to ∞. $\square$

Proposition 3.7: *Let $1 \leq p, q \leq \infty$ and $\gamma < \frac{2\alpha+4}{p}$. Then $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ is a Banach space.*

Let us first establish the following lemmas.

Lemma 3.8: *Let $h \in S_*(\mathbb{K})$ and $\psi \in S^1_{*,0}(\mathbb{K})$. Then, for all $k \in \mathbb{N}$, there exists $\psi_{[k]} \in S_{*,0}(\mathbb{K})$ such that*

$$h_{\#}\psi_r = r^{2k}(D_2^k h)_{\#}(\psi_{[k]})_r \tag{3.2}$$

where D_2 is the differential operator given in the introduction part. Furthermore there exists $C > 0$ such that

$$\|h_{\#}\psi_r\|_{L^1(\mathbb{K})} \leq Cr^{2k} \qquad \text{for all } 0 \leq r \leq 1 \tag{3.3}$$

and

$$\|h_{\#}\psi_r\|_{L^1(\mathbb{K})} \leq C \qquad \text{for all } r \geq 1. \tag{3.4}$$

Proof: For $k = 1$ one can write

$$\begin{aligned}\mathcal{F}(h_{\#}\psi_r)(\lambda, m) &= 4|\lambda|(m + \frac{\alpha+1}{2})r^2\mathcal{F}h(\lambda, m)\frac{\mathcal{F}\psi(r^2\lambda, m)}{4r^2|\lambda|(m + \frac{\alpha+1}{2})} \\ &= r^2\mathcal{F}(D_2h)(\lambda, m)\,\mathcal{F}\psi_{[1]}(r^2\lambda, m)\end{aligned}$$

where $\mathcal{F}\psi_{[1]}(\lambda, m) = \frac{\mathcal{F}\psi(\lambda,m)}{4|\lambda|(m+\frac{\alpha+1}{2})}$, which leads to

$$h_{\#}\psi_r = r^2(D_2h)_{\#}(\psi_{[1]})_r,$$

and hence we obtain (3.2) by induction on k. The inequalities (3.3) and (3.4) follow from (3.2) and Proposition 2.3. $\square$

Lemma 3.9: *Let $1 \leq p, q \leq \infty$ and $\gamma < \frac{2\alpha+4}{p}$. For ϕ in $S^1_{*,0}$, put*

$$\Phi(g) = \int_0^\infty r^\gamma g(r)_{\#}\phi_r \frac{dr}{r}. \tag{3.5}$$

Then Φ defines a linear and continuous mapping from $L^q(\mathbb{R}^*_+, L^p(\mathbb{K}), \frac{dr}{r})$ to $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$.

Proof: Let us first prove that, for $g \in L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})$, $\Phi(g)$ defines an element of $S'_*(\mathbb{K})$, that is for all $h \in S_*(\mathbb{K})$,

$$\int_0^\infty \left| r^\gamma < g(r)_\# \phi_r, h > \right| \frac{dr}{r} < \infty.$$

Take $\psi \in S_*(\mathbb{K})$ such that $\mathcal{F}(\psi) = 1$ on $Supp \mathcal{F}\phi$. Then, using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} | < g(r)_\#\phi_r, h > | &= | < g(r)_\#\phi_r, h_\#\psi_r > | \\ &\leq \|g(r)_\#\phi_r\|_{L^\infty(\mathbb{K})} \|h_\#\psi_r\|_{L^1(\mathbb{K})} \\ &\leq C r^{-\frac{2\alpha+4}{p}} \|g(r)\|_{L^p(\mathbb{K})} \|h_\#\psi_r\|_{L^1(\mathbb{K})}. \end{aligned}$$

On the other hand, using Lemma 3.8, we get

$$\begin{aligned} &\int_0^\infty \left| r^\gamma < g(r)_\#\phi_r, h > \right| \frac{dr}{r} \\ &\quad \leq C \left\{ \int_0^1 r^{2k+\gamma-\frac{2\alpha+4}{p}} \|g(r)\|_{L^p(\mathbb{K})} \frac{dr}{r} + \int_1^\infty r^{\gamma-\frac{2\alpha+4}{p}} \|g(r)\|_{L^p(\mathbb{K})} \right\} \frac{dr}{r} \\ &\quad \leq C \left(\int_0^1 \|g(r)\|^q_{L^p(\mathbb{K})} \frac{dr}{r} \right)^{1/q} \left(\int_0^1 r^{(2k+\gamma-\frac{2\alpha+4}{p})\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}} \\ &\qquad + C \left(\int_1^\infty \|g(r)\|^q_{L^p(\mathbb{K})} \frac{dr}{r} \right)^{1/q} \left(\int_1^\infty r^{(\gamma-\frac{2\alpha+4}{p})\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}}. \end{aligned}$$

where $\bar{q}$ is the conjugate exponent of q. Then, for k sufficiently large it holds

$$\int_0^\infty \left| r^\gamma < g(r)_\#\phi_r, h > \right| \frac{dr}{r} \leq C \|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})} < \infty.$$

Now, let ψ in $S^1_{*,0}$. We proceed as in Proposition 3.3 to obtain

$$\frac{\|\Phi(g)_\#\psi_\rho\|_{L^p(\mathbb{K})}}{\rho^\gamma} \leq C \int_\alpha^\beta r^\gamma \|g(r\rho)\|_{L^p(\mathbb{K})} \frac{dr}{r} = C \int_0^\infty r^\gamma \mathbb{1}_{[\alpha,\beta]}(r) \|g(r\rho)\|_{L^p(\mathbb{K})} \frac{dr}{r}$$

which leads to

$$\|\Phi(g)\|_{\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})} \leq C \|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})}.$$

The lemma is proved. □

Proof: (Proposition 3.7) Let ψ in $S^1_{*,0}$ and take $\phi = \psi$ in Lemma 3.9. Then Φ defined by (3.5) is a continuous linear mapping from $L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})$

to $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$. On the other hand the operator Ψ associating to f in $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ the function $\Psi(f)$ defined on $\mathbb{R}_+^*$ by:

$$\Psi(f)(r) = \frac{f_{\#}\psi_r}{r^\gamma}$$

is obviously a linear isometry from $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ to $L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})$ and using the decomposition (3.1), we obtain $\Phi \circ \Psi = Id_{\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})}$. This implies

$$(\Psi \circ \Phi) \circ \Psi = \Psi \qquad \text{on } \overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K}).$$

So $\Psi\Big(\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})\Big) = \ker\Big(\Psi \circ \Phi - Id_{L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})}\Big)$ is a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})$. Since Ψ is an isometry, then $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ can be identified with a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{K}), \frac{dr}{r})$. The completeness of $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ follows. □

Remark 3.10: From the Proof of Lemma 3.9, the result of Proposition 3.7 remains valid, for $q = 1$, if $\gamma = \frac{2\alpha+4}{p}$.

To introduce some embedding results of the spaces $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ with respect to their parameters p, q and γ we begin by the following lemma which will be useful.

Lemma 3.11: *Let $f \in S'(\mathbb{K})$ satisfying (3.1) and let*

$$\tilde{f} = \left(\int_0^\infty \left|f_{\#}\psi_r\right|^2 \frac{dr}{r}\right)^{1/2}. \qquad (3.6)$$

Then, for all $1 < p < \infty$, we have

$$f \in L^p(\mathbb{K}) \Longleftrightarrow \tilde{f} \in L^p(\mathbb{K}).$$

Moreover, there exists $C_p > 0$ such that

$$\frac{1}{C_p}\|\tilde{f}\|_{L^p(\mathbb{K})} \leq \|f\|_{L^p(\mathbb{K})} \leq C_p\|\tilde{f}\|_{L^p(\mathbb{K})}.$$

Proof: The proof of the above lemma is the same as in [21] p. 46. □

Proposition 3.12: **1)** *Let $1 \leq q \leq \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 \leq p_1 \leq p_2 \leq \infty$ such that $d(p_1, \gamma_1) = d(p_2, \gamma_2)$ where $d(p, \gamma) = \frac{2\alpha+4}{p} - \gamma$. Then we have*

$$\dot{\Lambda}^{\gamma_1}_{p_1,q}(\mathbb{K}) \subseteq \dot{\Lambda}^{\gamma_2}_{p_2,q}(\mathbb{K}) \qquad \text{(with continuous embedding).}$$

2) *Let $p \geq 2$. Then*

$$\dot{\Lambda}^{0}_{p,2}(\mathbb{K}) \subseteq L^p(\mathbb{K}) \qquad \text{(with continuous embedding).}$$

3) *Let $1 \leq p \leq \infty$. Then we have*

$$\dot{\Lambda}^{0}_{p,1}(\mathbb{K}) \subseteq L^p(\mathbb{K}) \qquad \text{(with continuous embedding).}$$

PROOF: **1)** Let $f \in \dot{\Lambda}^{\gamma}_{p_1,q}(\mathbb{K})$ and let $1 \leq p_3 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_3} = 1 + \frac{1}{p_2}$. We consider $\phi \in S_*(\mathbb{K})$ satisfying $\mathcal{F}\phi = 1$ on $Supp\mathcal{F}\psi$. Then it holds for all $r > 0$

$$\begin{aligned} \|f \# \psi_r\|_{L^{p_2}(\mathbb{K})} &= \|f \# \psi_r \# \phi_r\|_{L^{p_2}(\mathbb{K})} \\ &\leq \|f \# \psi_r\|_{L^{p_1}(\mathbb{K})} \|\phi_r\|_{L^{p_3}(\mathbb{K})} = C \|f \# \psi_r\|_{L^{p_1}(\mathbb{K})} r^{(2\alpha+4)(\frac{1}{p_3}-1)}. \end{aligned}$$

Hence, we obtain

$$\frac{\|f \# \psi_r\|_{L^{p_2}(\mathbb{K})}}{r^{\gamma_2}} \leq C \frac{\|f \# \psi_r\|_{L^{p_1}(\mathbb{K})}}{r^{\gamma_1}}.$$

2) Let $f \in S'(\mathbb{K})$ and let $\tilde{f}$ defined in (3.6). Then,

$$\begin{aligned} \|\tilde{f}\|_{L^p(\mathbb{K})} &= \left\| \left(\int_0^\infty \left| f \# \psi_r \right|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mathbb{K})} \\ &= \left\| \int_0^\infty \left| f \# \psi_r \right|^2 \frac{dr}{r} \right\|_{L^{p/2}(\mathbb{K})} \\ &\leq \left(\int_0^\infty \left\| (f \# \psi_r)^2 \right\|_{L^{p/2}(\mathbb{K})} \frac{dr}{r} \right)^{1/2} \\ &= \left(\int_0^\infty \left\| f \# \psi_r \right\|^2_{L^p(\mathbb{K})} \frac{dr}{r} \right)^{1/2} \\ &= \|f\|_{\dot{\Lambda}^{0}_{p,2}(\mathbb{K})}. \end{aligned}$$

The desired continuous embedding holds using Lemma 3.11.
3) Let $f \in S'(\mathbb{K})$ satisfying (3.1). Then

$$\|f\|_{L^p(\mathbb{K})} = \left\| \int_0^\infty f_{\#}\psi_r{}_{\#}\psi_r \frac{dr}{r} \right\|_{L^p(\mathbb{K})} \leq C \int_0^\infty \|f_{\#}\psi_r\|_{L^p(\mathbb{K})} \frac{dr}{r} = C\|f\|_{\dot{\Lambda}^0_{p,1}(\mathbb{K})}.$$

The proof is finished. □

To obtain more general inclusion properties we introduce a discrete norm on $\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})$ replacing the group $\mathbb{R}^*_+ =]0, +\infty[$ by the 2-powers group $\mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\}$.

Theorem 3.13: *Let $1 \leq p, q \leq \infty$, $\gamma \in \mathbb{R}$ and $\theta \in S_*(\mathbb{K})$ such that $\mathcal{F}\theta \in \mathcal{D}(\mathbb{R}^* \times \mathbb{N})$ and, for fixed $\lambda_1, \lambda_2 \in \mathbb{R}$; $\lambda_2 > 4\lambda_1 > 0$, $\mathcal{F}\theta(\lambda, m) \neq 0$ on $\mathcal{C}_{\lambda_1,\lambda_2}$, where*

$$\mathcal{C}_{\lambda_1,\lambda_2} = \Big\{(\lambda, m) \in \mathbb{R} \times \mathbb{N};\ \lambda_1 \leq \lambda \leq \lambda_2\Big\}. \tag{3.7}$$

For f in $\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})$ put

$$\mathbf{D}^{\gamma,\theta}_{p,q}(f) = \begin{cases} \left(\sum_{j \in \mathbb{Z}} \left(\frac{\|f_{\#}\theta_{2^j}\|_p}{2^{\gamma j}} \right)^q \right)^{\frac{1}{q}}, & \text{if} \quad 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} \left(\frac{\|f_{\#}\theta_{2^j}\|_p}{2^{\gamma j}} \right), & \text{if} \quad q = \infty. \end{cases}$$

Then $\mathbf{D}^{\gamma,\theta}_{p,q}$ is a norm on $\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})$ equivalent to $\|.\|_{\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})}$.

Remarks 3.14: 1) An immediate consequence of the above theorem is the independence of the norm $\mathbf{D}^{\gamma,\theta}_{p,q}$ with respect to θ that will be denoted with or without θ.
2) The case $1 < q < \infty$ could be proved by interpolation with the extreme cases ($q = 1$ and $q = \infty$), but a direct proof is presented in this paper.

Proof: Taking into account the fact that $\mathcal{F}\theta \neq 0$ on $\mathcal{C}_{\lambda_1,\lambda_2}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$; $\lambda_2 > 4\lambda_1$, where $\mathcal{C}_{\lambda_1,\lambda_2}$ is defined as in (3.7), then there exists $\mathcal{F}\sigma$ in $\mathcal{D}(\mathbb{R}^* \times \mathbb{N})$ such that $\mathcal{F}\theta(\lambda, m)\mathcal{F}\sigma(\lambda, m) = 1$ on $\mathcal{C}_{\lambda_1,\lambda_2}$. Let $\psi \in S^1_{*,0}$ satisfying $Supp\mathcal{F}\psi \subset \mathcal{C}_{4\lambda_1,\lambda_2}$. This gives, for all $1 \leq r \leq 2$ and $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$

$$\mathcal{F}\psi(2^{2j}r^2\lambda, m) = \mathcal{F}\psi(2^{2j}r^2\lambda, m)\mathcal{F}\theta(2^{2j}\lambda, m)\mathcal{F}\sigma(2^{2j}\lambda, m). \tag{3.8}$$

So, using the fact that $\mathcal{F}(\psi_r)(\lambda, m) = \mathcal{F}(\psi)(r^2\lambda, m)$ it holds

$$f\#\psi_{2^j r} = f\#\psi_{2^j r}\#\theta_{2^j}\#\sigma_{2^j}.$$

And, by the same reasoning giving in Proposition 3.3, we obtain for $1 \leq q < \infty$

$$\begin{aligned} \|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} &= \Big(\sum_{j\in\mathbb{Z}} 2^{-j\gamma q}\int_1^2 \Big(\frac{\|f\#\psi_{2^j r}\|_p}{r^\gamma}\Big)^q \frac{dr}{r}\Big)^{\frac{1}{q}} \\ &\leq C\Big(\sum_{j\in\mathbb{Z}} \Big(\frac{\|f\#\theta_{2^j}\|_p}{2^{j\gamma}}\Big)^q\Big)^{\frac{1}{q}} \\ &= C\mathbf{D}^{\gamma}_{p,q}(f). \end{aligned}$$

Conversely, let $\mathcal{F}\theta$ supported on $\mathcal{C}_{\lambda_1,\lambda_2}$ and let $\psi \in S^1_{*,0}(\mathbb{K})$ satisfying $\mathcal{F}\psi = 1$ on $\mathcal{C}_{\lambda_1,4\lambda_2}$. Then it holds, for $1 \leq r \leq 2$ and $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, that

$$\mathcal{F}\theta(2^{2j}\lambda, m) = \mathcal{F}\psi(2^{2j}r^2\lambda, m)\mathcal{F}\theta(2^{2j}\lambda, m). \tag{3.9}$$

The above reasoning leads to

$$\mathbf{D}^{\gamma}_{p,q}(f) \leq C\|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})}.$$

Now we consider the case $q = \infty$. Let us assume that $\mathbf{D}^{\gamma}_{p,q}(f) < \infty$ and let $r > 0$ and $j \in \mathbb{Z}$ be such that $2^j \leq r \leq 2^{j+1}$, then from (3.8) we get

$$\|f\#\psi_r\|_p \leq C\|f\#\theta_{2^j}\|_p \leq C2^{\gamma j} \leq Cr^\gamma$$

which implies that $\|f\|_{\dot{\Lambda}^{\gamma}_{p,\infty}(\mathbb{K})} < \infty$.
Conversely let us take $\|f\|_{\dot{\Lambda}^{\gamma}_{p,\infty}(\mathbb{K})} < \infty$, then it holds from (3.9) that for $1 \leq r \leq 2$ the following estimation

$$\|f\#\theta_{2^j}\|_p \leq C\|f\#\psi_{2^j r}\|_p \leq C(2^j r)^\gamma \leq (2^\gamma C)2^{\gamma j}.$$

This completes the proof. □

Remark 3.15: Equipped with the norm $\mathbf{D}^{\gamma}_{p,q}$, the space $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ $(1 \leq p, q \leq \infty,\ \gamma \in \mathbb{R})$ is homogeneous in a weaker sense: there exists $c_1, c_2 > 0$ such that for all $f \in \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$

$$c_1 r^{\frac{2\alpha+4}{p}-\gamma}\mathbf{D}^{\gamma}_{p,q}(f) \leq \mathbf{D}^{\gamma}_{p,q}(d_r f) \leq c_2 r^{\frac{2\alpha+4}{p}-\gamma}\mathbf{D}^{\gamma}_{p,q}(f), \qquad \textit{for all } r > 0.$$

Proposition 3.16: *Let $1 \leq p \leq \infty$ and $\gamma \in \mathbb{R}$. Then, for $1 \leq q_1 \leq q_2 \leq \infty$ we have*

$$\dot{\Lambda}^{\gamma}_{p,q_1}(\mathbb{K}) \subseteq \dot{\Lambda}^{\gamma}_{p,q_2}(\mathbb{K}) \qquad \text{(with continuous embedding).}$$

Proof: The result holds using the discrete norm and the fact that $l^{q_1} \subset l^{q_2}$ for all $q_1 \leq q_2$ and we have

$$\Big(\sum |u_j|^{q_2}\Big)^{1/q_2} \leq \Big(\sum |u_j|^{q_1}\Big)^{1/q_1}$$

for all $(u_j) \in l^{q_1}$. □

4 Generalized non homogeneous Besov-Laguerre type spaces

In this section we study the non homogeneous Besov-Laguerre type spaces defined as $L^p(\mathbb{K})$ subspaces and we give some characterizations using equivalent norms. The main tool used here is the Calderón formula on the Laguerre hypergroup introduced in Lemma 4.7. In what follows we equip the spaces $\mathbb{K}$ by the invariant measure $\frac{dxdt}{N^3(x,t)}$.

Definition 4.1: Let $1 \leq p, q \leq \infty$ and $\gamma > 0$. The non homogeneous Besov-Laguerre type space is $\Lambda^{\gamma}_{p,q}(\mathbb{K}) = \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K}) \cap L^p(\mathbb{K})$ endowed with the norm $\|.\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})}$

Theorem 4.2: *Let $1 \leq p, q \leq \infty$ and $0 < \gamma < 2$. For f in $\Lambda^{\gamma}_{p,q}(\mathbb{K})$ put*

$$\mathbf{B}^{\gamma}_{p,q}(f) = \begin{cases} \left(\displaystyle\int_{\mathbb{K}} \Big(\frac{\|\Delta_{(x,t)}f\|_p}{N^{\gamma}(x,t)}\Big)^q \frac{dxdt}{N^3(x,t)}\right)^{\frac{1}{q}}, & \textit{if} \quad 1 \leq q < \infty, \\ \underset{(x,t)\in\mathbb{K}}{\text{esssup}} \Big(\frac{\|\Delta_{(x,t)}f\|_p}{N^{\gamma}(x,t)}\Big), & \textit{if} \quad q = \infty. \end{cases}$$

Then $\mathbf{B}^{\gamma}_{p,q}$ is a norm on $\Lambda^{\gamma}_{p,q}(\mathbb{K})$ equivalent to $\|.\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})}$.

Remark 4.3: This characterization is similar to the results obtained by T. Coulhon, E. Russ and V. Tardivel-Nachef in [9].

Let us first prove the following lemmas that will be useful in the sequel.

Lemma 4.4: *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{C}$ be a measurable function. Then it holds*

$$f \in L^1(\mathbb{R}_+, \frac{dr}{r}) \qquad \textit{if and only if} \qquad f \circ N \in L^1\Big(\mathbb{K}, \frac{dxdt}{N^3(x,t)}\Big)$$

and we have

$$\int_{\mathbb{K}} f(N(x,t)) \frac{dxdt}{N^3(x,t)} = 4 \int_0^\infty f(r) \frac{dr}{r}. \tag{4.1}$$

Proof: We obtain the equality (4.1) by a polar decomposition formula. □

In the following lemma we give a Delsarte type development (see [10]) on the Laguerre hypergroup of the function $T^{(\alpha)}_{(x,t)}\psi$ using the differential operators D_1 and D_2.

Lemma 4.5: *Let ψ in $S_*(\mathbb{K})$ and $(x,t) \in \mathbb{K}$. Then, for all $(y,s) \in \mathbb{K}$, there exist $0 \leq \eta, \mu \leq 1$ such that*

$$\begin{aligned} T^{(\alpha)}_{(x,t)}\psi(y,s) &= \psi(x,t) + sT^{(\alpha)}_{(x,t)}(D_1\psi)(0,\mu s) \\ &+y^2\Big[T^{(\alpha)}_{(x,t)}(D_2\psi)(\eta y,s) - (2\alpha+1)\int_0^1 T^{(\alpha)}_{(x,t)}(D_2\psi)(\eta uy,s)u^{2\alpha+1}du\Big] \\ &-y^4\Big[\eta^2 T^{(\alpha)}_{(x,t)}(D_1^2\psi)(\eta y,s) - (2\alpha+1)\eta^2\int_0^1 u^2 T^{(\alpha)}_{(x,t)}(D_1^2\psi)(\eta uy,s)u^{2\alpha+1}du\Big]. \end{aligned} \tag{4.2}$$

Furthermore we have

$$\|\Delta_{(y,s)}\psi\|_1 \leq C(N^2(y,s) + N^4(y,s)), \qquad \textit{for all } (y,s) \in \mathbb{K}. \tag{4.3}$$

Proof: For $\psi \in S_*(\mathbb{K})$ we have (see [1, Proposition 2.2]) $T^{(\alpha)}_{(y,s)}\psi \in S_*(\mathbb{K})$. And using Proposition 2.3 together with Taylor's formula we get

$$T^{(\alpha)}_{(y,s)}\psi(x,t) = \psi(x,t) + s\frac{\partial}{\partial s}\Big(T^{(\alpha)}_{(x,t)}\psi\Big)(0,\mu s) + y^2\frac{\partial^2}{\partial y^2}\Big(T^{(\alpha)}_{(x,t)}\psi\Big)(\eta y,s)$$

with $0 \leq \mu, \eta \leq 1$. On the other hand we have (see [19])

$$\begin{cases} \dfrac{\partial\psi}{\partial s}(y,s) &= D_1\psi(y,s) \\ \dfrac{\partial^2\psi}{\partial y^2}(y,s) &= D_3\psi(y,s) - (2\alpha+1)\displaystyle\int_0^1 D_3\psi(yu,s)u^{2\alpha+1}du \end{cases}$$

where the operator D_3 is given by $D_3\psi(y,s) = (D_2 - y^2 D_1^2)\psi(y,s)$. So we obtain the development (4.2) from Propositions 2.3 and 2.4. Also from (4.2) we deduce that

$$\begin{aligned} |\Delta_{(y,s)}\psi(x,t)| \le |s| \Big| T^{(\alpha)}_{(0,\mu s)}(D_1\psi)(x,t) \Big| \\ +y^2 \left\{ \Big| T^{(\alpha)}_{(\eta y,s)}(D_2\psi)(x,t) \Big| + (2\alpha+1) \int_0^1 \Big| T^{(\alpha)}_{(\eta u y,s)}(D_2\psi)(x,t) \Big| du \right\} \\ +y^4 \left\{ \Big| T^{(\alpha)}_{(\eta y,s)}(D_1^2\psi)(x,t) \Big| + (2\alpha+1) \int_0^1 \Big| T^{(\alpha)}_{(\eta u y,s)}(D_1^2\psi)(x,t) \Big| du \right\}. \end{aligned} \tag{4.4}$$

So by integration of (4.4) over $\mathbb{K}$ with respect to the measure $d\mu_\alpha$ we obtain (4.3). □

Lemma 4.6: *Let $\psi \in S_*(\mathbb{K})$. Then, for all $(x,t) \in \mathbb{K}$ and $r > 0$ we have*

$$\|\Delta_{(x,t)}\psi_r\|_1 \le C \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big). \tag{4.5}$$

Proof: From the expression of the kernel W_α and using (4.3), one can see easily that

$$\|\Delta_{(x,t)}\psi_r\|_1 = \|\Delta_{(x,t)_r}\psi\|_1 \le C\Big(\Big(\frac{N(x,t)}{r}\Big)^2 + \Big(\frac{N(x,t)}{r}\Big)^4\Big). \tag{4.6}$$

The contraction property of the translation operators $T^{(\alpha)}_{(y,s)}$ on $L^1(d\mu_\alpha)$ (see Proposition 2.3, 2), (ii)) leads to

$$\|\Delta_{(x,t)}\psi_r\|_1 \le C \min\Big(2, \Big(\frac{N(x,t)}{r}\Big)^2 + \Big(\frac{N(x,t)}{r}\Big)^4\Big) \le 2C \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big).$$

□

The following lemma gives a version of Calderón-Zygmund formula ([14], [7]) on the Laguerre hypergroup.

Lemma 4.7: *Let* g *in* $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K}) \cap L^p(d\mu_\alpha)$*,* $1 \leq p, q \leq \infty$*,* $0 < \gamma < 2$ *and* $\psi \in S^1_{*,0}(\mathbb{K})$*. For* $1 < \varepsilon < \infty$*, put*

$$g_\varepsilon(y,s) = \int_{1/\varepsilon}^{\varepsilon} (g_\# \psi_{r\#} \psi_r)(y,s) \frac{dr}{r}, \qquad \textit{for } (y,s) \in \mathbb{K}.$$

Then, for all $(x,t) \in \mathbb{K}$*,* $\Delta_{(x,t)} g_\varepsilon$ *converges to* $\Delta_{(x,t)} g$ *in* $L^p(d\mu_\alpha)$ *as* $\varepsilon \to +\infty$*.*

Proof: Using the fact that $\int_0^\infty (\mathcal{F}\psi_r(\lambda,m))^2 \frac{dr}{r} = \int_0^\infty (\mathcal{F}\psi(r^2\lambda,m))^2 \frac{dr}{r} = 1$, we deduce easily that, for $g \in S_*(\mathbb{K})$, $(x,t) \in \mathbb{K}$ and $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$,

$$\begin{aligned}
&\mathcal{F}(\Delta_{(x,t)} g_\varepsilon)(\lambda,m) - \mathcal{F}(\Delta_{(x,t)} g)(\lambda,m) \\
&= \Big(\varphi_{\lambda,m}(x,t) - 1\Big) \mathcal{F}g(\lambda,m) \left(\int_{1/\varepsilon}^{\varepsilon} (\mathcal{F}\psi_r(\lambda,m))^2 \frac{dr}{r} - \int_0^\infty (\mathcal{F}\psi_r(\lambda,m))^2 \frac{dr}{r} \right) \\
&= \mathcal{F}(\Delta_{(x,t)} g)(\lambda,m) \left(\int_{1/\varepsilon}^{\varepsilon} (\mathcal{F}\psi_r(\lambda,m))^2 \frac{dr}{r} - \int_0^\infty (\mathcal{F}\psi_r(\lambda,m))^2 \frac{dr}{r} \right).
\end{aligned}$$

Using the fact $\mathcal{F}(\Delta_{(x,t)} g) \in S(\mathbb{R} \times \mathbb{N})$, we obtain, for all $k, p, q \in \mathbb{N}$

$$\mathcal{V}_{k,p,q}\Big(\mathcal{F}(\Delta_{(x,t)} g_\varepsilon) - \mathcal{F}(\Delta_{(x,t)} g)\Big) \xrightarrow[\varepsilon \longrightarrow \infty]{} 0.$$

And so, using the fact that ([19, Theorem II.1]) the Fourier-Laguerre transform is a topological isomorphism from $S_*(\mathbb{K})$ onto $S(\mathbb{R} \times \mathbb{N})$, one can conclude that $\Delta_{(x,t)} g_\varepsilon$ tends to $\Delta_{(x,t)} g$ in $S_*(\mathbb{K})$ as ε tends to ∞. Let us now take $g \in L^p(d\mu_\alpha)$ considered as an element of $S'_*(\mathbb{K})$ the topological dual of $S_*(\mathbb{K})$ then an elementary calculation leads to

$$< \Delta_{(x,t)} g_\varepsilon, f > = < g, \Delta_{(x,-t)} f_\varepsilon > .$$

This leads to, for all $g \in L^p(d\mu_\alpha)$

$$\Delta_{(x,t)} g_\varepsilon \xrightarrow[\varepsilon \longrightarrow \infty]{} \Delta_{(x,t)} g \qquad \textit{in } S'_*(\mathbb{K}). \tag{4.7}$$

So, using Lemma (4.6) it holds for $\varepsilon < \varepsilon'$ large that

$$\big\|\Delta_{(x,t)} g_\varepsilon - \Delta_{(x,t)} g_{\varepsilon'}\big\|_p \leq \Big(\int_{1/\varepsilon'}^{1/\varepsilon} + \int_{\varepsilon}^{\varepsilon'} \Big) \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \|g * \psi_r\|_p \frac{dr}{r}.$$

On the other hand, using Hölder's inequality and the fact that g belongs to $\Lambda^\gamma_{p,q}(\mathbb{K})$ one can prove easily that the right hand side of the above inequality tends to 0 as $\varepsilon, \varepsilon'$ tend to ∞. This implies that the family $(\Delta_{(x,t)} g_\varepsilon)_\varepsilon$ is a Cauchy net in $L^p(d\mu_\alpha)$. We get the desired result using (4.7). $\square$

Lemma 4.8: *Let $1 \le p, q \le \infty$, $0 < \gamma < 2$ and $\psi \in S^1_{*,0}$. Then*
1) *For all $\beta > 0$ there exists $C > 0$ such that for all $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$ we have, for a.e. $r > 0$,*

$$\|f \,{}_\#\psi_r\|_p \le C \int_{\mathbb{K}} \min\left(\left(\frac{N(x,t)}{r}\right)^{2\alpha+4}, \left(\frac{r}{N(x,t)}\right)^{\beta}\right) \|\Delta_{(x,t)} f\|_p \frac{dxdt}{N^3(x,t)}. \quad (4.8)$$

2) *There exists $C > 0$ such that for all $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$ we have, for a.e. $(x,t) \in \mathbb{K}$,*

$$\|\Delta_{(x,t)} f\|_p \le C \int_0^\infty \min\left(1, \left(\frac{N(x,t)}{r}\right)^2\right) \|f \,{}_\#\psi_r\|_p \frac{dr}{r}. \quad (4.9)$$

Proof: **1)** Let us take ψ in $S_{*,0}(\mathbb{K})$ and $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$. Then, using (2.1) and the fact that $\int_{\mathbb{K}} \psi(x,t) d\mu_\alpha(x,t) = 0$, we get

$$\begin{aligned} (f \,{}_\#\psi_r)(y,s) &= \int_{\mathbb{K}} \psi_r(x,-t) T^{(\alpha)}_{(x,t)} f(y,s) d\mu_\alpha(x,t) \\ &= \int_{\mathbb{K}} \psi_r(x,-t) \Delta_{(x,t)} f(y,s) d\mu_\alpha(x,t). \end{aligned}$$

From Minkowski's inequality we get

$$\begin{aligned} \|f \,{}_\#\psi_r\|_p &\le \int_{\mathbb{K}} |\psi_r(x,-t)| \, \|\Delta_{(x,t)} f\|_p \, d\mu_\alpha(x,t) \\ &= C \int_{\mathbb{K}} r^{-(2\alpha+4)} |\psi((x,-t)_r)| \, \|\Delta_{(x,t)} f\|_p \, x^{2\alpha+1} dxdt \\ &\le C \int_{\mathbb{K}} (N((x,t)_r))^{2\alpha+4} |\psi((x,-t)_r)| \, \|\Delta_{(x,t)} f\|_p \frac{dxdt}{N^3(x,t)}. \end{aligned}$$

Taking into account that ψ belongs to $S_*(\mathbb{K})$ then, for all $\beta > 0$, there exists $C > 0$ such that

$$N^{2\alpha+4}((x,t)_r)|\psi((x,-t)_r)| \leq CN^{-\beta}((x,t)_r)$$

and

$$|\psi((x,-t)_r)| \leq C.$$

Hence (4.8) follows from the above estimations.
2) Let us take f in $\Lambda^{\gamma}_{p,q}(\mathbb{K})$ and $1 < \varepsilon < \infty$. Then, for all $(x,t),(y,s) \in \mathbb{K}$, we have

$$\Delta_{(x,t)} f_\varepsilon(y,s) = \int_{1/\varepsilon}^{\varepsilon} \Big((\Delta_{(x,t)}\psi_r)_{\#} f_{\#}\psi_r\Big)(y,s)\frac{dr}{r}.$$

Minkowski's inequality together with Young's inequality imply

$$\|\Delta_{(x,t)} f_\varepsilon\|_p \leq \int_{1/\varepsilon}^{\varepsilon} \|\Delta_{(x,t)}\psi_r\|_1 \|f_{\#}\psi_r\|_p \frac{dr}{r}.$$

Now using Lemma 4.7 we obtain

$$\|\Delta_{(x,t)} f\|_p \leq \int_0^{\infty} \|\Delta_{(x,t)}\psi_r\|_1 \|f_{\#}\psi_r\|_p \frac{dr}{r}. \tag{4.10}$$

Inequality (4.9) follows immediately from (4.5) and (4.10). □

Proof: (Theorem 4.2.) Let $1 \leq p,q \leq \infty$ and $0 < \gamma < 2$. We shall prove the desired results in different cases $q = \infty$, $q = 1$ and $1 < q < \infty$.

(i) Let us start with the case $q = \infty$ which follows immediately from Lemma 4.8. Assume $\mathbf{B}^{\gamma}_{p,q}(f) < \infty$. Using (4.8) with $\beta > \gamma$ and Lemma (4.4), we get

$$\begin{aligned}
\|f_{\#}\psi_r\|_p &\leq C\int_{\mathbb{K}} \min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^{\beta}\Big)\|\Delta_{(x,t)} f\|_p \frac{dxdt}{N^3(x,t)}\\
&\leq C\int_{\mathbb{K}} N^{\gamma}(x,t)\min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^{\beta}\Big)\frac{dxdt}{N^3(x,t)}\\
&= C\int_0^{\infty} \rho^{\gamma}\min\Big(\Big(\frac{\rho}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{\rho}\Big)^{\beta}\Big)\frac{d\rho}{\rho}\\
&= Cr^{\gamma}.
\end{aligned}$$

That is $\|f\|_{\overset{\bullet}{\Lambda}^{\gamma}_{p,\infty}(\mathbb{K})}$ is finite.
Take now $f \in \Lambda^{\gamma}_{p,\infty}(\mathbb{K})$. Then from (4.9) it holds, using $0 < \gamma < 2$, that

$$\begin{aligned}\|\Delta_{(x,t)} f\|_p &\leq C \int_0^{\infty} \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \|f \# \psi_r\|_p \frac{dr}{r} \\ &\leq C \int_0^{\infty} r^{\gamma} \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \frac{dr}{r} \\ &= C N^{\gamma}(x,t).\end{aligned}$$

(ii) Let us prove the case $q = 1$. Assume $\mathbf{B}^{\gamma}_{p,1}(f) < \infty$. We shall prove that $\|f\|_{\overset{\bullet}{\Lambda}^{\gamma}_{p,1}(\mathbb{K})} \leq C\mathbf{B}^{\gamma}_{p,1}(f)$. Hence from (4.8) with $\beta > \gamma$ we obtain

$$\begin{aligned}\|f\|_{\overset{\bullet}{\Lambda}^{\gamma}_{p,1}(\mathbb{K})} &= \int_0^{\infty} \frac{\|f \# \psi_r\|_p}{r^{\gamma}} \frac{dr}{r} \\ &\leq C \int_0^{\infty} r^{-\gamma} \int_{\mathbb{K}} \min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^{\beta}\Big) \|\Delta_{(x,t)} f\|_p \frac{dxdt}{N^3(x,t)} \frac{dr}{r} \\ &= C \int_{\mathbb{K}} \|\Delta_{(x,t)} f\|_p \int_0^{\infty} r^{-\gamma} \min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^{\beta}\Big) \frac{dr}{r} \frac{dxdt}{N^3(x,t)} \\ &= C \int_{\mathbb{K}} \frac{\|\Delta_{(x,t)} f\|_p}{N^{\gamma}(x,t)} \frac{dxdt}{N^3(x,t)} \\ &= C\mathbf{B}^{\gamma}_{p,1}(f).\end{aligned}$$

Conversely, let us take f in $\Lambda^{\gamma}_{p,1}(\mathbb{K})$. Then using (4.9), we get

$$\begin{aligned}\mathbf{B}^{\gamma}_{p,1}(f) &= \int_{\mathbb{K}} \frac{\|\Delta_{(x,t)} f\|_p}{N^{\gamma}(x,t)} \frac{dxdt}{N^3(x,t)} \\ &\leq C \int_{\mathbb{K}} \int_0^{\infty} N^{-\gamma}(x,t) \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \|f \# \psi_r\|_p \frac{dr}{r} \frac{dxdt}{N^3(x,t)} \\ &= C \int_0^{\infty} \|f \# \psi_r\|_p \int_{\mathbb{K}} N^{-\gamma}(x,t) \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \frac{dxdt}{N^3(x,t)} \frac{dr}{r} \\ &= C \int_0^{\infty} \|f \# \psi_r\|_p \int_0^{\infty} \rho^{-\gamma} \min\Big(1, \Big(\frac{\rho}{r}\Big)^2\Big) \frac{d\rho}{\rho} \frac{dr}{r} \\ &= C \int_0^{\infty} \frac{\|f \# \psi_r\|_p}{r^{\gamma}} \frac{dr}{r} \\ &= C\|f\|_{\overset{\bullet}{\Lambda}^{\gamma}_{p,1}(\mathbb{K})}.\end{aligned}$$

(iii) Let us now prove the case $1 < q < \infty$. Using inequality (4.8) we obtain, for $\beta > \gamma$

$$\begin{aligned}
\frac{\|f_{\#}\psi_r\|_p}{r^\gamma} &\leq C\int_{\mathbb{K}}\Big(\frac{N(x,t)}{r}\Big)^\gamma \min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^\beta\Big)\times \\
&\qquad\qquad \frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}\frac{dxdt}{N^3(x,t)} \\
&= C\int_{\mathbb{K}} F((x,t),r)\frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}\frac{dxdt}{N^3(x,t)} \\
&= CT_F\Big(\frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}\Big)(r)
\end{aligned}$$

where $F((x,t),r) = (\frac{N(x,t)}{r})^\gamma \min((\frac{N(x,t)}{r})^{2\alpha+4}, (\frac{r}{N(x,t)})^\beta)$. By Lemma A with $h_1 = h_2 = 1$, we obtain the boundedness of the operator T_F from $L^q(\mathbb{K}, \frac{dxdt}{N^3(x,t)})$ into $L^q(\mathbb{R}_+, \frac{dr}{r})$. Moreover the condition $\mathbf{B}^\gamma_{p,q}(f) < \infty$ means that the function $(x,t) \longmapsto \frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}$ belongs to $L^q(\mathbb{K}, \mathcal{B}(\mathbb{K}), \frac{dxdt}{N^3(x,t)})$. Hence

$$\begin{aligned}
\|f\|_{\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})} &= \Big\|\frac{\|f_{\#}\psi_r\|_p}{r^\gamma}\Big\|_{L^q(\frac{dr}{r})} \\
&\leq C\Big\|T_F\Big(\frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}\Big)\Big\|_{L^q(\frac{dr}{r})} \\
&\leq C\Big\|\frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)}\Big\|_{L^q(\frac{dxdt}{N^3(x,t)})} \\
&= C\mathbf{B}^\gamma_{p,q}(f).
\end{aligned}$$

Conversely, let us take $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$. From (4.9) we get

$$\begin{aligned}
\frac{\|\Delta_{(x,t)}f\|_p}{N^\gamma(x,t)} &\leq C\int_0^\infty \Big(\frac{r}{N(x,t)}\Big)^\gamma \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big)\frac{\|f_{\#}\psi_r\|_p}{r^\gamma}\frac{dr}{r} \\
&= C\int_0^\infty F(r,(x,t))\frac{\|f_{\#}\psi_r\|_p}{r^\gamma}\frac{dr}{r} \\
&= C\,T_F\Big(\frac{\|f_{\#}\psi_r\|_p}{r^\gamma}\Big)(x,t)
\end{aligned}$$

where $F(r,(x,t)) = (\frac{r}{N(x,t)})^\gamma \min(1, (\frac{N(x,t)}{r})^2)$. We proceed as above to obtain

$$\begin{aligned} \mathbf{B}^\gamma_{p,q}(f) &\leq C \left\| T_F\Big(\frac{\|f_\#\psi_r\|_p}{r^\gamma}\Big)\right\|_{L^q(\frac{dxdt}{N^3(x,t)})} \\ &\leq C \left\| \frac{\|f_\#\psi_r\|_p}{r^\gamma}\right\|_{L^q(\frac{dr}{r})} \\ &= C\|f\|_{\overset{\bullet}{\Lambda}{}^\gamma_{p,q}(\mathbb{K})} \end{aligned}$$

which gives the desired result. □

Theorem 4.9: *Let $1 \leq p, q \leq \infty$ and $0 < \gamma < 2$. For f in $\Lambda^\gamma_{p,q}(\mathbb{K})$ put*

$$\mathbf{C}^\gamma_{p,q}(f) = \begin{cases} \left(\displaystyle\int_{\mathbb{K}} \Big(\frac{m_p(f,(x,t))}{N^\gamma(x,t)}\Big)^q \frac{dxdt}{N^3(x,t)}\right)^{\frac{1}{q}}, & \text{if} \quad 1 \leq q < \infty, \\ \underset{(x,t)\in\mathbb{K}}{\operatorname{esssup}} \Big(\dfrac{m_p(f,(x,t))}{N^\gamma(x,t)}\Big), & \text{if} \quad q = \infty \end{cases}$$

$m_p(f,(x,t)) = \sup\limits_{\substack{0\leq y\leq x\\ 0\leq|s|\leq|t|}} \|\Delta_{(y,s)} f\|_p$ being the generalized modulus of continuity on the Laguerre hypergroup. Then $\mathbf{C}^\gamma_{p,q}$ is a norm on $\Lambda^\gamma_{p,q}(\mathbb{K})$ equivalent to $\|.\|_{\overset{\bullet}{\Lambda}{}^\gamma_{p,q}(\mathbb{K})}$.

Proof: To compare $\|f\|_{\overset{\bullet}{\Lambda}{}^\gamma_{p,q}(\mathbb{K})}$ and $\mathbf{C}^\gamma_{p,q}(f)$ we proceed as in the proof of the Theorem 4.2 using the following lemma instead of Lemma 4.8. □

Lemma 4.10: *Let $1 \leq p, q \leq \infty$, $0 < \gamma < 2$ and $\psi \in S_{*,0}$. Then*
1) *For all $\beta > 0$ there exists $C > 0$ such that for all $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$ we have, for a.e. $r > 0$,*

$$\|f_\#\psi_r\|_p \leq C\int_{\mathbb{K}} \min\Big(\Big(\frac{N(x,t)}{r}\Big)^{2\alpha+4}, \Big(\frac{r}{N(x,t)}\Big)^\beta\Big) m_p(f,(x,t)) \frac{dxdt}{N^3(x,t)}. \tag{4.11}$$

2) *There exists $C > 0$ such that for all $f \in \Lambda^\gamma_{p,q}(\mathbb{K})$ we have, for a.e. $(x,t) \in \mathbb{K}$,*

$$m_p(f,(x,t)) \leq C \int_0^\infty \min\Big(1, \Big(\frac{N(x,t)}{r}\Big)^2\Big) \|f_\#\psi_r\|_p \frac{dr}{r}. \tag{4.12}$$

Proof: **1)** (4.11) follows immediately from (4.8) and the fact that

$$\|\Delta_{(x,t)} f\|_p \leq m_p\,(f,(x,t)), \qquad \forall\,(x,t) \in \mathbb{K}.$$

2) To prove (4.12) we use the inequality

$$N(y,s) \leq N(x,t)\ ; \qquad 0 \leq y \leq x \text{ and } 0 \leq |s| \leq |t|.$$

So it holds

$$\begin{aligned} \|\Delta_{(y,s)} f\|_p &\leq C\int_0^\infty \min\Big(1,\Big(\frac{N(y,s)}{r}\Big)^2\Big)\|f\#\psi_r\|_p\frac{dr}{r} \\ &\leq C\int_0^\infty \min\Big(1,\Big(\frac{N(x,t)}{r}\Big)^2\Big)\|f\#\psi_r\|_p\frac{dr}{r}. \end{aligned}$$

And hence

$$m_p\,(f,(x,t)) \leq C\int_0^\infty \min\Big(1,\Big(\frac{N(x,t)}{r}\Big)^2\Big)\|f\#\psi_r\|_p\frac{dr}{r}.$$

The proof is finish. □

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