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Notes on prequantization of moduli of G-bundles with connection on Riemann surfaces

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Abstract

Let $\mathcal{X} \to S$ be a smooth proper family of complex curves (i.e. family of Riemann surfaces), and $\mathcal{F}$ a G-bundle over $\mathcal{X}$ with connection along the fibres $\mathcal{X} \to S$. We construct a line bundle with connection $(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})$ on S (also in cases when the connection on $\mathcal{F}$ has regular singularities). We discuss the resulting $(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})$ mainly in the case $G = \mathbb{C}^*$. For instance when S is the moduli space of line bundles with connection over a Riemann surface X, $\mathcal{X} = X \times S$, and $\mathcal{F}$ is the Poincaré bundle over $\mathcal{X}$, we show that $(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})$ provides a prequantization of S.

1 Introduction

Of special interest in physics are line bundles with connection over various moduli spaces of G-bundles with connection over Riemann surfaces. Such line bundles are used to construct conformal field theories, which for example produce interesting 3-manifold (topological) invariants. We consider the problem of constructing such line bundles as a problem of constructing Deligne cohomology classes.

Recall the Deligne complexes on an algebraic variety S, roughly

$$\mathbb{Z}(n) = ...0 \to \mathbb{Z} \to \mathcal{O} \to \Omega^1 \to ... \to \Omega^n \to 0 \to ...$$

In particular $\mathbb{Z}(1) = ...0 \to \mathbb{Z} \to \mathcal{O} \to 0... \simeq \mathcal{O}^*[1]$, and hence classes in $H^2(S, \mathbb{Z}(1))$ correspond to isomorphism classes of line bundles on S. Further, classes in $H^2(S, \mathbb{Z}(2))$ correspond to isomorphism classes of line bundles with connection on S. So our objective is to construct classes in $H^2(S, \mathbb{Z}(2))$ where S is one of the moduli spaces being considered.

In topology, the theory of characteristic classes constructs cohomology classes on spaces which are equiped with say G-bundles over them. Let us review how this is done. Recall that a G-bundle on a space Y is described by a classifying map $Y \to BG$ (defined up to homotopy). Also $H^{\cdot}(BG, \mathbb{C}) \cong (Sym^{\cdot}\mathfrak{g}^*)^G$ as graded algebras, with elements of $\mathfrak{g}^*$ being of degree 2. So a G-bundle on Y yields for instance classes in $H^4(Y, \mathbb{C})$ corresponding to G-invariant bilinear forms on $\mathfrak{g}$.

In the setting of objects with algebraic structure, one may carry out an analogous procedure to construct classes in Deligne cohomology.

Suppose Y is an algebraic variety, G a reductive algebraic group, and $\mathcal{F}$ a G-bundle over Y. Instead of a classifying map we have

$$Y \leftarrow \mathcal{F} \times_G \Delta G \to BG,$$

where ΔG is the standard simplicial G-scheme model of EG and $BG := \Delta G/G$. But since $\Delta G \sim pt.$ as a simplicial scheme,

$$H^{2m}(Y, \mathbb{Z}(m)) \xrightarrow{\sim} H^{2m}(\mathcal{F} \times_G \Delta G, \mathbb{Z}(m)) \leftarrow H^{2m}(BG, \mathbb{Z}(m)).$$

And there is a natural map $H^{2m}(BG, \mathbb{Z}(m)) \to H^{2m}_{top}(BG, \mathbb{Z})$ which is actually an isomorphism [1]. So again, we have classes in $H^4(Y, \mathbb{Z}(2))$ corresponding to certain bilinear G-invariant forms on $\mathfrak{g}$.

Consider the particular situation of a proper smooth family of complex curves (i.e. family of compact Riemann surfaces) $\mathcal{X} \to S$, and a G-bundle $\mathcal{F}$ on $\mathcal{X}$. We may construct classes in $H^4(\mathcal{X}, \mathbb{Z}(2))$ as above. And further, classes in $H^4(\mathcal{X}, \mathbb{Z}(2))$ may be integrated down to a obtain classes in $H^2(S, \mathbb{Z}(1))$.

We consider arbitrary $\mathcal{X} \to S$ as above, and $\mathcal{F}$ with (relative) connection; and fix a class in $H^4(BG, \mathbb{Z}(2))$. By following the same construction with slightly different complexes we produce a class in $H^2(S, \mathbb{Z}(2))$. i.e. a line bundle with connection on S. In the cases relevant to physics the curvature is the natural 2-form which the considered moduli spaces carry; which makes our objects prime candidates for physical applications.

Here we discuss the complexes involved in the construction, and compute the curvature of the resulting objects in the case when $G = \mathbb{C}^*$.

The ideas discussed here grew out of conversations with A. Beilinson, who in particular suggested a version of our main construction.

2 Construction.

2.1 Regular case.

Consider $\mathcal{X} \xrightarrow{\pi} S$ a family of proper smooth curves, $\mathcal{F}$ a G-bundle on $\mathcal{X}$ with connection along the fibres. Put $\Delta\mathcal{F} := \mathcal{F} \times_G \Delta G$; and denote $q : \Delta\mathcal{F} \to \mathcal{X}$, $p : \Delta\mathcal{F} \to BG$.

We will construct complexes $\mathbb{Z}^{\Delta}(n)$, $\mathbb{Z}_{\pi}(n)$ on $\Delta\mathcal{F}$, $\mathcal{X}$ respectively, for which a diagram of the form

$$H^4(BG, \mathbb{Z}(2)) \to H^4(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(2)) \xleftarrow{\sim} H^4(\mathcal{X}, \mathbb{Z}_{\pi}(2)) \xrightarrow{tr} H^2(S, \mathbb{Z}(2))$$

holds.

First of all recall that for any complex $C^{\cdot}$ on S, there is a natural

$$H^n(\mathcal{X}, \pi^*(C^{\cdot})) \xrightarrow{tr} H^{n-2}(S, C^{\cdot}),$$

so put $\mathbb{Z}_{\pi}(n) = \pi^*(\mathbb{Z}(n))$.

We shall now construct $\mathbb{Z}^{\Delta}$ on $\Delta\mathcal{F}$ such that $p : \Delta\mathcal{F} \to BG$, $q : \Delta\mathcal{F} \to \mathcal{X}$ induce

$$H^4(BG, \mathbb{Z}(2)) \to H^4(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(2)) \xleftarrow{\sim} H^4(\mathcal{X}, \mathbb{Z}_{\pi}(2)).$$

Suppose U is a neighbourhood in $\mathcal{X}$ over which the relative connection $\mathcal{F}$ can be extended to a total flat connection and further there is a (flat) trivialization $\mathcal{F}_U \xrightarrow{\sim} U \times G$. Which yields a trivialization $\Delta\mathcal{F}_U \xrightarrow{\sim} U \times \Delta G$, and a map $t_U : \Delta\mathcal{F}_U \to S \times \Delta G$. Put $\mathbb{Z}^{\Delta}_{\Delta\mathcal{F}_U}(n) = t_U^*(\mathbb{Z}(n))$, which defines a complex $\mathbb{Z}^{\Delta}(n)$ on $\Delta\mathcal{F}$.

$\Delta\mathcal{F} \xrightarrow{p} BG$ induces $p^* : p^*(\mathbb{Z}(n)) \to \mathbb{Z}^{\Delta}(n)$ because p factors through t_U. Also $\Delta\mathcal{F} \xrightarrow{q} \mathcal{X}$ induces $q^* : q^*(\mathbb{Z}_{\pi}(n)) \to \mathbb{Z}^{\Delta}(n)$.

Claim: $H^4(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(2)) \xleftarrow{q^*} H^4(\mathcal{X}, \mathbb{Z}_{\pi}(2))$ is an isomorphism.

We will first check that $q^*(\mathbb{Z}_{\pi})(n) \to \mathbb{Z}^{\Delta}(n)$ is a quasi-isomorphism by showing that that is the case locally.

Consider U as in the construction of $\mathbb{Z}^{\Delta}(n)$, and $t_U : \Delta\mathcal{F}_U \to S \times \Delta G$ as before. Since $\Delta G \simeq pt.$, $\mathbb{Z}(n) \simeq p_1^*(\mathbb{Z}(n))$ on $S \times \Delta G$. Hence

$$\begin{aligned} t_U^*(\mathbb{Z}(n)) &\simeq t_U * (p_1^*(\mathbb{Z}(n))) \simeq (p_1 \circ t_U)^*(\mathbb{Z}(n)) = (\pi \circ q)^*(\mathbb{Z}(n)) \\ &\simeq q^*(\pi^*(\mathbb{Z}(n))) = q^*(\mathbb{Z}_{\pi}(n)). \end{aligned}$$

Finally, the map induced by p^* on cohomology is

$$H^m(\mathcal{X}, \mathbb{Z}_\pi(n)) \xrightarrow{\sim} H^m(\Delta\mathcal{F}, q^*(\mathbb{Z}(n))) \xrightarrow{\sim} H^m(\Delta\mathcal{F}, \mathbb{Z}^\Delta(n)),$$

with the first map being an isomorphism again because $\Delta G \simeq pt$. $\square$

$$H^4(BG, \mathbb{Z}(2)) \xrightarrow{p^*} H^4(\Delta\mathcal{F}, \mathbb{Z}^\Delta(2)) \xleftarrow{\sim} H^4(\mathcal{X}, \mathbb{Z}_\pi(2)) \xrightarrow{tr} H^2(S, \mathbb{Z}(2))$$

now yields an isomorphism class $(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})$.

2.2 Case of regular singularities.

We shall discuss the case of G compact, and then comment about the case of general G.

$\mathcal{X} \to S, \mathcal{F}, \Delta\mathcal{F}$ as before, and non-intersecting sections $S \xrightarrow{\sigma_i} \mathcal{X}$, along which the connection on $\mathcal{F}$ has regular singularities. Assume that the isomorphism type of the $\mathcal{F}_s$ at the marked points is constant, and fix trivializations of the underlying bundle of $\mathcal{F}$ at the $\sigma_i(s)$'s.

Notation: $\Omega^\cdot(n) := 0 \to \Omega^0 \to ... \to \Omega^{n-1} \to 0...$ ($\Omega^0 = \mathcal{O}$ being in position 0).
$\Omega^\cdot_\pi(n) := \pi^*(\Omega^\cdot_S(n))$,
$\Omega^{\cdot,G}_{\Delta G}(n) := (\Omega^\cdot_{\Delta G}(n))^G \simeq \Omega^\cdot_{\Delta G}(n)$.

Remark: Let Y have a G-torsor $\mathcal{H}$, and suppose G acts freely on E. Then there is a canonical map $\Omega^{\cdot,G}_E \hookrightarrow \Omega^*_{Y\times_\mathcal{H} E}$. A special case of this is when $E = G$, in which case $\mathfrak{g}^* \xrightarrow{\sim} \Omega^{1,G}_G \to \Omega^1_{Y\times_\mathcal{H} G}$ maps $c \in \mathfrak{g}^*$ to $c(\sigma)$, with σ being the connection 1-form of $\mathcal{H}$. Let $\Omega^\cdot_{Y,\mathcal{H},E} (\simeq \Omega^{\cdot,G}_E)$ denote the image.

Put $\Omega^{\cdot,\Delta}(n) := q^*(\Omega^\cdot_\pi(n)) \otimes_\mathbb{C} \Omega^\cdot_{\mathcal{X},\mathcal{F},\Delta G} (\hookrightarrow \Omega^\cdot_{\Delta\mathcal{F}})$,
and

$$\begin{aligned}\mathbb{Z}_\pi(n) := \pi^*(\mathbb{Z}_S(n)) &\simeq 0 \to \pi^*(\mathbb{Z}) \to \Omega^0_\pi \to ... \to \Omega^n_\pi \to 0...\\ &\simeq 0 \to \mathbb{Z} \to \Omega^0_\pi \to ... \to \Omega^n_\pi \to 0...,\end{aligned}$$

$$\mathbb{Z}^\Delta(n) := 0 \to \mathbb{Z} \to \Omega^{0,\Delta} \to ... \to \Omega^{n,\Delta} \to 0 \to ...$$

For general G, replace $\Omega^\cdot$ complexes for BG, ΔG, by appropriate $\Omega^\cdot[logD]$ complexes.

3 Curvature in the regular case with $G = \mathbb{C}^*$

Let S= infinitesimal point with closed point 0, $\mathcal{X} = X \times S$ for X a smooth proper complex curve. Denote $B\mathbb{C}^*$ by $\mathbb{P}$.

Consider

$$H^2(\mathbb{P}, \mathbb{Z}) \overset{\sim}{\leftarrow} H^2(\mathbb{P}, \mathbb{Z}(1)) \overset{p^*}{\to} H^2(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(1)) \overset{q^*}{\leftarrow} H^2(\mathcal{X}, \mathbb{Z}_{\pi}(1)).$$

Let $c \in H^2(\mathbb{P}, \mathbb{Z})$ be the canonical generator, and c^{Δ}, c^{π} the corresponding classes in $H^2(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(1)), H^2(\mathcal{X}, \mathbb{Z}_{\pi}(1))$ respectively.

Consider $\mathbb{Z}_{\pi}(1) \simeq \pi^*(\mathcal{O}_S^*)[1]$. Then $\mathcal{F}$ is described by a class $[\mathcal{F}]_{\pi} \in H^2(\mathcal{X}, \mathbb{Z}_{\pi}(1))$.

Claim: $c_{\pi} = [\mathcal{F}]_{\pi}$.

Let $\{U_i\}$ be open cover of $\mathbb{P}$ s.t. (U_i, f_{ij}) describe c; and $\{V_k\}$ open cover of $\mathcal{X}$ s.t. over each V_k there is a flat trivialization of $\mathcal{F}$, so it is described by (V_k, g_{kl}) with the g_{kl} being locally constant along fibres $\mathcal{X} \to S$. Denote $\phi : \Delta\mathbb{C}^* \to \mathbb{P}$. Since $\phi^*(c)$ is trivial, there are F_i on $\phi^{-1}(U_i)$ such that $f_{ij} = F_j F_i^{-1}$ and $a^*(F_i) = a^{-1} F_i$ for any $a \in \mathbb{C}^*$ acting on $\phi^{-1}(U_i)$.

Note that $(V_k \times U_i, F_i)$ describes a chain with values in $\mathbb{Z}^{\Delta}(1)$. But the boundary of $(V_k \times U_i, F_i)$ is exactly $\{(V_k \times U_i, g_{kl})\} - p^*(c)$, so $p^*(c) = q^*([\mathcal{F}]_{\pi})$ in $H^2(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(1))$. $\square$

Notation: $\mathfrak{d} : \mathbb{Z}(n) \to \Omega^n[n]$, and the same letter will denote the induced map on cohomology.

For $v \in T_0 S$, $\centerdot_v : H^2(\mathcal{X}, \pi^*(\Omega^1)[1]) \to H^2(\mathcal{X}, \pi^*(\mathbb{C})[1])$ is that induced by contraction by v.

Claim: $\mathfrak{d}(c_{\pi})_v \in H^2(X, \mathbb{C}[1]) \cong H^1(X, \mathbb{C})$ is actually the class describing the infinitesimal deformation of $\mathcal{F}_0$ along S in the direction v.

By considering classes in $H^2(\mathcal{X}, \mathbb{Z}_{\pi}(1)) \cong H^1(\mathcal{X}, \pi^*(\mathcal{O}_S^*))$ as represented by Čech cocycles, the current claim follows from the previous one. $\square$

Recall that $c^2 \in H^4(BG, \mathbb{Z}(2))$ is the canonical generator. Then $(c^{\Delta})^2, c_{\pi}^2$ are the corresponding classes in $H^4(\Delta\mathcal{F}, \mathbb{Z}^{\Delta}(2)), H^4(\mathcal{X}, \mathbb{Z}_{\pi}(2))$ respectively.

Denote the curvature of $(\mathcal{L}_{\mathcal{F}}, \nabla_{\mathcal{F}})$ by $\omega_{\mathcal{F}}$.

Claim: for $v, w \in T_0 S$, $\omega_{\mathcal{F}}(v, w) = tr(\mathfrak{d}(c_{\pi})_v \centerdot \mathfrak{d}(c_{\pi})_w)$.

$\mathfrak{d} : \mathbb{Z}_\pi(2) \to \pi^*(\Omega^2)$ on $\mathcal{X}$ is the pullback of $\mathfrak{d} : \mathbb{Z}(2) \to \Omega^2$ on S, so the naturality of tr implies that

$$\mathfrak{d}([(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})])(v,w) = tr(\mathfrak{d}(c_\pi^2)_{v,w}).$$

But $\mathfrak{d}$ actually induces a ring map on cohomology, then

$$tr(\mathfrak{d}(c_\pi^2)_{v,w}) = tr(\mathfrak{d}(c_\pi)_v \centerdot \mathfrak{d}(c_\pi)_w).$$

$(\centerdot_{v,w} : \pi^*(\Omega^2) \to \mathbb{C}$ is induced by contraction by $v, w)$. □

Consider S being the moduli space of line bundles with connection over X and $\mathcal{F}$ the Picard bundle over S. For any $s \in S$, T_sS can be identified with $H^1(X,\mathbb{C})$. So $H^1(X,\mathbb{C}) \otimes H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C}) \to \mathbb{C}$ defines a non-degenerate bilinear form on T_sS, which actually endows S with a symplectic structure. Denote the symplectic form by ω_S.

Proposition. $\omega_S = \omega_\mathcal{F}$. i.e. $(\mathcal{L}_\mathcal{F}, \nabla_\mathcal{F})$ is a prequantization of S.

This is a direct consequence of the previous two claims. □

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