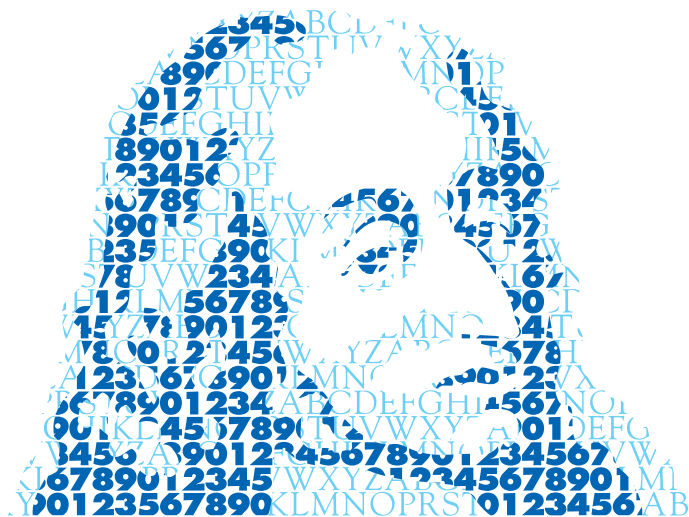


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On an integral formula of Berndtsson related to the inversion of the Fourier-Laplace transform of $\bar{\partial}$ -closed $(n, n - 1)$ -forms

Telemachos Hatziafratis

Abstract

We give a proof of an integral formula of Berndtsson which is related to the inversion of Fourier-Laplace transforms of $\bar{\partial}$ -closed $(n, n - 1)$ -forms in the complement of a compact convex set in \mathbb{C}^n .

1 Introduction

Let K be a compact and convex subset of \mathbb{C}^n and $F(\zeta)$ an entire analytic function of the following exponential type: For every $\delta > 0$ there exists a constant $C_\delta > 0$ so that

$$|F(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|) \quad (\zeta \in \mathbb{C}^n), \tag{1.1}$$

where $H_K(\zeta) = \sup\{\text{Re}\langle z, \zeta \rangle : z \in K\}$ and $\langle z, \zeta \rangle = \sum_{j=1}^n z_j \zeta_j$. One way to produce functions $F(\zeta)$, which satisfy (1.1), is to take a $\bar{\partial}$ -closed $(n, n - 1)$ -form $\theta(z)$ in $\mathbb{C}^n - K$ and consider its Fourier-Laplace transform $F_\theta(\zeta)$ defined by the integral

$$F_\theta(\zeta) = \int_{z \in S} e^{\langle z, \zeta \rangle} \theta(z),$$

where S is a smooth $(2n - 1)$ -dimensional surface surrounding K . Then it is easy to see that F_θ does not depend on the choice of S and that it satisfies (1.1).

In [2], we showed that, conversely, any entire function $F(\zeta)$, which satisfies (1.1), is $F_\theta(\zeta)$ for some $\theta \in Z_{\bar{\partial}}^{(n, n-1)}(\mathbb{C}^n - K)$. (Notation: $Z_{\bar{\partial}}^{(n, n-1)}$ denotes sets of $\bar{\partial}$ -closed $(n, n - 1)$ -forms.) The proof uses an integral of Berndtsson, which is defined as follows:

$$\theta_F^\rho(z) = a_n \left(\int_{t=0}^{\infty} t^{n-1} e^{-t\langle z, \partial\rho/\partial z \rangle} F(t\partial\rho/\partial z) dt \right) \partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1},$$

for $z \in \mathbb{C}^n - \{\rho \leq 1\}$,

where $\partial\rho/\partial z = (\partial\rho/\partial z_1, \dots, \partial\rho/\partial z_n)$ and $a_n = 1/[(n-1)!(2\pi i)^n]$. Here we assume that $0 \in K$ and that the function ρ is chosen to be convex, positively homogeneous (i.e., $\rho(sz) = s\rho(z)$ for $s > 0$) and such that $\{\rho < 1\}$ is a strictly convex neighborhood of K . ((1.1) is needed for the convergence of the above integral.)

Then Berndtsson proved (in [1]) that if the entire function $F(\zeta)$ satisfies (1.1) then

$$\int_{\{\rho(z)=1\}} e^{\langle z, \zeta \rangle} \theta_F^\rho(z) = F(\zeta), \quad \text{for } \zeta \in \mathbb{C}^n. \quad (1.2)$$

The proof given in [1] was based on an integral formula with weights and a change of variables, using some facts from convex analysis concerning the polar set of the convex set $\{\rho \leq 1\}$.

In this note we will give a proof of (1.2) by a direct computation of the integral which is based on the following observations: First, the differential form $\theta_F^\rho(z)$ is $\bar{\partial}$ -closed in the set $\mathbb{C}^n - \{\rho \leq 1\}$ (see [2, Lemma 1]) and therefore

$$\int_{\{\rho(z)=1\}} e^{\langle z, \zeta \rangle} \theta_F^\rho(z) = \int_{\{|z|=R\}} e^{\langle z, \zeta \rangle} \theta_F^\rho(z) \quad (1.3)$$

when the sphere $\{|z| = R\}$ surrounds the compact set $\{\rho \leq 1\}$, and second, if we expand the entire function $F(\zeta)$ in power series

$$F(\zeta) = \sum_{k_1, \dots, k_n \geq 0} c_{k_1, \dots, k_n} \zeta_1^{k_1} \dots \zeta_n^{k_n}$$

and we if we substitute this expansion in the integral which defines $\theta_F^\rho(z)$, then we may interchange the order of summation and integration, provided that R is sufficiently large.

After this interchange we see that $\theta_F^\rho(z)$ is a combination of terms of the form

$$a_n \left(\int_{t=0}^{\infty} t^{n+k_1+\dots+k_n-1} e^{-t\langle z, \partial\rho/\partial z \rangle} dt \right) \times \left(\frac{\partial\rho}{\partial z_1} \right)^{k_1} \dots \left(\frac{\partial\rho}{\partial z_n} \right)^{k_n} \partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1}. \quad (1.4)$$

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Since $\int_{t=0}^{\infty} t^N e^{-t\sigma} dt = \frac{N!}{\sigma^{N+1}}$ (for $\operatorname{Re}\sigma > 0$), we see that

$$\int_{t=0}^{\infty} t^{n+k_1+\dots+k_n-1} e^{-t\langle z, \partial\rho/\partial z \rangle} dt = \frac{(n+k_1+\dots+k_n-1)!}{\langle z, \partial\rho/\partial z \rangle^{n+k_1+\dots+k_n}}.$$

It follows that (1.4) is the following derivative of the Cauchy-Fantappiè kernel:

$$(n-1)! a_n \frac{\partial^{k_1+\dots+k_n}}{\partial w_1^{k_1} \dots \partial w_n^{k_n}} \Big|_{w=0} \left(\frac{\partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1}}{\langle z-w, \partial\rho/\partial z \rangle^n} \right). \quad (1.5)$$

Now recall the Cauchy-Fantappiè formula: For entire functions f ,

$$\frac{1}{(2\pi i)^n} \int_{\{|z|=R\}} f(z) \frac{\partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1}}{\langle z-w, \partial\rho/\partial z \rangle^n} = f(w) \quad (|w| < R).$$

Differentiating both sides of this equation with respect to w , we obtain that

$$\int_{\{|z|=R\}} f(z) \mathfrak{F}_{k_1, \dots, k_n}^\rho(z) = \frac{\partial^{k_1+\dots+k_n} f}{\partial w_1^{k_1} \dots \partial w_n^{k_n}}(0), \quad (1.6)$$

where $\mathfrak{F}_{k_1, \dots, k_n}^\rho(z)$ is the kernel (1.5) (which, as we pointed out, is equal to (1.4)).

These observations lead to a proof of the following theorem.

Theorem 1.1: *If the entire function $F(\zeta) = \sum c_k \zeta^k$ satisfies (1.1) then*

$$\int_{\{\rho(z)=1\}} f(z) \theta_F^\rho(z) = \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} \frac{\partial^{k_1+\dots+k_n} f}{\partial w_1^{k_1} \dots \partial w_n^{k_n}}(0), \text{ for every entire function } f. \quad (1.7)$$

Notice that (1.2) is the formula (1.7) when $f(z) = e^{\langle z, \zeta \rangle}$. Since the set of the functions $e^{\langle z, \zeta \rangle}$, $\zeta \in \mathbb{C}^n$, is dense in the space of entire functions (with the topology of uniform convergence on compact sets), (1.2) is actually equivalent to (1.7).

2 The proof of the Theorem

First (1.1) guarantees the convergence of the integral which defines $\theta_F^\rho(z)$, for $\rho(z) \geq 1$ (see [2, p.910]) and, as we pointed out before, $\theta_F^\rho \in Z_{\bar{\partial}}^{(n,n-1)}(\mathbb{C}^n - \{\rho \leq 1\})$. Therefore, by Stokes's theorem, the integral $\int_{\{\rho=1\}} f \theta_F^\rho$ is equal to

$$a_n \int_{\{|z|=R\}} f(z) \int_{t=0}^{\infty} t^{n-1} e^{-t(z, \partial \rho / \partial z)} \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} t^{k_1 + \dots + k_n} \\ \times \left(\frac{\partial \rho}{\partial z_1} \right)^{k_1} \dots \left(\frac{\partial \rho}{\partial z_n} \right)^{k_n} dt \partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}. \quad (2.1)$$

We want to show that we may interchange the order of integration and summation in (2.1), provided that R is sufficiently large. By Lebesgue's dominated convergence theorem, it suffices to choose R so that

$$\sum_{k_1, \dots, k_n} |c_{k_1, \dots, k_n}| \int_{\{|z|=R\}} \int_{t=0}^{\infty} \left| f(z) t^{n-1} e^{-t(z, \partial \rho / \partial z)} t^{k_1 + \dots + k_n} \right. \\ \left. \times \left(\frac{\partial \rho}{\partial z_1} \right)^{k_1} \dots \left(\frac{\partial \rho}{\partial z_n} \right)^{k_n} \right| dt |\partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}| < \infty. \quad (2.2)$$

For this purpose, we will need an estimate for the coefficients c_k , which follows from (1.1). First (1.1) implies that $|F(\zeta)| \leq A e^{B|\zeta|}$ for $\zeta \in \mathbb{C}^n$, where A and B are positive constants. Using this and Cauchy's formula in the polydisc, we see that the coefficients

$$c_{k_1 \dots k_n} = \frac{1}{k_1! \dots k_n!} \frac{\partial^{k_1 + \dots + k_n} F}{\partial \zeta_1^{k_1} \dots \partial \zeta_n^{k_n}}(0)$$

satisfy the inequality

$$|c_{k_1 \dots k_n}| \leq A \frac{e^{B(r_1 + \dots + r_n)}}{r_1^{k_1} \dots r_n^{k_n}}, \quad \text{for every } r_1, \dots, r_n > 0.$$

Applying this with $r_1 = k_1/B, \dots, r_n = k_n/B$, we obtain that

$$|c_{k_1 \dots k_n}| \leq A \frac{(eB)^{k_1 + \dots + k_n}}{k_1^{k_1} \dots k_n^{k_n}}, \quad \text{for every } k_1, \dots, k_n. \quad (2.3)$$

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On the other hand $(\partial\rho/\partial z_j)(sz) = (\partial\rho/\partial z_j)(z)$ for $s > 0$, and therefore

$$\left| \frac{\partial\rho}{\partial z_j}(z) \right| \leq \beta \stackrel{def}{=} \max \left\{ \left| \frac{\partial\rho}{\partial z_j}(\xi) \right| : |\xi| = 1, j = 1, \dots, n \right\} \quad (z \neq 0).$$

Also the function $\gamma(z) \stackrel{def}{=} \operatorname{Re}\langle z, \partial\rho/\partial z \rangle$ has the property $\gamma(sz) = s\gamma(z)$ ($s > 0$), and therefore

$$\gamma(z) = |z|\gamma(z/|z|) \geq \epsilon|z| \quad \text{for } z \neq 0, \quad \text{where } \epsilon \stackrel{def}{=} \min\{\gamma(\xi) : |\xi| = 1\} > 0.$$

It follows that

$$\begin{aligned} \int_{t=0}^{\infty} t^{n+k_1+\dots+k_n-1} |e^{-t\langle z, \partial\rho/\partial z \rangle}| dt &= \frac{(n+k_1+\dots+k_n-1)!}{[\gamma(z)]^{n+k_1+\dots+k_n}} \\ &\leq \frac{(n+k_1+\dots+k_n-1)!}{(\epsilon|z|)^{n+k_1+\dots+k_n}}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\{|z|=R\}} \int_{t=0}^{\infty} &\left| f(z) t^{n-1} e^{-t\langle z, \partial\rho/\partial z \rangle} t^{k_1+\dots+k_n} \left(\frac{\partial\rho}{\partial z_1} \right)^{k_1} \dots \left(\frac{\partial\rho}{\partial z_n} \right)^{k_n} \right| dt \\ &\times |\partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1}| \\ &\leq \frac{(n+k_1+\dots+k_n-1)!}{(\epsilon R)^{n+k_1+\dots+k_n}} \beta^{k_1+\dots+k_n} \int_{\{|z|=R\}} |f(z)| |\partial\rho(z) \wedge [\bar{\partial}\partial\rho(z)]^{n-1}|. \end{aligned}$$

This inequality together with (2.3) imply that, in order to have (2.2), it suffices to choose R so that

$$\sum_{k_1, \dots, k_n} \frac{(n+k_1+\dots+k_n-1)!}{(\epsilon R)^{k_1+\dots+k_n}} \frac{(\beta e B)^{k_1+\dots+k_n}}{k_1^{k_1} \dots k_n^{k_n}} < \infty. \quad (2.4)$$

But (2.4) holds, if $R > n\beta e B/\epsilon$, since

$$\begin{aligned} \sum_{k_1, \dots, k_n} \frac{(k_1+\dots+k_n)!}{k_1! \dots k_n!} \tau_1^{k_1} \dots \tau_n^{k_n} &= \frac{1}{1 - (\tau_1 + \dots + \tau_n)} \\ &\text{for } \tau_1 + \dots + \tau_n < 1, \tau_j > 0. \end{aligned}$$

Thus, working with $R > n\beta eB/\epsilon$, we may interchange the order of integration and summation in (2.1). The result is that the integral $\int_{\{\rho=1\}} f\theta_F^\rho$ is equal to the sum

$$\sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} \int_{\{|z|=R\}} f(z) \mathfrak{F}_{k_1, \dots, k_n}^\rho(z),$$

which, by (1.6), is equal to $\sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} \frac{\partial^{k_1 + \dots + k_n} f}{\partial w_1^{k_1} \dots \partial w_n^{k_n}}(0)$.

This proves (1.7) and completes the proof of the Theorem.

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