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On finitely generated birational flat extensions of integral domains

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1 Introduction

In this paper, all rings and their extensions are commutative with a unit element.

It is well-known that birational, integral, flat extensions of integral domains are trivial.

Our objective is to extend this fact to a result that birational, finitely generated, flat extensions of integral domains are open-immersions. In addition, we show that their complementary closed sets are of grade one if not empty.

We use the following notation unless otherwise specified: R is an integral domain with quotient field K and A is a birational extension of R in K.

2 Results

Lemma 2.1: *Assume that A is flat over R and that the canonical morphism* $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ *is surjective. Then $R = A$.*

Proof: Take $a \in A$ with $a = y/x$ $(x, y \in R)$. A is faithfully flat over R by [3, (7.2)]. So it follows that $y = ax \in xA \cap R = xR$ (cf. [3, (7.5)]). Hence $a = y/x \in R$. Therefore $R = A$.

□

Proposition 2.2: *Let A be a birational extension of R. If A is integral and flat over R, then $A = R$.*

Proof: Since A is integral over R, the canonical map $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ is surjective. Hence our conclusion follows from Lemma 2.1.

□

Lemma 2.3: *Assume that A is flat over R. Then for any $P \in \operatorname{Spec}(A)$, $A_P = R_{P \cap R}$. Moreover, $A_{P \cap R} = R_{P \cap R}$.*

Proof: Put $p = P \cap R$. Then $R_p \to A_P$ is flat. As a flat extension of rings satisfies the Going-Down Theorem (cf. [2, (5.D)]), $\operatorname{Spec}(A_P) \to \operatorname{Spec}(R_p)$ is surjective. Hence $A_P = R_p$ by Lemma 2.1. The last statement follows from the factorization $R_p \hookrightarrow A_p \hookrightarrow A_P$. $\square$

Theorem 2.4: *Let R be an integral domain with quotient field K and let A be a birational extension of R in K. Put*

$$I_R(A) := \{a \in R \mid a \neq 0, A[1/a] = R[1/a]\} \cup \{0\}.$$

Assume that A is finitely generated over R. Then

(i) *$I_R(A)$ is a radical ideal of R and $I_R(A) \neq (0)$.*

(ii) *For $p \in \operatorname{Spec}(R)$, $I_R(A) \not\subseteq p \iff A_p = R_p$.*

Proof: Put $A = R[\alpha_1, \ldots, \alpha_n]$ and let $\alpha_i = a_i/b$ $(a_i, b \in R, b \neq 0)$.

(i) Since $A[1/b] = R[1/b]$, we have $b \in I_R(A)$ and so $I_R(A) \neq (0)$. Let $a, b \in I_R(A)$. Since $A[1/a] = R[1/a]$ and $A[1/b] = R[1/b]$, there exists an integer $\ell >> 0$ such that $a^\ell \alpha_i \in R$ and $b^\ell \alpha_i \in R$ for every $1 \leq i \leq n$. Thus we have $(a+b)^{2\ell}\alpha_i \in R$ and hence $A[1/(a+b)] = R[1/(a+b)]$, which shows that $a + b \in I_R(A)$. For any $r(\neq 0) \in R$, it is obvious that $ra \in I_R(A)$. Therefore, $I_R(A)$ is a non-zero ideal of R. The ideal $I_R(A)$ is a radical ideal by definition.

(ii) If there exists $a \in I_R(A) \setminus p$ with $p \in \operatorname{Spec}(R)$, then $A_p = A[1/a]_p = R[1/a]_p = R_p$. Conversely, suppose that $A_p = R_p$ for $p \in \operatorname{Spec}(R)$. Put $\alpha_i = c_i/t_i$, $c_i \in R$, $t_i \in R \setminus p$ and let $t = t_1 \cdots t_n \in R \setminus p$. Since $\alpha_i \in R[1/t]$, we have $A \subseteq R[1/t]$, that is, $A[1/t] = R[1/t]$ with $t \notin p$, which means that $t \in I_R(A)$ but $t \notin p$. Thus $I_R(A) \not\subseteq p$.

$\square$

Theorem 2.5: *Let R be an integral domain with quotient field K and let A be a birational extension of R in K. If $(0) \neq I_R(A) \neq R$, then $\operatorname{grade}(I_R(A)) = 1$, i.e. $I_R(A)$ contains only a regular sequence of one element.*

Proof: Suppose that there exists a regular sequence $\{x, y\}$ in $I_R(A)$. Take an element $\alpha \in A \setminus R$ (such an element exists because $I_R(A) \neq R$). Then for a large integer $\ell \in \mathbb{N}$, we have $x^\ell \alpha = a \in R$ and $y^\ell \alpha = b \in R$. Then in K, $x^\ell/y^\ell = a/b$, that is, $x^\ell b = y^\ell a$ in R. Since $\{x^\ell, y^\ell\}$ is also a regular sequence, we have $a = x^\ell c$ for some $c \in R$. So we have $x^\ell \alpha = a = x^\ell c$. Since R is an integral domain, we have $\alpha = c \in R$, which is a contradiction.

□

Remark 2.6: A is finitely generated over $R \Longrightarrow I_R(A) \neq (0)$, as was seen above, but the reverse implication does not always hold. Let $R = k[X, Y]$ be a polynomial ring over a field k and let $A = k[Y, \{X/Y^\ell\}_{\ell \in \mathbb{N}}]$. Then it is obvious that A is a birational, infinitely generated extension of R. But $R[1/Y] = A[1/Y]$ and hence $I_R(A) \ni Y$.

Theorem 2.7: *Let R be an integral domain with quotient field K and let A be an extension of R. Assume that A is a birational, finitely generated extension of R in K and that A is flat over R. Then the canonical morphism* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is an open-immersion. Moreover,* $\mathrm{Spec}(A) \cong \mathrm{Spec}(R) \setminus V(I_R(A))$ *is canonically an isomorphism of schemes.*

Proof: We claim that $I_R(A)A = A$. In fact, suppose that there exists $P \in \mathrm{Spec}(A)$ such that $I_R(A)A \subseteq P$. Put $p = P \cap R$ so that $p \supseteq I_R(A)$. Now A_P is faithfully flat over R_p. Thus $A_P = A_p = R_p$ by Lemma 2.3. Hence $I_R(A) \not\subseteq p$ by Theorem 2.4, a contradiction. Therefore, we have shown $I_R(A)A = A$ and $\mathrm{Spec}(A) \to \mathrm{Spec}(R) \setminus V(I_R(A))$ is defined. Next we will show that this map is surjective by using the fact that a flat birational extension of integral domains $R \to A$ verifies the following property: if p is a prime ideal of R, then either $pA = A$ or $R_p \to A_p$ is an isomorphism. Suppose that there exists $p \in \mathrm{Spec}(R) \setminus V(I_R(A))$ such that $pA = A$. Then $I_R(A) \not\subseteq p$ implies that there exists $a(\neq 0) \in I_R(A) \setminus p$. So $R[1/a] = A[1/a]$. Thus $R_p = R[1/a]_p = A[1/a]_p = A_p$, which is a contradiction. Thus $\mathrm{Spec}(A) \to \mathrm{Spec}(R) \setminus V(I_R(A))$ is surjective. Now let $P, P' \in \mathrm{Spec}(A)$ with $P \cap R = P' \cap R := p$. Then $A_P = R_p = A_{P'}$, all of which are local rings. Hence $P = P'$. So $\mathrm{Spec}(A) \to \mathrm{Spec}(R) \setminus V(I_R(A))$ is injective. Since for any $P \in \mathrm{Spec}(A)$, $A_P = R_{P \cap R}$, $\mathrm{Spec}(A) \to \mathrm{Spec}(R) \setminus V(I_R(A))$ is a homeomorphism. Hence $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is an open-immersion.

□

Corollary 2.8: *Let R be an integral domain with quotient field K and let A be an extension of R. Assume that A is a birational, finitely generated extension of R in K and that A is flat over R. Let* **(P)** *be any local-global property* (*e.g. regular, normal, ...*). *If R has* **(P)**, *so does A.*

PROOF: Since $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is an open-immersion by Theorem 2.7, our conclusion is obvious.

□

Lemma 2.9: *Let R be a UFD, and let P be a prime ideal of R with* $\mathrm{grade}(P) = 1$. *Then* $\mathrm{ht}(P) = 1$.

PROOF: Suppose that $\mathrm{ht}(P) \geq 2$. Then there exists a prime element $x \in P$. Since $P \neq (x)$, take $y \in P \setminus (x)$. Then $\{x, y\}$ is a regular sequence in P, which means that $\mathrm{grade}(P) \geq 2$, a contradiction. Hence $\mathrm{ht}(P) = 1$. □

Proposition 2.10: *Let R be a UFD, and let I be an ideal of R with* $\mathrm{grade}(I) = 1$. *Then* $\mathrm{ht}(I) = 1$ *and hence* $V(I) = V(a) \subsetneq \mathrm{Spec}(R)$ *for some $a \in R$.*

PROOF: Let P be a minimal prime ideal containing I. Then $\sqrt{I}R_P = PR_P$. Since $\mathrm{grade}(I) = 1$ implies $\mathrm{grade}(\sqrt{I}) = 1$, we have $1 = \mathrm{grade}(\sqrt{I}R_P) = \mathrm{grade}(PR_P)$. Noting that R_P is a UFD, $\mathrm{ht}(PR_P) = 1$ by Lemma 2.9. Since $I \subseteq P$, $\mathrm{ht}(I) = 1$. Since R is a UFD, $\sqrt{I} = P_1 \cap \cdots \cap P_n$ for some $\mathrm{ht}(P_i) = 1$. Indeed, if I is an ideal whose minimal prime ideals are finitely generated, then I has only finitely many minimal prime ideals [1, Theorem]. Put $P_i = (a_i)$ with $a_i \in I$. Hence in $\mathrm{Spec}(R)$, $V(I) = V(\sqrt{I}) = V(P_1 \cap \cdots \cap P_n) = V(P_1 \cdots P_n) = V(a_1 \cdots a_n) = V(a)$, where $a = a_1 \cdots a_n$. □

Theorem 2.11: *Let R be an integral domain with quotient field K and let A be an extension of R. Assume that A is a birational, finitely generated extension of R in K and that A is flat over R. If R is a UFD (a unique factorization domain), then $A = R[1/a]$ for some $a \in R$.*

PROOF: We may assume that $I_R(A) \neq R$. Then $\mathrm{grade}(I_R(A)) = 1$ by Theorem 2.5. Since R is a UFD, $V(I_R(A)) = V(a)$ for some $a \in R$. So by the last statement of Theorem 2.7, $\mathrm{Spec}(A) \cong \mathrm{Spec}(R) \setminus V(a) = \mathrm{Spec}(R[1/a])$. Therefore, $A = R[1/a]$.

□

Theorem 2.12: *Let R be an integral domain with quotient field K and let A be an extension of R. Assume that A is a birational, finitely generated extension of R in K and that A is flat over R. If R is a UFD, then A is also a UFD.*

PROOF: If R is a UFD, then a localization $R[1/a]$ with $a \in R \setminus (0)$ is a UFD. So our conclusion follows from Theorem 2.11. □

Added in Proof.

Professor Gabriel Picavet informed the author of the following in a letter. We write it here with his permission:

“ During the sixties, Pr. Samuel organized in Paris a seminar about epimorphisms of the category of commutative rings. Daniel Lazard was finishing his thesis about flatness whose reference is “ Autour de la platitude, Bull. Soc. Math. France, (97), 1969, 81-128 ”.

A classical example of flat epimorphism is a localization with respect to a multiplicative subset. Hence, if A is an integral domain with quotient field K, then $A \to K$ is a flat epimorphism.

Now there is a fundamental result (Corollaire 3.2) in chapter IV of Lazard's paper: let $A \to C \to B$ a composite of ring morphisms be such that $A \to B$ is a flat epimorphism and $C \to B$ is injective. Then $C \to B$ is a flat epimorphism and if $A \to C$ is flat, then $A \to C$ is an epimorphism.

Hence, if $R \to A$ is birational and flat, $R \to A$ is a flat epimorphism.

Now faithfully flat epimorphisms are isomorphisms (Lazard, Lemme 1.2) and we find Lemma 2.1 and Proposition 2.2. Moreover, if $A \to B$ is a ring morphism, then $A \to B$ is a flat epimorphism if and only if $A_P \to B_Q$ is an isomorphism for each $Q \in \mathrm{Spec}(B)$ and $P := f^{-1}(Q)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is injective.

Thus the local-global principle (Corollary 2.8) is an easy consequence.

If we read the E.G.A of Dieudonné and Grothendieck, we may see that an open immersion of affine schemes is nothing but a flat epimorphism of finite presentation (as an algebra). Moreover, by a paper of Michel Raynaud and Laurent Gruson,“ Critères de platitude et projectivité, Invent. Math. (13), 1971, 1-89 ”, a flat ring morphism of finite type $A \to B$ where A is an integral domain is of finite presentation.

Now, Chevalley's theorem states that a flat morphism of finite presentation $A \to B$ is Zariski open (even if A and B are not noetherian).

It follows that we have a result more general than Theorem 2.7. More generally, if $A \to B$ is a flat epimorphism, $\mathrm{Spec}(B) \to \mathrm{Spec(A)}$ is a homeomorphism onto its image (Lazard, Corollaire 2.2). ”

Thus the part of Theorem 2.7 that $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is an open-immersion has been known. But we would like to emphasize that our proof is elementary and simpler because we do not use the notion of epimorphism.

Moreover, note the explicit computation of the open set introduced in Theorem 2.7 and the interesting property of its complement $V(I_R)$ (cf. Theorem 2.5).

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