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Standard Subalgebras of Semisimple Lie Algebras and Computer-Aided for Enumeration

B. Es Saadi
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Abstract

The aim of this work is to enumerate the standard subalgebras of a semisimple Lie algebra. The computations are based on the approach developed by Yu. Khakimdjanov in 1974. In this paper, we give a general formula for the number of standard subalgebras not necessarily nilpotent of a semisimple Lie algebra of type $A_p$ and the exceptional semisimple Lie algebras. With computer aided, we enumerate this number for the other types of small rank. Therefore, We deduce the number in the nilpotent case and describe a family of complete nilpotent standard subalgebras, these algebras are the nilradical of their normalizer.

1 Introduction

The motivation for this study can be found in the theory of complex homogeneous spaces: Let $M$ be a compact homogeneous space $M = G/H$, $G$ being a complex Lie group and $H$ a closed subgroup. Tits has established the following result: if $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras corresponding to $G$ and $H$, the normalizer of $\mathfrak{t}$ in $\mathfrak{g}$ is parabolic, such a Lie algebra $\mathfrak{t}$ is called standard. Then, one may translate the study of homogeneous complex spaces into study of standard subalgebras. The computations of standard Lie algebra, in the nilpotent case, were done by different authors (G. Favre and L. Santharoubane [2], P. Cellini and P. Papi [1], L. Orsina [5]).

This paper is organized as follows. In section 2, we summarize the basic facts about semisimple Lie algebras and standard subalgebras. In section 3, we characterize the complete nilpotent standard subalgebras and prove that
if the semisimple Lie algebra has rank \( p \) then their number is \( 2^p \). The section 4 is devoted to the semisimple Lie algebras \( A_p \), we give a recursive formula of the number of standard subalgebras (not necessarily nilpotent). We prove that for \( A_p \) this number is

\[
ST(p) = \frac{1}{p} \left( \frac{2p - 2}{p - 1} \right) + \frac{1}{p+2} \left( \frac{2p + 2}{p + 1} \right) + \\
\sum_{1 \leq j \leq p-1} \left( \frac{2j-2}{j-1} \right) + \sum_{1 \leq j \leq p-2} \frac{1}{j} \left( \frac{2j-2}{j-1} \right) \cdot \left( \sum_{1 \leq i \leq p-j-1} ST(i) \right)
\]

In the last section, we enumerate for the exceptional semisimple Lie algebras and for \( A_p, B_p, C_p, D_p \) of small rank the number of nilpotent standard subalgebras, the sequence of nilpotent standard subalgebras for each value of nilindex, the number of complete nilpotent standard subalgebras and the number of standard subalgebras. The computations uses Mathematica package available in: http://www.math.uha.fr/publi2002.html.

Acknowledgements: The authors would like to thank the referee for his comments and suggestions.

2 Standard Lie algebras

2.1 Parabolic and Borel subalgebras of semisimple Lie algebra

Let \( \mathfrak{g} \) be a finite dimensional semisimple complex Lie algebra of rank \( p \). A Borel subalgebra of \( \mathfrak{g} \) is a maximal solvable subalgebra of \( \mathfrak{g} \), and a parabolic subalgebra of \( \mathfrak{g} \) is a subalgebra containing a Borel subalgebra of \( \mathfrak{g} \).

We fix the following notations: \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \), \( \Delta \) is the set of roots corresponding to \( \mathfrak{h} \), \( S \) is a basis of \( \Delta \) (the simple roots), \( \Delta_+ \) (respectively, \( \Delta_- \)) is the set of positive (respectively, negative) roots. Let \( \alpha \in \Delta \):

\[
\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : [X, H] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}
\]

This space \( \mathfrak{g}_\alpha \) has dimension one. A Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \) is conjugate, up to inner automorphism, to subalgebra of the following type:

\[
\mathfrak{b}' = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha
\]
The parabolic subalgebra may be characterized, up to inner automorphism, by a subset $T$ of $S$. Let $\Omega_1$ be the set of roots whose decomposition on $S$ contains only elements of $S \setminus T$. We set $\Omega_2 = \Delta \setminus \Omega_1$, $\Omega_2^+ = \Omega_2 \cap \Delta_+$. The Lie algebra
\[ \rho = \mathfrak{h} \oplus \sum_{\alpha \in \Omega_2^+ \cup \Omega_1} \mathfrak{g}_\alpha \] (2.1)
is a parabolic subalgebra and every parabolic subalgebra is conjugate to this Lie algebra.

We note that the reductive Levi subalgebra of $\rho$ is
\[ r = \mathfrak{h} \oplus \sum_{\alpha \in \Omega_1} \mathfrak{g}_\alpha \] (2.2)
and its nilradical part is
\[ \mathfrak{n} = \sum_{\alpha \in \Omega_2^+} \mathfrak{g}_\alpha \] (2.3)

### 2.2 Standard subalgebras

**Definition:** A subalgebra of a semisimple Lie algebra is called *standard* if its normalizer is a parabolic subalgebra.

In order to simplify terminology, we shall call standard algebra every standard subalgebra of a semisimple Lie algebra. These algebras have been studied, in the first time, by G.B. Gurevich [3], in the case where the simple Lie algebra is of type $A_p$.

In the following, we give the characterization of these algebras, using the roots, due to Yu. Khakimjanov [4].

We consider a partial order relation on the dual $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$: $\omega_1 \leq \omega_2$ if and only if $\omega_2 - \omega_1$ is linear combination of simple roots with non-negative coefficients.

**Proposition 2.1:** Let $\mathfrak{t}$ be a standard algebra such that its normalizer can be written $\rho(\mathfrak{t}) = \mathfrak{h} \oplus \sum_{\alpha \in \Omega_1 \cup \Omega_2^+} \mathfrak{g}_\alpha$. Suppose that $\gamma$ and $\beta$ are positive roots with $\gamma \leq \beta$. If the subspace $\mathfrak{g}_\gamma$ is included in $\mathfrak{t}$, then $\mathfrak{g}_\beta$ is also in $\mathfrak{t}$.

This proposition is useful for the study of nilpotent standard algebras.
2.3 Nilpotent standard algebras

Let $\mathcal{R}$ be a subset of $\Delta_+$ whose elements are pairwise noncomparable (for the previous ordering on $h^*$). We put:

$$\mathcal{R}_1 = \{\alpha \in \Delta_+ : \beta \leq \alpha \text{ for some } \beta \in \mathcal{R}\} \quad (2.4)$$

The subspace $m = \sum_{\alpha \in \mathcal{R}_1} g_{\alpha}$ is a nilpotent subalgebra of $g$.

The normalizer of $m$ contains a Borel subalgebra. Thus, $m$ is a nilpotent standard algebra of $g$. We say that $m$ is the nilpotent standard algebra associated to $\mathcal{R}$. This process permits to construct more easily such subalgebra. The following theorem due to Yu. Khakimdjanov [4] shows that every nilpotent standard algebra is of this type.

**Theorem 2.2:** Let $m$ be a nilpotent standard algebra whose normalizer has the form $\rho(m) = h \oplus \sum_{\alpha \in \Omega_1 \cup \Omega^+_2} g_{\alpha}$. Then, there is a subset $\mathcal{R} \subset \Delta_+$ of pairwise noncomparable roots such that $m$ is the nilpotent standard algebra associated to $\mathcal{R}$.

**Corollary 2.3:** Every nilpotent standard algebra is conjugate to a nilpotent standard algebra associated to a set $\mathcal{R} \subset \Delta_+$ of pairwise noncomparable roots.

**Remark:** Let $g^+ = \sum_{\alpha \in \Delta_+} g_{\alpha}$ be a nilpotent subalgebra of $g$. If $m$ is nilpotent standard algebra then the quotient $g^+/m$ is also nilpotent. These quotients contain all nilpotent Lie algebras of maximal rank studied by G. Favre and L. Santharoubane [2].

**Remark:** In their paper, P. Cellini and P. Papi [1] called these nilpotent standard algebras, ad-nilpotent ideals of a Borel subalgebra and gave the number for each type of semisimple Lie algebra. In another paper, L. Orsina and P. Papi [5] gave for $A_p$ the number of nilpotent standard algebras for each nilindex. We compute, in section 5, these numbers for exceptional semisimple Lie algebras and other type of semisimple Lie algebras of small rank $p$.

2.4 Structure and construction of standard algebras

In this section, we generalize the study to standard algebra not necessarily nilpotent.
Definition: The root $\alpha \in S$ is called extremal for $\beta \in \Delta_+$ if it satisfies $\alpha = \beta$ or $\beta - \alpha \in \Delta$.

In the following proposition, we characterize the normalizer of a nilpotent standard algebra using the extremal roots. Let $\mathfrak{R} \subset \Delta_+$ be a subsystem of pairwise noncomparable roots, $m$ be the nilpotent standard algebra defined by this subsystem and $S^\beta$ be the set of all extremal roots for $\beta \in \Delta_+$.

**Proposition 2.4:** The normalizer $\rho(m)$ of $m$ is defined by the subsystem $S_2 = \bigcup_{\beta \in \mathfrak{R}} S^\beta \subset S$.

We construct a standard algebra whose nilpotent part $m$ is given by a subsystem $\mathfrak{R} \subset \Delta_+$.

Consider the subsystems $S_1 = \mathfrak{R} \cap S$ and $S_2 = \bigcup_{\beta \in \mathfrak{R}} S^\beta$ of the system $S$.

These subsystems define respectively the parabolic subalgebras $\rho_1$ and $\rho_2$ of the form (2.1). Let $r_i$ denote the reductive Levi subalgebra of the form (2.2) and $n_i$ denote the nilradical of Lie algebra $\rho_i$ of the form (2.3), $i = 1, 2$.

Let $r_0$ be an ideal in $r_1$ contained in $r_2$.

**Lemma 2.5:** The semidirect sum $t = r_0 + m$ is a standard algebra.

**Theorem 2.6:** Given any ideal $r_0$ of the algebra $r_1$ lying in $r_2$, the subalgebra $t = r_0 + m$ is standard algebra, while $\rho(t) = \rho(m)$. Conversely, any standard algebra is conjugate to subalgebra $t$ for some subsystem $\mathfrak{R} \subset \Delta_+$ of pairwise noncomparable roots and for some ideal $r_0$.

### 3 Complete nilpotent standard Lie algebras

In this section, we characterize the complete nilpotent standard algebras and count their number.

**Definition:** A nilpotent standard algebra $m$ is called complete if it is the nilradical of its normalizer.

**Proposition 3.1:** Let $m$ be the nilpotent standard algebra defined by a subsystem $\mathfrak{R} \subset \Delta_+$ of pairwise noncomparable roots.

Then, $m$ is complete if and only if $\mathfrak{R}$ is formed by simple roots.

The proposition leads to the following result.
Corollary 3.2: Let $\mathfrak{g}$ be a semisimple Lie algebra of rank $p$, then it has exactly $2^p$ complete nilpotent standard algebras.

Remark: The number $2^p$ is independent from the type of $\mathfrak{g}$.

4 The general case for $A_p$

In this section, we establish a formula which gives the number of standard algebras (not necessarily nilpotent) of $A_p$. Let $\mathfrak{g}$ be a semisimple complex Lie algebra of type $A_p$.

Let $S = \{\alpha_1, \cdots, \alpha_p\}$ be a basis of $\Delta$ (the simple roots), then the set of positive roots is

$$\Delta_+ = \{\sum_{i \leq k \leq j} \alpha_k, \ 1 \leq i \leq j \leq p\}$$

Let $\alpha \in \Delta$ :

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [X, H] = \alpha(H)X \quad \text{for all} \ H \in \mathfrak{h}\}.$$  

This space $\mathfrak{g}_\alpha$ has dimension one. Let $e_\alpha$ be a non-null vector in $\mathfrak{g}_\alpha$.

Let $\mathfrak{R} \subset \Delta_+$ be a subsystem of pairwise noncomparable roots and $\mathfrak{m}$ be a nilpotent standard algebra defined by this subsystem. We fix the following notations : $S_1 = \mathfrak{R} \cap S$, $S_2 = \bigcup_{\beta \in \mathfrak{R}} S_\beta^\beta$ and $\Omega_1$(respectively, $\Omega_2$) the set of roots whose decomposition on $S$ contains only elements of $S \setminus S_1$ (respectively, $S \setminus S_2$).

From formula 2.2, we have two reductive Levi subalgebras $r_1$ and $r_2$ defined by the subsystems $S_1$ and $S_2$ :

$$r_1 = \mathfrak{h} + \sum_{\alpha \in \Omega_1} \mathfrak{g}_\alpha$$

and

$$r_2 = \mathfrak{h} + \sum_{\alpha \in \Omega_2} \mathfrak{g}_\alpha$$

Let $r_0$ be an ideal of $r_1$ contained in $r_2$. Since $r_2 = \sum_{\alpha \in \Delta_+ \setminus \Omega_2} \mathbb{C}[e_\alpha, e_{-\alpha}] + \sum_{\alpha \in \Omega_2} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ where $\mathbb{C}[e_\alpha, e_{-\alpha}]$ denotes the vector space generated by
Standard subalgebras

\[ [e_\alpha, e_{-\alpha}], \] then the ideal \( r_0 \) has the form \( h_0 + \sum_{\alpha \in \Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha) \) where \( h_0 \subset h \) and \( \Omega_0 \subset \Omega_2 \).

In the following, we consider an ideal \( r_0 \) of the form \( \sum_{\alpha \in \Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha) \) where \( \Omega_0 \subset \Omega_2 \) (the other cases would be obtained by adding a subalgebra \( h_0 \) of \( h \) such that \( h_0 + r_0 \) is an ideal of \( r_1 \) contained in \( r_2 \)).

Let \( \Pi_1 \) be a subsystem of \( S \).

Definition: The subsystem \( \Pi_1 \) is called connected if its diagram in the Dynkin diagram of \( A_p \) is connected.

Notation: Let \( \Phi \) be a subsystem of \( S \). The set of roots expressed only with elements of \( \Phi \) is denoted by \( < \Phi > \).

Let \( \Pi_1 \) be a connected subsystem of \( S \setminus S_1 \) and \( S \setminus S_2 \) and \( \Gamma_1 \) be a set of roots expressed only with elements of \( \Pi_1 \). We set \( I_1 = \sum_{\alpha \in \Gamma_1} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha) \).

**Lemma 4.1:** The subspace \( I_1 \) is an ideal of \( r_1 \) contained in \( r_2 \).

**Proof:** Since \( \Pi_1 \subset S \setminus S_2 \subset S \setminus S_1 \), then \( \Gamma_1 \subset \Omega_2 \subset \Omega_1 \) and \( I_1 \subset r_2 \subset r_1 \).

We can write \( S \setminus S_1 = \bigcup_{1 \leq j \leq m} C_j \) with \( m \in \mathbb{N}^* \), \( C_1 = \Pi_1 \) and \( \{C_j\}_{2 \leq j \leq m} \) is the family of connected subsystems of \( S \setminus S_1 \) such that \( C_j \) are pairwise disjoint. We have \( r_1 = h + \sum_{\alpha \in \Pi_1} \mathfrak{g}_\alpha \), then \( [I_1, r_1] = \sum_{\alpha \in \Pi_1} \mathfrak{g}_\alpha + \sum_{(\alpha, \beta) \in \Pi_1 \times \Omega_1} [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \).

Now, let \( j \in [2, m] \) and \( \beta \in < C_j > \cap \Delta_+ \). For all \( \alpha \in \Gamma_1 \cap \Delta_+ \), we have \( \alpha - \beta \notin \Delta \) and \( \alpha + \beta \notin \Delta \). Therefore, for \( (\alpha, \beta) \in \Gamma_1 \times \Omega_1 \), we have \([\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\) equal to 0 or \( \mathbb{C}[e_\alpha, e_{-\alpha}] \) or \( \mathfrak{g}_{\alpha+\beta} \) with \( \alpha + \beta \in \Gamma_1 \).

It follows \( [I_1, r_1] \subset I_1 \). \( \square \)

Let \( \Pi = \{\Pi_i\}_{1 \leq i \leq k} \) be a family of connected subsystems of \( S \setminus S_1 \) and \( S \setminus S_2 \) such that \( \Pi_i \) are pairwise disjoint. Let \( I_i \) be the ideal associated to \( \Pi_i \), \( 1 \leq i \leq k \). We set \( I = \sum_{1 \leq i \leq k} I_i \) and it is called ideal associated to family \( \Pi \).

**Lemma 4.2:** The semidirect sum \( I = \sum_{1 \leq i \leq k} I_i \) is an ideal of \( r_1 \) contained in \( r_2 \).

**Proof:** Let \( i \in [1, k] \). We have \( I_i \subset r_2 \) and \([I_i, r_1] \subset I_i \). Then \( I \subset r_2 \) and \([I, r_1] = \sum_{1 \leq i \leq k} [I_i, r_1] \subset \sum_{1 \leq i \leq k} I_i = I \). \( \square \)

**Proposition 4.3:** Let \( r_0 = \sum_{\alpha \in \Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha) \) be an ideal of \( r_1 \) contained in \( r_2 \), then there exists a family \( \Pi = \{\Pi_i\}_{1 \leq i \leq k} \) of connected subsystems.
of $S/S_1$ and $S/S_2$ such that $\Pi_i$ are pairwise disjoint, $I = \sum_{1 \leq i \leq k} I_i$ the ideal associated to $\Pi$ and $r_0 = I$.

**Proof:** First we show that $\Omega_0 \cap S \neq \emptyset$. Let $\beta \in \Omega_0$, $\beta = \beta_1 + \beta_2 + \cdots + \beta_m$ with $m \in \mathbb{N}^*$, $\beta_i \in S$, for $1 \leq i \leq m$. The partial sum $\beta_1 + \cdots + \beta_h$ is a root, for $1 \leq h \leq m$. Since $\Omega_0 \subseteq \Omega_1$, therefore $\beta' = \beta_1 + \cdots + \beta_{m-1}$ is a root in $\Omega_1$. Then $\beta_m = \beta - \beta' \in \Omega_0$ because $[r_0, r_1] \subset r_0$.

Now, we construct the family $\Pi$. We set $S_0 = \Omega_0 \cap S$, $S_0$ may be written $S_0 = \bigcup_{1 \leq i \leq k} \Pi_i$ where $\Pi = \{\Pi_i\}_{1 \leq i \leq k}$ is a family of connected subsystems of $S_0$ such that $\Pi_i$ are pairwise disjoint. It remains to prove that $\Pi$ is a family of connected subsystems of $S/S_1$ and $S/S_2$.

Let $i \in [1, k]$ and $\gamma \in (S/S_1) \setminus S_0$. We suppose that $\Pi_i \cup \{\gamma\}$ is a connected subsystem of $S/S_1$. Then, there exists a family $\{\gamma_j\}_{1 \leq j \leq s}$ of simple roots of $\Pi_i$ where :

1. $\gamma_1 + \cdots + \gamma_s + \gamma$ is a root and $\gamma' = \gamma_1 + \cdots + \gamma_s$ is a root of $\Omega_0$. Since $[r_0, r_1] \subset r_0$ then $[g_{\gamma'} + g_{-\gamma}, g_{-\gamma}] = g_{\gamma} \subset r_0$ i.e $\gamma \in S_0$, contradiction.

Or

2. $\gamma + \gamma_1 + \cdots + \gamma_s$ is a root. We have $[[g_{\gamma + \gamma_1 + \cdots + \gamma_s}, g_{-\gamma}], \cdots, g_{-\gamma_1}] = g_{\gamma} \subset r_0$ i.e $\gamma \in S_0$, contradiction.

Finally, for each $i \in [1, k]$, $\Pi_i$ is a connected subsystem of $S/S_1$. Since $S/S_2 \subset S/S_1$, then $\Pi_i$ is also a connected subsystem of $S/S_2$. Therefore, the ideal $I = \sum_{1 \leq i \leq k} \sum_{\alpha \in \Pi_i, >} (C[e_\alpha, e_{-\alpha}] + g_\alpha)$ associated to the family $\Pi$ is equal to $r_0$, because $\sum_{1 \leq i \leq k} <\Pi> = \Omega_0$. \qed

Let $\Pi = \{\Pi_i\}_{1 \leq i \leq k}$ be a family of connected subsystems of $S$ such that $\Pi_i$ are pairwise disjoint and $I = \sum_{1 \leq i \leq k} I_i$ the ideal associated to $\Pi$. We introduce the following property : Let $m$ be a nilpotent standard algebra such that $I$ is an ideal of $r_1$ contained in $r_2$. Let $\mathfrak{F}$ denote the set of nilpotent standard algebras satisfying the above property. Let $C(\Pi)$ be the cardinal of $\mathfrak{F}$. Then, the number of standard algebras is a sum of the number $C(\Pi)$ for different family $\Pi$.

Let $ST(j)$ be the number of standard algebras of semisimple Lie algebra of type $A_j$, for $j = 1, \cdots, p$.

If $k = 1$, there is 3 cases :

1. $\Pi = \Pi_1 = \emptyset$ then $C(\Pi) = \frac{1}{p+2} \begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix}$ (It’s the number of nilpotent
standard subalgebras)

2. $\Pi = \Pi_1 = \{\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+t}\}$ for $t$ in $\{p-j-1, p-j\}$ then $C(\Pi) = \frac{1}{j} \binom{2j-2}{j-1}$ for $j = 1, \ldots, p-1$.

3. $\Pi_1 = \{\alpha_p\}$ then $C(\Pi) = \frac{1}{p} \binom{2p-2}{p-1}$.

If $k > 1$, there is one case :

1. $\Pi_1 = \{\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+t}\}$ for $t = 0, \ldots, p-j-2$ then $C(\Pi) = \frac{1}{j} \binom{2j-2}{j-1} ST(p-j-t-1)$ for $j = 1, \ldots, p-2$.

Then, the number of standard algebras is :

$$ST(p) = \frac{1}{p+2} \binom{2p+2}{p+1} + \sum_{1 \leq j \leq p-1} \left( \sum_{p-j-1 \leq t \leq p-j} \frac{1}{j} \binom{2j-2}{j-1} \right) + \frac{1}{p} \binom{2p-2}{p-1} +$$

$$\sum_{1 \leq j \leq p-2} \left( \sum_{0 \leq t \leq p-j-2} \frac{1}{j} \binom{2j-2}{j-1} ST(p-j-t-1) \right).$$

After developing this formula, we obtain :

**Theorem 4.4:** The number of standard algebras (not necessarily nilpotent) of semisimple Lie algebra $\mathfrak{g}$ of type $A_p$ is given by the following recursive function :

$$ST(p) = \frac{1}{p} \binom{2p-2}{p-1} + \frac{1}{p+2} \binom{2p+2}{p+1} +$$

$$\sum_{1 \leq j \leq p-1} \frac{2}{j+1} \binom{2j-2}{j-1} + \sum_{1 \leq j \leq p-2} \frac{1}{j} \binom{2j-2}{j-1} \left( \sum_{1 \leq i \leq p-j-1} ST(i) \right).$$

5 Computer aided for enumeration of standard algebras.

In this section, we compute the number of standard algebras and nilpotent standard algebras of a semisimple Lie algebra. We give also for each value of the nilindex the number of corresponding standard subalgebras. These results are obtained using a Mathematica package available in http://www.math.uha.fr/publi2002.html.
5.1 Exceptional semisimple Lie algebras

Let $NS$ denotes the number of nilpotent standard algebras and the sequence $IND_p$ denotes the sequence of the number of nilpotent standard algebras for each value of the nilindex, the number in $IND_p$ at position $j$ corresponds to the number of nilpotent standard algebras of nilindex $j$. The number $CNS$ denotes the number of complete nilpotent standard algebras as defined in section 3 and the number $ST$ denotes the number of standard algebras.

<table>
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<tr>
<th>Algebra</th>
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<th>$CNS$</th>
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<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

5.2 Algebras $A_p$, $B_p$, $C_p$, $D_p$ for $p \leq 7$

Let $NS_p$ be the number of nilpotent standard algebras. The sequence $IND_p$ denotes the sequence of the number of nilpotent standard algebras for each value of the nilindex. Let $CNS_p$ be the number of complete nilpotent standard algebras and $ST_p$ be the number of standard algebras.

5.2.1 Algebra $A_p$

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### 5.2.2 Algebras $B_p$

<table>
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<tr>
<th>$p$</th>
<th>$NS_p$</th>
<th>$IND_p$</th>
<th>$CNS_p$</th>
<th>$ST_p$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>6</td>
<td>${1,3,1,1}$</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>${1,7,6,4,1,1}$</td>
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<td>29</td>
</tr>
<tr>
<td>4</td>
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<td>${1,15,23,16,7,6,1,1}$</td>
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<td>4499</td>
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### 5.2.3 Algebras $C_p$

<table>
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<th>$ST_p$</th>
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<tbody>
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<tr>
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### 5.2.4 Algebra $D_p$

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<th>$CNS_p$</th>
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**Remark:** The values of $NS_p$ for any semisimple Lie algebra and the $IND_p$ for $A_p$ correspond to the formula given in [1] and [5]. The values $CNS_p$ for any semisimple Lie algebra and $ST_p$ in $A_p$’s case correspond to the formula given in section 3 and section 4 of this paper.

**Remark:** In forthcoming paper we will give a formulas for $ST_p$ for the other semisimple Lie algebras.
References


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