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Product Theorems for Certain Summability Methods in Non-archimedean Fields

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Abstract

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. Sequences and infinite matrices have entries in K. The main purpose of this paper is to prove some product theorems involving the methods M and (N, p_n) in such fields K.

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Throughout the present paper, K denotes a complete, non-trivially valued, non-archimedean field. Sequences and infinite matrices have entries in K.

Given an infinite matrix $A = (a_{nk}), n, k = 0, 1, 2, \cdots$ and a sequence $x = \{x_k\}, k = 0, 1, 2, \cdots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, n = 0, 1, 2, \cdots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = l$, we say that $x = \{x_k\}$ is A-summable to l. If $\lim_{n\to\infty} (Ax)_n = l$ whenever $\lim_{k\to\infty} x_k = l$, we say that the matrix method A is regular. Necessary and sufficient conditions for A to be regular in terms of its entries are well-known (see [2]).

The (N, p_n) methods (or Nörlund methods) were introduced in K and some of their properties were studied earlier by Srinivasan (see [6]). A more detailed study of the (N, p_n) methods was taken up by the author later and published in a series of articles (for instance, see [4], [5]).

The (N, p_n) method is defined by the matrix $A = (a_{nk})$, where

$$\begin{aligned} a_{nk} &= \frac{p_{n-k}}{P_n}, k \leq n; \\ &= 0, k > n, \end{aligned}$$

where $p_0 \neq 0, |p_j| < |p_0|, j = 1, 2, \cdots$ and $P_n = \sum_{k=0}^{n} p_k, n = 0, 1, 2, \cdots$. The following result is known ([4], Theorem 1).

Theorem 1.1: *The (N, p_n) method is regular if and only if*

$$\lim_{n \to \infty} p_n = 0.$$

Let $\{\lambda_n\}$ be a sequence in K such that $\lim_{n \to \infty} \lambda_n = 0$. Let $M = (b_{nk})$, where

$$\begin{aligned} b_{nk} &= \lambda_{n-k}, k \leq n; \\ &= 0, \qquad k > n. \end{aligned}$$

In this context, we note that the M method reduces to the Y method of Srinivasan (see [6]), when $K = Q_p$, the p-adic field for a prime $p, \lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_n = 0, n \geq 2$.

We need the following definition in the sequel.

Definition: Two matrix methods $A = (a_{nk}), B = (b_{nk})$ are said to be consistent, if whenever $x = \{x_k\}$ is A-summable to s and B-summable to t, then $s = t$.

It is clear that the relation "matrices A and B are consistent" is an equivalence relation.

We now recall that a product theorem means the following: given regular methods A, B, does $x = \{x_k\} \in (A)$ imply $B(x) \in (A)$, limits being the same, where (A) is the convergence field of A? i.e., does "$A(x)$ converges" imply "$A(B(x))$ converges to the same limit"?

The main purpose of this paper is to prove some product theorems involving the $M, (N, p_n)$ methods in K. In the sequel, we suppose that the (N, p_n) methods are regular.

Theorem 1.2: *If $(N, p_n)(x)$ converges to l, then $(N, p_n)(M(x))$ converges to $l \left(\sum_{n=0}^{\infty} \lambda_n \right)$.*

Proof: Let

$$\begin{aligned} \tau_n &= \frac{p_0 x_n + p_1 x_{n-1} + \cdots + p_n x_0}{P_n}, \\ t_n &= \lambda_n x_0 + \lambda_{n-1} x_1 + \cdots + \lambda_0 x_n, n = 0, 1, 2, \cdots . \end{aligned}$$

By hypothesis, $\lim_{n\to\infty} \tau_n = l$. Since $\lim_{n\to\infty} p_n = 0$ and $p_0 \neq 0$, $\lim_{n\to\infty} P_n = P,\ P \neq\ 0$. Now,

$$\begin{aligned} \tau'_n &= (N, p_n)\ (\{t_n\}) \\ &= \frac{p_0 t_n + p_1 t_{n-1} + \cdots + p_n t_0}{P_n} \\ &= \frac{1}{P_n} [p_0(\lambda_n x_0 + \lambda_{n-1} x_1 + \cdots + \lambda_0 x_n) \\ &\qquad + p_1(\lambda_{n-1} x_0 + \lambda_{n-2} x_1 + \cdots + \lambda_0 x_{n-1}) \\ &\qquad + \cdots + p_n(\lambda_0 x_0)] \\ &= \frac{1}{P_n} [\lambda_0(p_0 x_n + p_1 x_{n-1} + \cdots + p_n x_0) \\ &\qquad + \lambda_1(p_0 x_{n-1} + p_1 x_{n-2} + \cdots + p_{n-1} x_0) \\ &\qquad + \cdots + \lambda_n(p_0 x_0)] \\ &= \frac{1}{P_n} [\lambda_0 P_n \tau_n + \lambda_1 P_{n-1} \tau_{n-1} + \cdots + \lambda_n P_0 \tau_0] \\ &= \frac{1}{P_n} [\{\lambda_0 P_n(\tau_n - l) + \lambda_1 P_{n-1}(\tau_{n-1} - l) + \cdots \\ &\qquad + \lambda_n P_0(\tau_0 - l)\} + l\,\{\lambda_0 P_n + \lambda_1 P_{n-1} + \cdots + \lambda_n P_0\}] \\ &= \frac{1}{P_n} [\{\lambda_0 P_n(\tau_n - l) + \lambda_1 P_{n-1}(\tau_{n-1} - l) + \cdots + \lambda_n P_0(\tau_0 - l)\} \\ &\qquad + l\,\{\lambda_0(P_n - P) + \lambda_1(P_{n-1} - P) + \cdots + \lambda_n(P_0 - P)\} \\ &\qquad + lP\,\{\lambda_0 + \lambda_1 + \cdots + \lambda_n\}]\,. \end{aligned}$$

Using Theorem 1 of [3] and the fact that $|P_n| = |P_0|, n = 0, 1, 2, \cdots$, it follows that

$$\lim_{n\to\infty} \frac{1}{P_n} [\lambda_0 P_n(\tau_n - l) + \lambda_1 P_{n-1}(\tau_{n-1} - l) + \cdots + \lambda_n P_0(\tau_0 - l)] = 0$$

and

$$\lim_{n\to\infty}\frac{1}{P_n}[\lambda_0(P_n-P)+\lambda_1(P_{n-1}-P)+\cdots+\lambda_n(P_0-P)]=0,$$

so that

$$\lim_{n\to\infty}\tau_n'=l\sum_{n=0}^{\infty}\lambda_n,$$

i.e., $(N,p_n)(M(x))$ converges to $l\sum_{n=0}^{\infty}\lambda_n$, completing the proof of the theorem. □

Corollary 1.3: *If we want to get the same limit l, we have to choose the sequence $\{\lambda_n\}$ such that $\sum_{n=0}^{\infty}\lambda_n=1$, an example being the Y method of Srinivasan.*

Corollary 1.4: *The Y and (N,p_n) methods are consistent.*

We make use of well-known properties of analytic elements (a general reference in this direction is [1]) to prove our next result.

Theorem 1.5: *Let $|\lambda_n|\le|\lambda_0|, n=0,1,2,\cdots$. If $M(\{a_n\})$ converges to l, then $M((N,p_n)(\{a_n\}))$ converges to l too.*

Proof: Let F be a complete, algebraically closed extension of K; let U be the disk $|x|\le 1$ in F; let $H(U)$ be the F-algebra of analytic elements in U, which is known as the set of restricted power series with coefficients in F. Let $\mathcal{A}$ be the algebra of analytic functions in the disk D of $F:|x|<1$.

Let $\phi(x)=\sum_{n=0}^{\infty}\lambda_n x^n$. We note that $\phi\in H(U)$ and ϕ in invertible in $\mathcal{A}$, since $|\lambda_n|\le|\lambda_0|, n=0,1,2,\cdots$.

Let $\widehat{M}$ be the linear mapping defined by M in the space of power series: if $M(\{a_n\})=\{c_n\}$, then

$$\widehat{M}(f)=\sum_{n=0}^{\infty}c_n x^n.$$

It is easily seen that

$$\widehat{M}(f)=\phi(x)f(x).$$

Since $M(\{a_n\})$ has limit l,

$$\begin{aligned}\phi(x)f(x) &= \sum_{n=0}^{\infty}(l+\epsilon_n)x^n \\ &= \frac{l}{1-x}+\epsilon,\end{aligned}$$

where $\epsilon = \sum_{n=0}^{\infty}\epsilon_n x^n \in H(U)$, since $\lim_{n\to\infty}\epsilon_n = 0$. Since ϕ is invertible in $\mathcal{A}$, we have

$$f(x) = \frac{l}{(1-x)\phi} + \frac{\epsilon}{\phi}.$$

But $\frac{l}{(1-x)\phi} + \frac{\epsilon}{\phi} \in \mathcal{A}$. Thus $f \in \mathcal{A}$ and therefore is bounded because so are $\frac{l}{1-x}, \frac{1}{\phi}, \epsilon$ and so $\frac{l}{(1-x)\phi} + \frac{\epsilon}{\phi}$. Consequently the sequence $\{a_n\}$ is also bounded (see, for instance, [1]).

Let $\pi(x) = \sum_{n=0}^{\infty}p_n x^n$. $\pi \in H(U)$, since $\lim_{n\to\infty}p_n = 0$.

Let $\widehat{(N,p_n)}(f) = \sum_{n=0}^{\infty}c_n x^n$. Since $\{P_n\}$ converges to a limit $P \neq 0$ and since $\{a_n\}$ is bounded,

$$\sum_{n=0}^{\infty}c_n x^n = \frac{1}{P}\{\pi(x)f(x)+\theta(x)\},$$

where $\theta(x) = \sum_{n=0}^{\infty}\theta_n x^n, \theta_n = \left(\frac{1}{P_n}-\frac{1}{P}\right)\sum_{k=0}^{n}p_k a_{n-k}, n = 0,1,2,\cdots$.
Noting that $\lim_{n\to\infty}\theta_n = 0$, we have $\theta \in H(U)$.
Thus, we have,

$$\begin{aligned}P\widehat{M}(\widehat{(N,p_n)}(f)) &= \phi(x)(\pi(x)f(x)+\theta(x)) \\ &= \frac{l\pi(x)}{1-x}+\epsilon(x)\pi(x)+\phi(x)\theta(x) \\ &= \frac{l\pi(1)}{1-x}+\frac{l(\pi(x)-\pi(1))}{1-x}+\epsilon(x)\pi(x)+\phi(x)\theta(x).\end{aligned}$$

It is well-known (see [1]) that $x-1$ divides $\pi(x)-\pi(1)$ in $H(U)$ and so

$$P\widehat{M}((\widehat{N,p_n})(f)) = \frac{l\pi(1)}{1-x} + \tau(x),$$

where $\tau(x) \in H(U)$. Since $\pi(1) = P$,

$$\widehat{M}((\widehat{N,p_n})(f)) = \frac{l}{1-x} + \frac{1}{P}\tau(x),$$

which proves that $M((N,p_n)(\{a_n\}))$ has limit l. This completes the proof of the theorem. $\square$

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