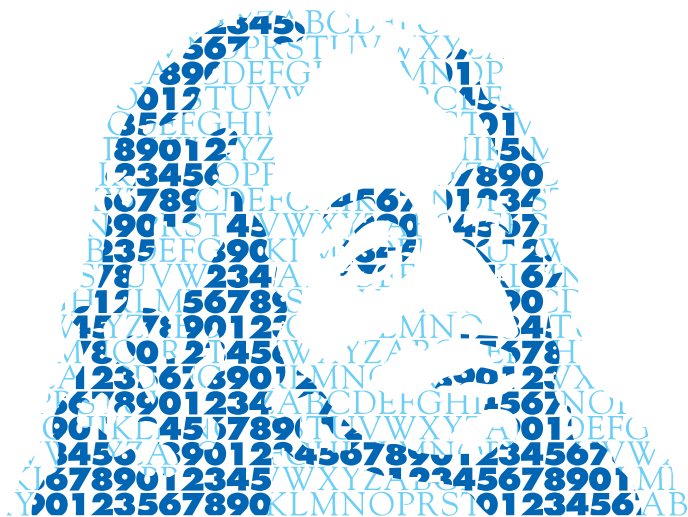


ANNALES MATHÉMATIQUES



BLAISE PASCAL

DAVID E. DOBBS, GABRIEL PICAUVET

On Strong Going-Between, Going-Down, And Their Universalizations, II

Volume 10, n°2 (2003), p. 245-260.

http://ambp.cedram.org/item?id=AMBP_2003__10_2_245_0

© Annales mathématiques Blaise Pascal, 2003, tous droits réservés.

L'accès aux articles de la revue « Annales mathématiques Blaise Pascal » (<http://ambp.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://ambp.cedram.org/legal/>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France*

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

On Strong Going-Between, Going-Down, And Their Universalizations, II

David E. Dobbs
Gabriel Picavet

Abstract

We consider analogies between the logically independent properties of strong going-between (SGB) and going-down (GD), as well as analogies between the universalizations of these properties. Transfer results are obtained for the (universally) SGB property relative to pullbacks and Nagata ring constructions. It is shown that if $A \subseteq B$ are domains such that A is an LFD universally going-down domain and B is algebraic over A , then the inclusion map $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ satisfies GB for each $n \geq 0$. However, for any nonzero ring A and indeterminate X over A , the inclusion map $A \hookrightarrow A[X]$ is not universally (S)GB.

1 Introduction

All rings considered below are commutative with identity; all ring extensions and ring homomorphisms are unital. Our goal is to further the work that was begun in [9] on the interplay between the going-down (GD) and the strong going-between (SGB) properties of ring homomorphisms. To ease the discussion, we adapt the notation in [15, p. 28], by letting GD and GU denote the going-down and going-up properties, respectively, for ring homomorphisms; we also let $\text{Spec}(A)$ denote the set of prime ideals of a ring A .) Following [18], we say that a ring homomorphism $f : A \rightarrow B$ satisfies SGB in case the following condition is satisfied: if $P_1 \subseteq P_2 \subseteq P_3$ in $\text{Spec}(A)$ and $Q_1 \subseteq Q_3$ in $\text{Spec}(B)$ are such that $f^{-1}(Q_1) = P_1$ and $f^{-1}(Q_3) = P_3$, then there exists Q_2 in $\text{Spec}(B)$ such that $Q_1 \subseteq Q_2 \subseteq Q_3$ and $f^{-1}(Q_2) = P_2$. This terminology was chosen in order to avoid confusion with the different "going-between" (GB) property introduced by Ratliff [19]: a ring homomorphism $f : A \rightarrow B$ satisfies GB in case, whenever $Q_1 \subset Q_3$ are prime ideals in B such that there exists a prime ideal P_2 in A such that $f^{-1}(Q_1) \subset P_2 \subset f^{-1}(Q_3)$,

then there exists a prime ideal Q_2 in B such that $Q_1 \subset Q_2 \subset Q_3$. For motivational purposes, it is useful to observe that $\text{SGB} \Rightarrow \text{GB}$.

The most basic fact regarding the interplay between GD and SGB is that these two properties are logically independent. Explicit examples were given in [9, Proposition 2.4] to show this logical independence. In view of the somewhat *ad hoc* nature of those examples, we think it appropriate to re-establish this fact by appealing to the fundamental structure of the category of commutative rings. For this reason, the logical independence of GD and SGB is proved in Proposition 2.2 by using a realization theorem of Hochster [14, Theorem 6 (a)] for surjective spectral maps of spectral sets.

One of the examples in [9] depended on a result [9, Remark 2.5 (e)] on the transfer behavior of the SGB property relative to the classical $D + M$ construction (in the sense of [12]). This is generalized in Theorem 2.3, which gives a transfer result for SGB in a pullback context that is in the spirit of [10, Section 2] and is, thus, more general than that of the classical $D + M$ construction. Section 2 adds further to the SGB/GD analogy, by giving an additional transfer result (Proposition 2.4) for SGB in the context of Nagata rings. As explained there, this result is motivated by known transfer results for going-down domains [1] and for GB-rings ([19], [20]).

In view of the preceding comment, it is timely to recall from [4] and [8] that GD has been used to introduce a class of (commutative integral) domains that includes all Prüfer domains and all domains of (Krull) dimension at most 1. Indeed, according to [8, Theorem 1], a domain A , with quotient field K , is a going-down domain if and only if $A \hookrightarrow B$ satisfies GD for all domains B that contain A (resp., for all overrings B of A ; resp., for all valuation overrings B of A ; resp., for all rings of the form $B = A[u]$, $u \in K$). It was shown in [9, Corollary 2.3] that the analogy between SGB and GD can be extended as follows: if one replaces "GD" with "SGB" in each of the preceding four conditions, the resulting conditions are still equivalent and still characterize going-down domains. In other words, what might be called an "SGB-domain" is precisely the same as a going-down domain. (For additional motivation, note that the class of GB-rings was introduced in [19] and studied further in papers such as [20].)

It was shown in [7, Theorem 2.6] that if one replaces "GD" with "universally going-down" in each of the above four conditions from [8, Theorem 1], the resulting conditions are still equivalent and characterize what is there called "universally going-down domains". It thus seems natural to ask if the "universally SGB" property can be used to similarly characterize the

universally going-down domains. The answer, as given in [9, Corollary 3.2, Corollary 3.9], is essentially “yes and no”: “yes” if one restricts attention to overrings B of A ; “no” if one considers test extension domains B that are not algebraic over A . As explained below, the work in Section 3 deepens these results from [9] in two ways.

First, [9, Corollary 3.9] is re-obtained in Corollary 3.5, as a consequence of a result (Lemma 3.3) that seems more direct, albeit somewhat intricate, than the method used in [9]. As a pleasant bonus, we note, after the proof of Corollary 3.5, that our present approach actually establishes more, in particular, that there does not exist a domain A such that $A \leftrightarrow B$ is universally GB for each domain B containing A . This is apparently the first result on “universally GB” in the literature. Second, continuing in the “universally GB” vein, we develop a positive result in Theorem 3.6 that addresses the following question: if A is a universally going-down domain, is it possible to enlarge (beyond the arena of overrings) the class of “test domains” B appearing in the statement of [9, Corollary 3.2]? The proof of Theorem 3.6 depends, in part, on a result concerning the “universally catenarian” property [3, Corollary 6.3].

Our final result, Proposition 3.8, merges several of the above themes by establishing that the universally SGB property transfers to the induced map on Nagata rings.

For the sake of motivation and ease of reference, some results from [9] are restated in this paper. Finally, in addition to the notational conventions indicated above, we mention the following. If A is a ring, then $\dim(A)$ denotes the Krull dimension of A ; $\text{ht}_A(P) = \text{ht}(P)$ denotes the height (in A) of $P \in \text{Spec}(A)$; and A_{red} denotes A/\sqrt{A} , the reduced ring canonically associated to A . If A is a domain, with quotient field K , then A' denotes the integral closure of A (in K); and by an *overring* of A , we mean any ring B such that $A \subseteq B \subseteq K$. If $h : A \rightarrow B$ is a ring homomorphism, then ${}^a h$ denotes the canonical map $\text{Spec}(B) \rightarrow \text{Spec}(A)$, $Q \mapsto h^{-1}(Q)$; and h_{red} denotes the induced ring homomorphism $A_{\text{red}} \rightarrow B_{\text{red}}$. Any unexplained material is standard, as in [12], [13], [15].

2 Logical independence and transfer results

We begin by stating some known connections between the SGB and GD properties. The following definitions are needed for Proposition 2.1 (b), (c).

As recalled in the Introduction, a domain A is called a *going-down domain* in case $A \subseteq B$ satisfies GD for each domain B containing A . Following [5], a ring A is called a *going-down ring* in case A/P is a going-down domain for each $P \in \text{Spec}(A)$.

Proposition 2.1: *Let $f : A \rightarrow B$ be a ring homomorphism.*

- (a) [18, Propositions 5.2 and 5.7] *The following conditions are equivalent:*
- (1) *f satisfies SGB;*
 - (2) *The induced map $A_P \rightarrow B_Q$ satisfies GU for all $Q \in \text{Spec}(B)$ and $P := f^{-1}(Q)$;*
 - (3) *The induced map $A/P \hookrightarrow B/Q$ satisfies GD for all $Q \in \text{Spec}(B)$ and $P := f^{-1}(Q)$;*
 - (4) *The induced map $f_{\text{red}} : A_{\text{red}} \rightarrow B_{\text{red}}$ satisfies SGB;*
 - (5) *The induced map $A/P \hookrightarrow B/Q$ satisfies SGB for all $Q \in \text{Spec}(B)$ and $P := f^{-1}(Q)$.*
- (b) [9, Corollary 2.2 (a)] *If B is a domain and f is injective and satisfies SGB, then f satisfies GD.*
- (c) [9, Corollary 2.2 (b)] *If A is a going-down ring, then f satisfies SGB.*

Despite the assertions in parts (b) and (c) of Proposition 2.1, we show next that neither SGB nor GD implies the other, even for ring extensions of a domain.

Proposition 2.2: *The SGB and GD properties are logically independent. In fact:*

- (a) *There exists an inclusion map $A \hookrightarrow B$ that satisfies SGB but does not satisfy GD. It can be arranged that A is a quasilocal one-dimensional domain and that B is a zero-dimensional reduced ring with exactly two prime ideals (that is, B is isomorphic to the direct product of two fields).*
- (b) *There exists an inclusion map $A \hookrightarrow B$ that satisfies GD but does not satisfy SGB. It can be arranged that A is a quasilocal two-dimensional treed domain and that B is a quasilocal two-dimensional ring with exactly two minimal prime ideals.*

PROOF: (a) An *ad hoc* approach, in the spirit of [9, Proposition 2.4 (a)], is available. Indeed, let X be an indeterminate over a field k , and consider $A := k[X]_{(X)}$ and $B := k \times k(X)$. Of course, A is a (quasi)local one-dimensional

domain with unique maximal ideal $M := XA$, and since $A = k + M$, one can identify $A/M = k$. Also, the only prime ideals of B are $k \times 0$ and $0 \times k(X)$, which are incomparable. Then the monomorphism $i : A \hookrightarrow B$, given by $a \mapsto (a + M, a)$, is easily seen to have the asserted properties, in view of the facts that ${}^a i(k \times 0) = 0$ and ${}^a i(0 \times k(X)) = M$. We next present a more categorical approach to the assertion in (a).

Consider a two-element set $Y := \{Q, N\}$ with the trivial partial order; that is, Q and N are unrelated. Also, consider a two-element linearly ordered set $X := \{P, M\}$ with $P < M$. It is easy to see directly that both Y and X are spectral sets (in the sense that each is order-isomorphic to the prime spectrum of some ring). However, we appeal here to a useful more general method, namely, the fact that any finite partially ordered set is a spectral set [16, Theorem 2.10]. Identify $Y = \text{Spec}(D)$ and $X = \text{Spec}(C)$ for some rings D and C ; equip the spectral spaces Y and X with the corresponding Zariski topology.

Consider the surjective function $g : Y \rightarrow X$ given by $Q \mapsto P$ and $N \mapsto M$. We claim that g is a spectral map (in the sense of [14, p. 43]), namely, that g is continuous and $g^{-1}(U)$ is quasi-compact open in Y for each quasi-compact open set U in X . As X and Y are each finite, the “quasi-compact” conditions can be ignored here. It suffices to consider the only nontrivial open subset of X , namely $U := \{P\}$. (According to the usual description of the open sets in the Zariski topology [2, p. 99], this set is the basic open set determined by any element of $M \setminus P$.) Evidently, $g^{-1}(\{P\}) = \{Q\}$, which is Zariski-open in Y , since $Y \setminus \{Q\} = V(N)$ is closed by virtue of the maximality of N [2, Definition 4, p. 99]. This proves the claim.

According to a realization result of Hochster [14, Theorem 6 (a)], Spec is invertible on the subcategory of spectral spaces and surjective spectral maps. Since we have shown that g is a surjective spectral map, there exist a ring homomorphism $h : E \rightarrow F$ and homeomorphisms $\alpha : X \rightarrow \text{Spec}(E)$, $\beta : Y \rightarrow \text{Spec}(F)$ such that $({}^a h) \circ \beta = \alpha \circ g$. As noted in [14], homeomorphisms of spectral spaces induce order-isomorphisms. Therefore, it is now easy to see that h satisfies SGB but does not satisfy GD (the point being that g was constructed to satisfy the corresponding order-theoretic conclusion).

Next, note via standard homomorphism theorems (and the surjectivity of g) that there is a natural bijection between $\text{Spec}(\text{im}(h))$ and $\text{Spec}(E)$. Thus, by *abus de langage*, we can now replace $h : E \rightarrow F$ with $\text{im}(h) \hookrightarrow F$; that is, $h : E \rightarrow F$ is now an inclusion map. Therefore, $h_{red} : A \rightarrow B$ is also an injection, where $A := E_{red}$ and $B := F_{red}$. Moreover, h_{red} inherits from h

the property of satisfying SGB (resp., of not satisfying GD) by condition (4) in the statement of Proposition 2.1 (a) (resp., by [6, Lemma 3.2 (a)]). Then h_{red} has the asserted properties. Indeed, since $\text{Spec}(A)$ is order-isomorphic to X , we see that A is quasilocal and one-dimensional; and since A is a reduced ring with a unique minimal prime ideal, A is a domain. Finally, since $\text{Spec}(B)$ is order-isomorphic to Y , we see that B is a zero-dimensional reduced ring with exactly two (incomparable) prime ideals, say q and n . Hence, $q + n = B$ and $q \cap n = 0$, and so the Chinese Remainder Theorem yields that $B \cong B/q \times B/n$, a direct product of two fields, to complete the proof.

(b) An *ad hoc* example is available: see [9, Proposition 2.4 (b)]. We next sketch how to use the order-theoretic machinery featured in the proof of (a) in order to prove (b).

To that end, consider a four-element set $Y := \{P_1, P_2, Q_1, M_1\}$ with the partial order induced by requiring that $P_1 < M_1$ and $P_2 < Q_1 < M_1$. Also, consider a three-element linearly ordered set $X := \{P, Q, M\}$ with $P < Q < M$. By [16, Theorem 2.10], Y and X are spectral sets, and so we can identify $Y = \text{Spec}(D)$ and $X = \text{Spec}(C)$ for some rings D and C ; equip the spectral spaces Y and X with the corresponding Zariski topology.

Consider the surjective function $g : Y \rightarrow X$ given by $P_1 \mapsto P$, $P_2 \mapsto P$, $Q_1 \mapsto Q$, and $M_1 \mapsto M$. We claim that g is a spectral map. As in the proof of (a), it is enough to show that $g^{-1}(U)$ is open in Y for each open set U in X . It suffices to treat the nontrivial Zariski-open subsets U of X , namely $U_1 := \{P\}$ and $U_2 := \{P, Q\}$. Evidently, $g^{-1}(U_1) = \{P_1, P_2\}$, which is Zariski-open in Y , since $Y \setminus \{P_1, P_2\} = V(Q_1)$ is Zariski-closed. It remains only to observe that $g^{-1}(U_2) = \{P_1, P_2, Q_1\} = Y \setminus V(M_1)$ is Zariski-open in Y . This proves the claim.

The rest of the proof now proceeds essentially as in the proof of (a). In other words, one appeals to the realization result of Hochster, then replaces h with $\text{im}(h) \hookrightarrow F$, then replaces h with $h_{red} : A \rightarrow B$, and then concludes via the order-isomorphisms between $\text{Spec}(A)$ (resp., $\text{Spec}(B)$) and X (resp., Y). □

We close the section by considering the transfer of the SGB property in certain pullback constructions and for Nagata rings. To motivate Theorem 2.3, note that [9, Remark 2.5 (e)] concluded with a transfer result on SGB for the special pullbacks of the classical $D + M$ form, while going-down-theoretic results have been obtained for more general pullbacks (cf. [5, Proposition 2.2]). Finally, to motivate Proposition 2.4, recall that Ratliff has obtained several transfer results for the GB-ring property relative to Nagata rings (cf. [19, Proposition 5.1], [20, Corollary 2.7 and Proposition 6.3]).

Theorem 2.3 generalizes the above-mentioned result on the classical $D + M$ construction [9, Remark 2.5 (e)] by treating the more general type of pullback featured in [10, Section 2].

Theorem 2.3: *Let V be a quasilocal ring with nonzero maximal ideal M and residue class field $K := V/M$; let $\varphi : V \rightarrow K$ denote the canonical projection map. Let $D \subseteq E$ be subrings of K , and consider the pullbacks $A := \varphi^{-1}(D)$ and $B := \varphi^{-1}(E)$. Then $A \hookrightarrow B$ satisfies SGB if and only if $D \hookrightarrow E$ satisfies SGB.*

PROOF: Observe the canonical isomorphisms $D \cong A/M$ and $E \cong B/M$. Therefore, the “only if” assertion is immediate from condition (5) in Proposition 2.1 (a). For the converse, suppose that $D \hookrightarrow E$ satisfies SGB. Our task is to show that if $P_1 \subseteq P_2 \subseteq P_3$ in $\text{Spec}(A)$ and $Q_1 \subseteq Q_3$ in $\text{Spec}(B)$ are such that $f^{-1}(Q_1) = P_1$ and $f^{-1}(Q_3) = P_3$, then there exists Q_2 in $\text{Spec}(B)$ such that $Q_1 \subseteq Q_2 \subseteq Q_3$ and $f^{-1}(Q_2) = P_2$.

Before proceeding further, we next summarize the order-theoretic impact of the “amalgamated sum” topological description [10, Theorem 1.4] of the prime spectra of the pullbacks A and B . For instance, $\text{Spec}(A)$ can be viewed order-theoretically as the quotient space of the disjoint union of $\text{Spec}(V)$ and $\text{Spec}(D)$ in which $M \in \text{Spec}(V)$ is identified with $0 \in \text{Spec}(D)$. In particular, each prime ideal of A is comparable with M , and the set of prime ideals of A that contain (resp., are contained in) M is order-isomorphic to the prime spectrum of D (resp., V). Of course, $\text{Spec}(B)$ admits a similar description. There are now three cases to consider.

If $P_1 \in \text{im}(\text{Spec}(D) \rightarrow \text{Spec}(A))$, then all the P_i (resp., Q_j) are canonical images of prime ideals of D (resp., E) in view of the above comments, and so the existence of a suitable Q_2 follows since $D \hookrightarrow E$ satisfies SGB. In the second case, $P_3 \in \text{im}(\text{Spec}(V) \rightarrow \text{Spec}(A))$; then all the P_i and Q_j are canonical images of prime ideals of V in view of the above comments, and so the existence of a suitable Q_2 follows from the triviality that the identity

map on V satisfies SGB. In the final case, $P_1 \subseteq M \subseteq P_3$. If $M \subseteq P_2$ (resp., $P_2 \subseteq M$), argue as in the first (resp., second) case, attending to the chain $M \subseteq P_2 \subseteq P_3$ (resp., $P_1 \subseteq P_2 \subseteq M$) in $\text{Spec}(A)$. The proof is complete. \square

Recall that if A is a ring and X is an indeterminate over A , then the Nagata ring $A(X)$ is defined to be the localization of the polynomial ring $A[X]$ at the multiplicatively closed set consisting of all the polynomials in $A[X]$ with unit content. Observe that any ring homomorphism $A \rightarrow B$ induces a ring homomorphism $A[X] \rightarrow B[X]$ and, hence, a ring homomorphism $A(X) \rightarrow B(X)$. Proposition 2.4 records a pair of elementary transfer results for the SGB property that involve Nagata rings. For these assertions, recall that a domain is called a *quasi-Prüfer domain* in case its integral closure is a Prüfer domain. For additional background on quasi-Prüfer domains, see [11, Section 6.5, especially Corollary 6.5.14].

Proposition 2.4: (a) *Let $f : A \rightarrow B$ be a ring homomorphism, where A and B are each quasi-Prüfer domains. Let $g : A(X) \rightarrow B(X)$ be the ring homomorphism induced by f . Then f satisfies SGB (resp., GD) if and only if g satisfies SGB (resp., GD).*

(b) *If A is both a quasi-Prüfer domain and a going-down domain, then any ring homomorphism $A(X) \rightarrow B$ satisfies SGB.*

PROOF: (a) Since A and B are each quasi-Prüfer domains, it follows from [1, Theorem 2.7] that the canonical functions $\text{Spec}(A(X)) \rightarrow \text{Spec}(A)$ and $\text{Spec}(B(X)) \rightarrow \text{Spec}(B)$ are each order-isomorphisms (in fact, homeomorphisms). The assertions (for SGB, GD, and analogously for any other order-theoretic property of the prime spectra, such as GU) now follow by easy analyses of diagrams.

(b) By [1, Corollary 2.12], the hypothesis on A ensures (actually, is equivalent to the fact) that $A(X)$ is a going-down domain. Accordingly, an application of Proposition 2.1 (c) completes the proof. For an alternate proof, note first via Proposition 2.1 (c) that the composite function $A \rightarrow A(X) \rightarrow B$ satisfies SGB, and then invoke the fact [1, Theorem 2.7] that $\text{Spec}(A(X)) \rightarrow \text{Spec}(A)$ is an order-isomorphism. \square

3 On the universalizations of GB, SGB and GD

We devote the present section to analogies between “universally SGB” and “universally going-down,” with particular emphasis on homomorphisms involving domains. Propositions 3.1 and 3.2 collect some relevant information. Notice that Proposition 3.1 (a) shows that “universalization” of SGB-GD interplay does not always lead to valid results, for [9, Remark 2.5 (f)] provided an example of an overring extension of domains that satisfies GD but does not satisfy SGB.

Proposition 3.1: (a) [9, Theorem 3.1 (c)] *Let A be a domain and let B be an overring of A . Then $A \hookrightarrow B$ is universally SGB if and only if $A \hookrightarrow B$ is universally going-down.*

(b) [9, Theorem 3.1 (d)] *If B is a flat overring of a domain A , then $A \hookrightarrow B$ is universally SGB.*

Next, it is convenient to recall a definition from [7]. A domain A is called a *universally going-down domain* if $A \hookrightarrow B$ is universally going-down for each overring B of A . Proposition 3.2 (b) presents a partial “universalization” of Proposition 3.2 (a) and, thus, a partial analogue of [7, Theorem 2.6].

Proposition 3.2: *Let A be a domain with quotient field K . Then:*

(a) [9, Corollary 2.3] *The following conditions are equivalent:*

- (1) $A \subseteq A[u]$ satisfies SGB for each $u \in K$;
- (2) $A \subseteq B$ satisfies SGB for each valuation overring B of A ;
- (3) $A \subseteq B$ satisfies SGB for each domain B containing A ;
- (4) A is a going-down domain.

(b) [9, Corollary 3.2] *The following conditions are equivalent:*

- (1) $A \hookrightarrow A[u]$ is universally SGB for each $u \in K$;
- (2) $A \hookrightarrow B$ is universally SGB for each valuation overring B of A ;
- (3) $A \hookrightarrow B$ is universally SGB for each overring B of A ;
- (4) A is a universally going-down domain.

(c) [9, Corollary 3.3] *A is a Prüfer domain if and only if A is integrally closed and $A \hookrightarrow B$ is universally SGB for each overring B of A .*

It is natural to consider the “universal” analogue of condition (3) in Proposition 3.2 (a). However, despite Proposition 3.2 (b), it is *not* the case that a universally going-down domain A has the property that $A \hookrightarrow B$ is universally SGB for *each* domain B containing A . In fact, we establish in Corollary 3.5 that *no* domain A has this property! The crux of the argument is isolated in Lemma 3.3 below.

In view of the above remarks, one should emphasize that some non-overring extensions of domains *do* satisfy the universally SGB property. In fact, if a ring homomorphism $f : A \rightarrow B$ is radiciel (in the sense of [13, Définition 3.7.2, p. 248]) and universally going-down, then f is universally SGB.

For the proof of Lemma 3.3, we need to recall the following material from [15, p. 26] and [3]. A domain A is called an S (eidenberg)-domain in case $\text{ht}_{A[X]}(PA[X]) = 1$ for all $P \in \text{Spec}(A)$ such that $\text{ht}_A(P) = 1$. If A is a one-dimensional S -domain, then [3, Corollary 6.3] ensures that A is a stably strong S -domain (in the sense that for each nonnegative integer n , the polynomial ring $A[X_1, \dots, X_n]$ has the property that each of its factor domains is an S -domain). Combining this information with [15, Theorem 27], we see that if A is a one-dimensional S -domain, then $\dim(A[X_1, \dots, X_n]) = n + 1$ for each nonnegative integer n .

The proof of Lemma 3.3 also assumes familiarity with the basic facts about “uppers” in a polynomial ring (cf. [15, p. 25]) and the very convenient $\langle P, \alpha \rangle$ notation introduced for them in [17, Notation, p. 707].

Lemma 3.3: *Let A be a one-dimensional S -domain and X, Y algebraically independent indeterminates over A . Then $A[X] \hookrightarrow A[X, Y]$ does not satisfy SGB.*

PROOF: It suffices to produce prime ideals $P_1 \subset P_2 \subset P_3$ of $A[X]$ and adjacent prime ideals $Q_1 \subset Q_3$ of $A[X, Y]$ (identified with $(A[X])[Y]$) such that $Q_1 \cap A[X] = P_1$ and $Q_3 \cap A[X] = P_3$. Let K denote the quotient field of A . Fix a nonzero (necessarily height 1 and maximal) prime ideal M of A . We begin the construction of the desired prime ideals by letting $P_1 = 0$, $P_2 = \langle 0, X \rangle$ and $P_3 = \langle M, X \rangle$. It follows easily from the definition of the notation for uppers cited above that $P_2 = A[X] \cap XK[X]$ and $P_3 = M + XA[X]$. Therefore, since $M \neq 0$, we have arranged that $P_1 \subset P_2 = XA[X] \subset P_3$. Moreover, $\text{ht}(P_3) = 2$ since the above comments about one-dimensional S -domains ensure that $\dim(A[X]) = \dim(A) + 1 = 2$.

We turn next to the definition of Q_3 . We let $Q_3 := P_3[Y] = (M + XA[X])[Y] = MA[Y] + XA[X, Y]$. Of course, $Q_3 \cap A[X] = P_3$. In addition, $\text{ht}(Q_3) \geq \text{ht}(P_3) = 2$. In fact, the existence of uppers of P_3 yields that $\text{ht}(Q_3) = 2$, since Q_3 is not a maximal ideal of $A[X, Y]$ and the above comments about one-dimensional S -domains ensure that $\dim(A[X, Y]) = \dim(A[X]) + 1 = 3$.

The construction of Q_1 is more intricate. For this purpose, fix any nonzero

element $m \in M$, consider the polynomial $p := Y - m^{-1}X \in K(X)[Y]$, and let $Q_1 = \langle 0, p \rangle$. By the definition of the notation for uppers cited above, $Q_1 = \{g \in A[X][Y] \mid p|g \text{ in } K(X)[Y]\}$. Using the Factor Theorem, we obtain a convenient description of this prime ideal, namely, $Q_1 = \{g(X, Y) \in A[X, Y] \mid g(X, m^{-1}X) = 0\}$. Of course, $Q_1 \cap A[X] = 0 = P_1$ and $\text{ht}(Q_1) = 1$, since Q_1 is an upper of 0. It remains only to verify that $Q_1 \subseteq Q_3$, for then consideration of heights will guarantee that Q_1 and Q_3 are adjacent.

We proceed by an indirect argument. Supposing that the assertion fails, choose (a necessarily nonzero element) $f \in Q_1 \setminus Q_3$ whose degree in Y , say n , is minimal. Note that $n \geq 1$ since $f \neq 0$. Now, write

$$f = f(X, Y) = f_0(X)Y^n + f_1(X)Y^{n-1} + \cdots + f_{n-1}(X)Y + f_n(X)$$

with each $f_i \in A[X]$. Since $f \in Q_1$, we have that

$$0 = f(X, m^{-1}X) = \sum_{i=0}^n f_i(X)(m^{-1}X)^{n-i}.$$

Multiplying this expression by m^n and then solving for $f_0(X)X^n$, we infer that $f_0(X)X^n \in mA[X]$. Consequently, $f_0 \in mA[X]$, and so we can write $f_0 = mh_0$ for some uniquely determined $h_0 \in A[X]$.

Consider the polynomial

$$g := (Xh_0(X) + f_1(X))Y^{n-1} + f_2(X)Y^{n-2} + \cdots + f_{n-1}(X)Y + f_n(X) \in A[X, Y].$$

Observe that $g \in Q_1$. (This can be accomplished by a straightforward calculation, using the facts that $f(X, m^{-1}X) = 0$ and $h_0/m^{n-1} = f_0/m^n$.) As the degree in Y of g is at most $n - 1$, it follows from the minimality of n that $g \in Q_3$. In other words,

$$g = (Xh_0(X) + f_1(X))Y^{n-1} + f_2(X)Y^{n-2} + \cdots + f_n(X) \in Q_3$$

where $Q_3 = MA[Y] + XA[X, Y]$. Substituting $X \mapsto 0$ reveals that the constant terms of $Xh_0 + f_1, f_2, \dots, f_{n-1}$ and f_n are all in M . However, the constant term of $Xh_0 + f_1$ is the same as the constant term of f_1 . Moreover, the constant term of f_0 is also in M , since $f_0 \in mA[X] \subseteq MA[X]$. Thus, the constant terms of $f_0, f_1, f_2, \dots, f_{n-1}$ and f_n are in M . It follows from the way that we wrote f that $f \in MA[Y] + XA[X, Y] = Q_3$, contradicting the choice of f . The proof is complete. \square

Theorem 3.4: *If A is a nonzero ring and X is an indeterminate over A , then $A \hookrightarrow A[X]$ is not universally SGB.*

PROOF: Deny. Since $A \neq 0$, we can choose a maximal ideal M of A . Put $k := A/M$. If T is another indeterminate, algebraically independent from X , then the canonical map $k[T]_{(T)} \hookrightarrow k[T]_{(T)} \otimes_k k[X]$ is universally SGB. Thus, $R := k[T]_{(T)}$ is such that $R \hookrightarrow R[X]$ is universally SGB. This contradicts Lemma 3.3, since R , being a DVR, is a one-dimensional S -domain [15, Theorem 68 or Theorem 149]. \square

Next, we record a way in which the behavior of “universally SGB” is fundamentally different from that of “universally going-down.”

Corollary 3.5: *There does not exist a domain A such that $A \hookrightarrow B$ is universally SGB for each domain B that contains A .*

PROOF: Domains are nonzero rings. Apply Theorem 3.4. \square

In the Introduction, we raised the following question. If A is a universally going-down domain, is it possible to enlarge the class of “test domains” B [with universally SGB behavior] appearing in the statement of Proposition 3.2 (b)? Theorem 3.6 establishes some related behavior with a “universally GB” flavor. This is especially relevant, since a careful reading of the above material reveals that one can replace “does not satisfy SGB” (resp., “not ... universally SGB”) with the stronger conclusion “does not satisfy GB” (resp., “not ... universally GB”) in the statement(s) of Lemma 3.3 (resp., Theorem 3.4 and Corollary 3.5).

Recall from [3] that a domain A is said to be *locally finite-dimensional* (LFD) in case $\text{ht}_A(P) < \infty$ for each $P \in \text{Spec}(A)$.

Theorem 3.6: *Let $A \subseteq B$ be domains such that A is an LFD universally going-down domain and B is algebraic over A . Then the inclusion map $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ satisfies GB for each nonnegative integer n .*

PROOF: Suppose, for the moment, that the assertion is known in case A is integrally closed (i.e., an LFD Prüfer domain [7, Corollary 2.3]). We show next how this enables one to handle the general case. Since A is a universally going-down domain, [7, Theorem 2.4] ensures that $C := A'$ is a Prüfer domain; and by the “incomparable” property of integrality [15, Theorem 44], C inherits the LFD property from A . Next, note via the

“algebraic” hypothesis that C is A -algebra isomorphic to a subring of the algebraic closure of the quotient field of B . Consequently, we may view $D := CB$ as a well-defined extension domain of C (and of B). Moreover, D is algebraic over C (since B is algebraic over A); and D is integral over B (since C is integral over A). In particular, the case that has been assumed guarantees that, for each nonnegative integer n , $C[X_1, \dots, X_n] \hookrightarrow D[X_1, \dots, X_n]$ satisfies GB. Moreover, Proposition 3.2 (c) yields that $A \hookrightarrow C$ is universally SGB, and so $A[X_1, \dots, X_n] \hookrightarrow C[X_1, \dots, X_n]$ also satisfies (S)GB. Since it is evident that the GB property is preserved by composition (of inclusion maps), we have that $A[X_1, \dots, X_n] \hookrightarrow D[X_1, \dots, X_n]$ satisfies GB. In addition, since it is an integral extension, $B[X_1, \dots, X_n] \hookrightarrow D[X_1, \dots, X_n]$ satisfies the lying-over, going-up and incomparable properties [15, Theorem 44]. It is now straightforward, by inspecting the tower $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n] \hookrightarrow D[X_1, \dots, X_n]$, to conclude that the inclusion map $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ satisfies GB, as desired.

It remains to settle the case in which A is an LFD Prüfer domain. As above, it follows from the lying-over, going-up and incomparable properties that we may replace B with B' . In other words, without loss of generality, B is integrally closed. It follows that if we define E to be the integral closure of A in B , then E is actually the integral closure of A in the quotient field of B . Consequently, by Prüfer’s ascent result [15, Theorem 101], E is a Prüfer domain. In addition, by incomparability, E inherits the LFD property from A . Moreover, we see via algebraicity (by clearing denominators) that B is an overring of E . Hence, by Proposition 3.2 (c), $E \hookrightarrow B$ is universally SGB. In particular, for each nonnegative integer n , $E[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ satisfies GB. As GB is preserved by composition, it is therefore enough to show that $A[X_1, \dots, X_n] \hookrightarrow E[X_1, \dots, X_n]$ satisfies GB. In other words, without loss of generality, we have reduced to the case in which A and B are each LFD Prüfer domains and B is an integral extension of A . The key upshot of this reduction is a consequence of a result of Nagata, namely, that both A and B are universally catenarian integral domains (cf. [3, Theorem 6.2]).

Suppose that the assertion fails, so that $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ does not satisfy GB. Then (cf. [19, 2.2.1]), there exist prime ideals $P_1 \subset P_2 \subset P_3$ of $A[X_1, \dots, X_n]$ and adjacent prime ideals $Q_1 \subset Q_3$ of $B[X_1, \dots, X_n]$ such that $Q_i \cap A[X_1, \dots, X_n] = P_i$ for $i = 1, 3$. Now, since A is a Prüfer domain, B is A -flat, and so $A \hookrightarrow B$ is universally going-down. In particular, $A[X_1, \dots, X_n] \hookrightarrow B[X_1, \dots, X_n]$ satisfies GD. Being integral, this

extension also satisfies the incomparability property. It follows easily that $\text{ht}(Q_i) = \text{ht}(P_i)$ for $i = 1, 3$. However, universal catenarity of A (resp., B), ensures that $\text{ht}(Q_3/Q_1) = \text{ht}(Q_3) - \text{ht}(Q_1)$ (resp., $\text{ht}(P_3/P_1) = \text{ht}(P_3) - \text{ht}(P_1)$). Therefore, $1 = \text{ht}(Q_3/Q_1) = \text{ht}(Q_3) - \text{ht}(Q_1) = \text{ht}(P_3) - \text{ht}(P_1) = \text{ht}(P_3/P_1) \geq 2$. This (desired) contradiction completes the proof. \square

We do not know if it is possible to delete the ‘‘LFD’’ hypothesis from Theorem 3.6. On the other hand, one cannot delete the ‘‘algebraic’’ hypothesis in Theorem 3.6. To see this, let A be a one-dimensional S -domain that is also a universally going-down domain (for instance, take A to be any DVR), and let X and Y be algebraically independent indeterminates over A . Then, by the above comments sharpening Lemma 3.3, $B := A[Y]$ is such that $A[X] \hookrightarrow B[X]$ does not satisfy GB.

In closing, we show that the universally SGB property is inherited by maps induced on Nagata rings. First, it is convenient to record the following companion for Proposition 3.1 (b): any ring of fractions $A \rightarrow A_S$ is universally SGB.

Lemma 3.7: *Let A be a ring and let S be a multiplicatively closed subset of A . Then the canonical structure map $A \rightarrow A_S$ is universally SGB.*

PROOF: By [18, Corollary 4.11], the problem reduces to showing that $A[X_1, \dots, X_n] \rightarrow A_S[X_1, \dots, X_n] \cong A[X_1, \dots, X_n]_S$ satisfies SGB for each nonnegative integer n . This, in turn, follows from the fact [18, Lemma 5.6 (3)] that any ring of fractions $B \rightarrow B_T$ satisfies SGB. \square

Proposition 3.8: *Let $f : A \rightarrow B$ be a ring homomorphism, and let $g : A(X) \rightarrow B(X)$ be the ring homomorphism induced by f . If f is universally SGB, then g is universally SGB.*

PROOF: Let S (resp., T) denote the set of all the polynomials in $A[X]$ (resp., $B[X]$) with unit content. Then, by definition, $A(X) = A[X]_S$ and $B(X) = B[X]_T$. Let $h : A[X] \rightarrow B[X]$ denote the ring homomorphism induced by f . Since f is universally SGB, so is the induced map $A(X) = A[X]_S \rightarrow B[X]_S$. Of course, $B[X]_S \cong B[X]_{h(S)}$. Now, let U denote the canonical image of T in $B[X]_{h(S)}$. Since $h(S) \subseteq T$, it follows from a basic fact about rings of fractions [2, Proposition 7 (i), p. 65] that $B(X) = B[X]_T \cong (B[X]_{h(S)})_U$. As the universally SGB property is evidently preserved by composition (by [18, Corollary 4.11 and Lemma 4.5 (1)]), the assertion concerning g will follow if we show that the canonical map $B[X]_{h(S)} \rightarrow (B[X]_{h(S)})_U$ is universally SGB.

However, Lemma 3.7 established that *any* ring of fractions is universally SGB, and so the proof is complete. \square

References

- [1] D. F. Anderson, D. E. Dobbs, and M. Fontana. On treed Nagata rings. *J. Pure Appl. Algebra*, 61:107–122, 1989.
- [2] N. Bourbaki. *Commutative Algebra*. Addison-Wesley, Reading, 1972.
- [3] A. Bouvier, D. E. Dobbs, and M. Fontana. Universally catenarian integral domains. *Adv. in Math.*, 72:211–238, 1988.
- [4] D. E. Dobbs. On going-down for simple overrings, II. *Comm. Algebra*, 1:439–458, 1974.
- [5] D. E. Dobbs. Going-down rings with zero-divisors. *Houston J. Math.*, 23:1–12, 1997.
- [6] D. E. Dobbs and M. Fontana. Universally going-down homomorphisms of commutative rings. *J. Algebra*, 90:410–429, 1984.
- [7] D. E. Dobbs and M. Fontana. Universally going-down integral domains. *Arch. Math.*, 42:426–429, 1984.
- [8] D. E. Dobbs and I. J. Papick. On going-down for simple overrings, III. *Proc. Amer. Math. Soc.*, 54:35–38, 1976.
- [9] D. E. Dobbs and G. Picavet. On strong going-between, going-down, and their universalizations. In *Rings, Modules, Algebras and Abelian Groups*. Dekker, New York, to appear.
- [10] M. Fontana. Topologically defined classes of commutative rings. *Ann. Mat. Pura Appl.*, 123:331–355, 1980.
- [11] M. Fontana, J. A. Huckaba, and I. J. Papick. *Prüfer Domains*. Dekker, New York, 1997.
- [12] R. Gilmer. *Multiplicative Ideal Theory*. Dekker, New York, 1972.

- [13] A. Grothendieck and J. A. Dieudonné. *Eléments de Géométrie Algébrique*. Springer-Verlag, Berlin, 1971.
- [14] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.
- [15] I. Kaplansky. *Commutative Rings, rev. ed.* Univ. Chicago Press, Chicago, 1974.
- [16] W. J. Lewis. The spectrum of a ring as a partially ordered set. *J. Algebra*, 25:419–434, 1973.
- [17] S. McAdam. Going down in polynomial rings. *Can. J. Math.*, 23:704–711, 1971.
- [18] G. Picavet. Universally going-down rings, 1-split rings and absolute integral closure. *Comm. Algebra*, 31:4655–4685, 2003.
- [19] L. J. Ratliff, Jr. Going-between rings and contractions of saturated chains of prime ideals. *Rocky Mountain J. Math.*, 7:777–787, 1977.
- [20] L. J. Ratliff, Jr. $A(X)$ and GB-Noetherian rings. *Rocky Mountain J. Math.*, 9:337–353, 1979.

DAVID E. DOBBS
UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS
KNOXVILLE, TENNESSEE 37996-1300

U.S.A.
dobbs@math.utk.edu

GABRIEL PICAVET
UNIVERSITÉ BLAISE PASCAL
LABORATOIRE DE MATHÉMATIQUES PURES
63177 AUBIÈRE CEDEX

FRANCE
Gabriel.Picavet@math.univ-bpclermont.fr