Spectral density estimation for $p$-adic stationary processes


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Abstract

In this paper, we propose two asymptotically unbiased and consistent estimators for the spectral density of a stationary $p$-adic process $X = (X(t))_{t \in \mathbb{Q}_p}$. The first estimator is constructed from observations $X = (X(t))_{t \in U_n}$, $U_n$ being the $p$-adic ball with center 0 and radius $p^n$, $n \in \mathbb{Z}$, and the second, from observations $(X(\tau_k))_{k \in \mathbb{Z}}$, where $(\tau_k)_{k \in \mathbb{Z}}$ is a sequence of random variables taking their values in $\mathbb{Q}_p$, associated to a Poisson counting process $\mathcal{N}$.

Key Words: $p$-Adic analysis; Periodogram; Quadratic-mean consistency; Spectral density.

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1 Introduction

In mathematical physics, we use real and complex numbers, since space time coordinates are well described by means of real numbers. Recently, to answer many questions in physics, an increasing interest has been given to $p$-adic numbers: they are used in superstrings theory (using very small distances, of the order of Planck length) where there are no grounds for believing the usual ideas to be valid. $P$-adic numbers are going to be used, not only in mathematical physics, but also in other scientific grounds, where are met fractals and hierarchical structures (turbulence theory, dynamical physics, biology ...) (cf. [13] and [1]). Brillinger, in [2], was the first to introduce spectral estimation for stationary $p$-adic processes, and he constructed the periodogram analogously to the real case. This paper has two focuses developing a consistent estimates of the spectrum of a $p$-adic stationary process: observed on a $p$-adic ball and observed at the points process like in [8] and [9].

First we give some preliminaries about $p$-adic numbers.

2 Preliminaries

2.1 $p$-Adic numbers

Let $p$ be a prime number. The norm $|.|_p$ on the field $\mathbb{Q}$ of rational numbers is defined by:

$$|x|_p = \begin{cases} p^{-\nu(x)} & \text{if } x = p^{\nu(x)}a/b, \text{ where } p \text{ is divisor neither of } a \text{ nor of } b. \\ 0 & \text{if } x = 0 \end{cases}$$

where $\nu(x) \in \mathbb{Z}$.

$|.|_p$ is a norm on $\mathbb{Q}$ and is called $p$-adic norm. The completion of $\mathbb{Q}$ for that norm is denoted $\mathbb{Q}_p$, which called the field of $p$-adic numbers.
Theorem 2.1 (Ostrowski's theorem). The Euclidian norms and the p-adic norms (p being a prime number), are the only non trivial (non equivalent) possible norms on the field \( \mathbb{Q} \).

Let \( x \in \mathbb{Q}_p \), \( x \neq 0 \); then \( x \) can be represented in a unique manner under the canonical form (Hansel representation):

\[
x = p^{v(x)} \sum_{j=0}^{\infty} a_j p^j \quad \text{where} \quad 0 \leq a_j < p, \quad a_0 > 0, \quad j = 0, 1, 2, \ldots
\]

where the series (1) converges for the \( |.|_p \) norm.

Definition 2.1 The fractional part of a p-adic number \( x \), denoted \( (x)_p \), or \( (x) \), is the number:

\[
(x) = \begin{cases} 
0 & \text{if} \ v(x) > 0, \\
\frac{p^{v(x)} \sum_{i=0}^{v(x)-1} a_i p^i}{p^n} & \text{if} \ v(x) \leq 0.
\end{cases}
\]

Remark 2.1 For all \( x \in \mathbb{Q}_p \); \( 0 < (x) < 1 \), if \( v(x) \leq 0 \).

The ball with center \( x_0 \) and radius \( p^n \) is denoted by \( U_n(x_0) \), i.e.

\[
U_n(x_0) = \{ x \in \mathbb{Q}_p / |x - x_0|_p \leq p^n \}.
\]

We denote \( U_n = U_n(0) \), and we have the following properties:

1. \( U_n(x_0) \) is compact, open in \( \mathbb{Q}_p \).
2. If \( x_1 \in U_n(x_0) \), \( U_n(x_1) = U_n(x_0) \): Every point of the ball \( U_n(x_0) \) is its center.
3. If \( U_n(x_0) \cap U_{n_1}(x_1) \neq \emptyset \) and \( n_1 \leq n \), then \( U_{n_1}(x_1) \subset U_n(x_0) \): Two balls in \( \mathbb{Q}_p \) are either disjoint, or included one in the other.

\((\mathbb{Q}_p, +, \cdot)\) is a complete separable metric space, locally compact and disconnected.

2.2 Characters of the group \((\mathbb{Q}_p, +)\) and Fourier analysis on \( \mathbb{Q}_p \)

\((\mathbb{Q}_p, +)\) is an abelian locally compact group; from Haar’s theorem there exists a positive measure on \( \mathbb{Q}_p \), determined uniquely except for a constant, denoted \( \mu \), which verifies the following properties: for \( a \in \mathbb{Q}_p \), \( d(t + a) = dt \), \( d(at) = |a|_p dt \) and \( \mu(Z_p) = 1 \), where \( Z_p = U_0 \).

If \( A \in B_{\mathbb{Q}_p} \); \( \mu(A) \) is the Haar measure of \( A \), where \( B_{\mathbb{Q}_p} \) is the borelian σ-field on \( \mathbb{Q}_p \). This measure is explicited in [6] (pages 202-203).

The characters \( \gamma \) of \( \mathbb{Q}_p \) are defined by \( \gamma : (\mathbb{Q}_p, +) \rightarrow (\mathbb{C}, \cdot) \), continuous and verifying:

1. \( \gamma(t)\overline{\gamma(-t)} = 1 \).
2. \( \forall t, s \in \mathbb{Q}_p \), \( \gamma(t + s) = \gamma(t)\gamma(s) \).

From [4], [6] (pages 400-402), we get the following expression for characters of \( \mathbb{Q}_p \):

\[
\forall \gamma \in \mathbb{Q}_p, \exists \gamma \in \mathbb{Q}_p; \forall t \in \mathbb{Q}_p \quad \gamma(t) = e^{2\pi i \gamma(t)}, \quad \text{where} \quad \gamma(t) \text{ is the fractional part of the p-adic number } \gamma t \text{ and } Q_p \text{ denotes the dual group of } \mathbb{Q}_p.
\]

The Fourier transform \( \mathcal{F}f \) of \( f \) is given by: \( \forall u \in \mathbb{Q}_p, \quad \mathcal{F}f(u) = \int_{\mathbb{Q}_p} f(x) e^{2\pi i (ux)} dx \).

It is defined for all absolutely integrable functions \( f \in L^1(\mathbb{Q}_p) \).

If \( f \in L^2(\mathbb{Q}_p) \), we have the inverse relation: \( f(x) = c \int_{\mathbb{Q}_p} e^{-2\pi i (ux)} \mathcal{F}f(u) du \), where \( c \) is a positive constant.

Moreover, Plancherel’s formula is: \( \int_{\mathbb{Q}_p} |f(x)|^2 dx = c \int_{\mathbb{Q}_p} |\mathcal{F}f(u)|^2 du \).
Example 2.1 1. $D_n(\lambda) = \int_{U_n} e^{-2i\pi(\lambda t)} dt$, Dirichlet kernel. If we calculate $D_n(\lambda)$, we obtain:
$$D_n(\lambda) = \begin{cases} p^n & \text{if } |\lambda|_p \leq p^{-n} \\ 0 & \text{elsewhere} \end{cases}$$

2. $F\delta = 1$ and $F1 = \delta$

Usual Fourier transforms are calculated in [4], [12].

3 Spectral density estimation for a $p$-adic stationary process from non-random sampling

3.1 The periodogram

Let $X = \{X(t)\}_{t \in \mathbb{Q}_p}$ be a real valued $p$-adic stationary second order process, with mean zero, continuous covariance function $c_2 = \operatorname{cum}\{X(t + u), X(t)\}$ for all $t, u \in \mathbb{Q}_p$, element of $L^1(\mathbb{Q}_p)$, such that

$\mathcal{H}_1$) $X$ is stationary up to order 4, and the fourth cumulant function

$$c_4(u_1, u_2, u_3) = \operatorname{cum}\{X(t + u_1), X(t + u_2), X(t + u_3), X(t)\},$$

is absolutely integrable.

$\mathcal{H}_2$) there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, and $F_n = \mathbb{Q}_p \setminus U_n$

$$\int_{F_n} |c_2(t)| dt \leq \text{const}/p^n,$$

where const denotes a positive constant.

The covariance $c_2$ is semi definite positive and continuous; then from Bochner's theorem, there exists a measure $F_X$ with bounded variation on $\mathbb{Q}_p$, such that

$$c_2(u) = \int_{\mathbb{Q}_p} e^{2i\pi(uz)} dF_X(z),$$

where $F_X$ is the spectral measure of $X$, and is uniquely determined from $c_2$.

As $c_2 \in L^1(\mathbb{Q}_p)$, the spectral density $f_X$ is defined by

$$f_X(x) = \int_{\mathbb{Q}_p} c_2(t) e^{-2i\pi(tx)} dt, \quad \forall x \in \mathbb{Q}_p.$$ 

First, we study the periodogram. The studied process is observed for all $t$ belonging to $U_n$. The finite Fourier transform is then: $d_{X,n}(\lambda) = \int_{U_n} X(t) e^{-2i\pi(t\lambda)} dt$, $\forall \lambda \in \mathbb{Q}_p$, and the periodogram is:

$$I_{X,n}(\lambda) = \frac{1}{p^n} |d_{X,n}(\lambda)|^2 = \frac{1}{p^n} d_{X,n}(\lambda)d_{X,n}(-\lambda) = \frac{1}{p^n} \int_{U_n^2} X(t)X(s) e^{-2i\pi((t-s)\lambda)} dtds. \quad (2)$$

Lemma 3.1 Under $\mathcal{H}_2$, we have: $\operatorname{cov}\{d_{X,n}(u_1), d_{X,n}(u_2)\} = f_X(u_1)D_n(u_2 - u_1) + O(1)$.
Proof. Since, \( d_{X,n}(u) = \int_{U_n} e^{-2i\pi(tu)} X(t) dt \).

\[
\text{cov}\{d_{X,n}(u_1), d_{X,n}(u_2)\} = \int_{U_n} \int_{U_n} e^{-2i\pi(t_1u_1 - t_2u_2)} c_2(t_1 - t_2) dt_1 dt_2
\]

\[= \int_{U_n} \int_{Q_p} e^{-2i\pi(t_1u_1)} c_2(t) dt e^{2i\pi((u_2 - u_1)t_2)} dt_2 (3)\]

Moreover, from \( H_2 \), (4) is less than \( \int_{U_n} \int_{F_n} |c_2(t)| dt dt_2 \), which is bounded. Thus (4) is \( O(1) \), and

\[
\text{cov}\{d_{X,n}(u_1), d_{X,n}(u_2)\} = f_X(u_1) \int_{U_n} e^{2i\pi(t_2(u_2 - u_1))} dt_2 + O(1)
\]

\[= f_X(u_1) D_n(u_2 - u_1) + O(1).\]

**Proposition 3.1** Let \( X = \{X(t)\}_{t \in Q_p} \) be a real variate \( p \)-adic stationary second order process, with mean zero, continuous covariance function \( c_2 \), element of \( L^1(Q_p) \), such that \( H_2 \) is satisfied, then

\[E[I_{X,n}(\lambda)] = f_X(\lambda) + O\left(\frac{1}{p^n}\right).\]

Therefore \( I_{X,n}(\lambda) \) is an asymptotically unbiased estimator for \( f_X(\lambda) \).

Proof. From lemma 3.1, we get

\[E[I_{X,n}(\lambda)] = \frac{1}{p^n} [f_X(\lambda) D_n(0) + O(1)] = f_X(\lambda) + O\left(\frac{1}{p^n}\right).\]

**Proposition 3.2** Let \( X = \{X(t)\}_{t \in Q_p} \) be a real variate \( p \)-adic stationary second order process, with mean zero, continuous covariance function \( c_2 \), element of \( L^1(Q_p) \), such that \( H_1 \) is satisfied, then

\[\lim_{n \to +\infty} \text{var}[I_{X,n}(\lambda)] = f_X^2(\lambda) + f_X^2(0)\]

Therefore \( I_{X,n}(\lambda) \) is a non consistent estimator of \( f_X(\lambda) \).

Proof. We have

\[
\text{var}[I_{X,n}(\lambda)] = \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(\lambda)d_{X,n}(-\lambda), d_{X,n}(\lambda)d_{X,n}(-\lambda)\}
\]

\[= \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(\lambda), d_{X,n}(-\lambda), d_{X,n}(\lambda), d_{X,n}(-\lambda)\} + \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(\lambda), d_{X,n}(-\lambda)\} \text{cum}\{d_{X,n}(-\lambda), d_{X,n}(\lambda)\} + \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(\lambda), d_{X,n}(\lambda)\} \text{cum}\{d_{X,n}(-\lambda), d_{X,n}(-\lambda)\}. (5)\]

From [2], page 162, we can write:

\[
\text{cum}\{d_{X,n}(\lambda_1), \ldots, d_{X,n}(\lambda_k)\} = D_n(\lambda_1 + \ldots + \lambda_k) \int_{(U_n)^{k-1}} e^{-2i\pi(\lambda_1 u_1 + \ldots + \lambda_k u_k)}
\]

\[\times c_{k-1}(u_1, \ldots, u_{k-1}) du_1 \ldots du_{k-1}, k = 2, \ldots (8)\]
Then, the term (5), can be written
\[ \frac{1}{p^{2n}} D_n(0) \int_{U_n} \int_{U_n} \int_{U_n} e^{-2i\pi(\lambda u_1 - \lambda u_2 - \lambda u_3)} c_4(u_1, u_2, u_3) du_1 du_2 du_3, \]
thus,
\[ |(5)| \leq \frac{1}{p^n} \int \int \int_{Q_p^3} |c_4(u_1, u_2, u_3)| du_1 du_2 du_3. \]
Since \( c_4(., ., .) \in L^1(Q_p^3) \) (from \( \mathcal{H}_1 \)), we obtain that (5) is \( O(1/p^n) \).

From proposition 3.1, we have \( \lim_{n \to +\infty} \mathbb{E}[I_{X,n}(\lambda)] = \int_{Q_p} c_2(u) e^{-2i\pi(u\lambda)} du = f_X(\lambda) \),
then, the limit of (6) is \( f_X^2(\lambda) \). As for (6); (7) is
\[ \frac{1}{p^n} |D_n(2\lambda)|^2 \int_{U_n} e^{-2i\pi(u\lambda)} c_2(u) du \int_{U_n} e^{2i\pi(v\lambda)} c_2(v) dv 
= 1_{\{2\lambda \leq p^{-n}\}}(\lambda) \left[ \int_{U_n} e^{-2i\pi(u\lambda)} c_2(u) du \right]^2, \tag{9} \]
and (9) converges to \( f_X^2(0) \). Thus \( \lim_{n \to \infty} \text{var}[I_{X,n}(\lambda)] = f_X^2(\lambda) + f_X^2(0) \).

4 Smoothing the periodogram

The method is analogous to the smoothing in real time processes. Let \( (M_n)_{n \in \mathbb{N}} \) be a sequence of rational numbers, the terms of which are powers of \( p \), and such that:
\[ \lim_{n \to +\infty} M_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} p^n M_n = +\infty. \tag{10} \]
For example, we choose \( M_n = p^{-\lfloor x \rfloor} \), where \( \lfloor x \rfloor \) is the integer part of \( x \). That sequence verifies (10).

Let us consider a function \( W : Q_p \to \mathbb{R} \), which is continuous, positive and even, and verifies:
\[ W \in L^\infty(Q_p) \cap L^1(Q_p) \text{ and } \int_{Q_p} W(\lambda) d\lambda = 1. \tag{11} \]
For each \( t \in Q_p \), we denote by \( \mathcal{F}W(t) \) the Fourier transform of \( W \), that is
\[ \mathcal{F}W(t) = \int_{Q_p} W(\lambda) e^{-2i\pi(\lambda t)} d\lambda. \]
Let \( W_n(\lambda) = 1/M_n W(M_n\lambda) \) be the spectral window.

The smoothed estimator of \( f_X \) is
\[ \hat{f}_{X,n}(\lambda) = \int_{Q_p} W_n(\lambda - u) I_{X,n}(u) du. \]

Proposition 4.1 Let \( X = \{X(t)\}_{t \in Q_p} \) be a real valued \( p \)-adic stationary second order process, with mean zero, continuous covariance function \( c_2 \in L^1(Q_p) \) satisfying \( \mathcal{H}_2 \), then
\[ \mathbb{E} \left[ \hat{f}_{X,n}(\lambda) \right] = \int_{Q_p} W_n(\lambda - u) f_X(u) du + O\left(\frac{1}{p^n}\right). \]
Therefore \( \hat{f}_{X,n}(\lambda) \) is an asymptotically unbiased estimator for \( f_X(\lambda) \).
Proof. From proposition 3.1, and with the change of variates $v = M_n \lambda - M_n u$, we get:

$$E \left[ \hat{f}_{x,n}(\lambda) \right] = \int_{Q_p} W(v) f_X(\lambda - \frac{v}{M_n}) dv + O(1/p^n).$$

Since $c_2 \in L^1(Q_p)$, $f_X$ is continuous; and as $\int_{Q_p} W(u) du = 1$, then by dominated convergence theorem, we get the result.

In the sequel, for $x \in Q_p$, we denote by $\delta_x$ the $p$-adic Dirac delta function, given for all $\lambda \in Q_p$ by:

$$\delta_x(\lambda) = \begin{cases} 1 & \text{if } \lambda = x \\ 0 & \text{otherwise.} \end{cases}$$

and shortly, $\delta(\lambda) = \delta_0(\lambda)$.

Proposition 4.2 Let $X = \{X(t)\}_{t \in Q_p}$ be a real variate $p$-adic stationary second order process, with mean zero, continuous covariance function $c_2$ element of $L^1(Q_p)$, such that hypotheses $H_1$ and $H_2$ are satisfied, then for all $\lambda_1, \lambda_2$ in $Q_p$

$$p^n M_n \text{cov}\left[ \hat{f}_{x,n}(\lambda_1), \hat{f}_{x,n}(\lambda_2) \right]$$

$$= \left[ \delta(\lambda_1 + \lambda_2) + \delta(\lambda_1 - \lambda_2) \right] f^2_X(\lambda_1) \int_{Q_p} W^2(u) du + o(1) + O(M_n).$$

Therefore $\hat{f}_{x,n}(\lambda)$ is a consistent estimator for $f_X(\lambda)$.

To prove proposition 4.2, we need the following lemma:

Lemma 4.1 We denote $\Delta_n(u) = 1/p^n |D_n(u)|^2$ the $p$-adic Fejer kernel, and $J_n(\lambda) = \int_{Q_p} W_n(\lambda - u) \Delta_n(u) du - W_n(\lambda)$.

Then $J_n(\lambda) = o(1/M_n)$, uniformly in $\lambda$.

Proof of lemma 4.1. With $t = (1/p^n) \lambda$, we obtain

$$D_n(v p^n) = \int_{|\lambda|_p \leq 1} e^{-2i \pi (v \lambda)} |1/p^n|_p d\lambda = p^n D_0(v),$$

where $D_0(v) = \int_{|\lambda|_p \leq 1} e^{-2i \pi (v \lambda)} d\lambda$.

As, $\int_{Q_p} \Delta_n(u) du = p^n \mu \{|u|_p \leq p^{-n}\}$, moreover $\mu \{|u|_p \leq p^{-n}\} = p^{-n}$, then

$$\int_{Q_p} \Delta_n(u) du = 1.$$  With the change of variates $u = vp^n$, we have

$$J_n(\lambda) = \frac{1}{M_n} \int_{Q_p} \left[ W(M_n \lambda - p^n M_n v) - W(M_n \lambda) \right] \Delta_n(v p^n) \frac{dv}{p^n}$$

and

$$\int_{Q_p} \left[ W(M_n \lambda - p^n M_n v) - W(M_n \lambda) \right] \frac{dv}{M_n} \int_{Q_p} \left| D_0(v) \right| dv.$$

As, $W(\lambda) = \int_{Q_p} e^{2i \pi (\lambda t)} W(t) dt$, we have

$$|W(M_n \lambda - p^n M_n v) - W(M_n \lambda)| \leq \int_{Q_p} \left| e^{-2i \pi (t M_n v)} - 1 \right| |W(t)| dt,$$

thus, $|M_n J_n(\lambda)| \leq \int_{Q_p} \left[ \int_{Q_p} \left| e^{-2i \pi (t M_n p^n v)} - 1 \right| |W(t)| |D_0(v)| dv,$

$$\text{where } \theta(v) = 2 |D_0(v)|^2 \int_{Q_p} |W(t)| dt \in L^1(Q_p).$$

Indeed, $\int_{Q_p} |\theta(v)| dv = 2 |\mathcal{F}W|_{L^1(Q_p)} < \infty$. Then, by dominated convergence theorem, we get:

$$\lim_{n \to +\infty} |M_n J_n(\lambda)| \leq \int_{Q_p} \int_{Q_p} \lim_{n \to +\infty} \left| e^{-2i \pi (t M_n p^n v)} - 1 \right| |W(t)| |D_0(v)| dv.$$
Since, \( (tvM_n p_n)_{n \in \mathbb{N}} \) is a \( p \)-adic sequence, and, \( \lim_{n \to +\infty} p^n M_n = +\infty \),
\[
\lim_{n \to +\infty} |tvM_n p_n| = \lim_{n \to +\infty} |t|_p |v|_p |M_n|_p |p^n|_p = \lim_{n \to +\infty} |t|_p |v|_p / M_n p^n = 0,
\]
thus, \( \lim_{n \to +\infty} M_n J_n(\lambda) = 0 \) uniformly in \( \lambda \), i.e. \( J_n(\lambda) = o(1/M_n) \).

**Proof of proposition 4.2.** Let \( \lambda_1, \lambda_2 \in Q_p \). We have
\[
\text{cov}\{\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\} = \int_{Q_p} \int_{Q_p} W_n(\lambda_1 - u_1) W_n(\lambda_2 - u_2) \text{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\} du_1 du_2.
\]
Let us compute \( \text{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\} \).

For this, we have \( \text{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\} = A_1 + A_2 + A_3 \), where
\[
A_1 = \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(u_1), d_{X,n}(-u_1), d_{X,n}(u_2), d_{X,n}(-u_2)\} \tag{12}
\]
\[
A_2 = \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(u_1), d_{X,n}(u_2)\} \text{cum}\{d_{X,n}(-u_1), d_{X,n}(-u_2)\} \tag{13}
\]
\[
A_3 = \frac{1}{p^{2n}} \text{cum}\{d_{X,n}(u_1), d_{X,n}(-u_2)\} \text{cum}\{d_{X,n}(u_2), d_{X,n}(-u_1)\} \tag{14}
\]

Thus, \( \text{cov}\{\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\} = I_1 + I_2 + I_3 \), where \( I_i \) is the contribution of \( A_i \) for \( i = 1, 2, 3 \).

For the first term \( I_1 \) : From the proof of proposition 2.2, we have \( A_1 = O(1/p^n) \), thus
\[
|I_1| \leq \frac{1}{p^n} \int_{Q_p} |W_n(\lambda_1 - u_1)| du_1 \int_{Q_p} |W_n(\lambda_2 - u_2)| du_2 \int \int_{Q^3_p} |c_4(v_1, v_2, v_3)| dv_1 dv_2 dv_3.
\]

W is integrable, and from \( x = M_n \lambda - M_n u_i \), for \( i = 1, 2 \), we get
\[
\int_{Q_p} |W_n(\lambda_i - u_i)| du_i = \int_{Q_p} |W(x)| dx < \infty.
\]

From \( H_1 \), we get \( p^n I_1 = O(1) \) i.e. \( p^n M_n I_1 = O(M_n) \).

The second term \( I_2 \) : From lemma 3.1, we get
\[
A_2 = \frac{1}{p^{2n}} \{ [f_X(u_1)D_n(u_2 - u_1) + O(1)][f_X(u_1)D_n(u_1 - u_2) + O(1)] \} = \frac{1}{p^{2n}} |f_X(u_1)D_n(u_2 - u_1)|^2 + \frac{1}{p^{2n}} O(1)f_X(u_1)D_n(u_2 - u_1) + \frac{1}{p^{2n}} O(1/p^{2n}).
\]
Since \( 1/p^{2n} \to 0 \) and \( 1/p^{2n} f_X(u_1)D_n(u_2 - u_1) \to 0 \) as \( n \to +\infty \), we deduce
\[
A_2 = \frac{1}{p^{2n}} |f_X(u_1)D_n(u_2 - u_1)|^2 + O(1/p^n).
\]

As, \( \Delta_n(u) = 1/p^n |D_n(u)|^2 \), we have
\[
I_2 = \frac{1}{p^n} \int_{Q_p} |W_n(\lambda_1 - u_1)| f_X(u_1)|^2 [J_n(\lambda_2 - u_1) + W_n(\lambda_2 - u_1)] du_1 + O(1/p^n).
\]

From lemma 4.1, and by dominated convergence theorem, we get
\[
I_2 = \frac{1}{p^n} \int_{Q_p} |W_n(\lambda_1 - u_1)| f_X(u_1)|^2 [W_n(\lambda_2 - u_1)] du_1 + \frac{1}{p^n} \int_{Q_p} |W_n(\lambda_1 - u_1)| f_X(u_1)|^2 O(1/p^n) du_1 + O(1/p^n)
\]
\[
= I_{2,1} + I_{2,2} + O(1/p^n),
\]
where, \( p^nM_{nI2,2} = o(1) \int_{Q_p} W_n(\lambda_1 - u_1)|f_X(u_1)|^2du_1. \)

As, \( W \in L^1(Q_p) \) and \( f_X \in L^\infty(Q_p) \), we have, \( p^nM_{nI2,2} = o(1). \)

With \( v = M_n\lambda_1 - M_nu_1 \), we get:

\[
p^nM_{nI2,1} = \int_{Q_p} W(v)W(M_n(\lambda_1 - \lambda_2) - v) f^2_X(\lambda_1 - \frac{v}{M_n})dv.
\]

If \( \lambda_1 = \lambda_2 \): Note that we have, \( W \) is a even function, \( f_X \) is a continuous and bounded function and \( \forall v \in Q_p; \lim_{n \to +\infty} v/M_n = 0 \) in \( Q_p \), then

\[
p^nM_{nI2,1} = \int_{Q_p} W(v)^2 \left[ f^2_X(\lambda_1 - \frac{v}{M_n}) - f^2_X(\lambda_1) \right] dv + \int_{Q_p} W(v)^2 f^2_X(\lambda_1)dv.
\]

Thus, by dominated convergence theorem, \( p^nM_{nI2,1} = o(1) + f^2_X(\lambda_1) \int_{Q_p} W(v)^2dv. \)

If \( \lambda_1 \neq \lambda_2 \): Since \( W(x) = \int_{Q_p} e^{-2i\pi(x)} FW(t)dt \), the Fourier transform of \( FW \), it is uniformly continuous; and since \( \lim_{n \to +\infty} |M_n(\lambda_2 - \lambda_1)|_p = \lim_{n \to +\infty} |(\lambda_2 - \lambda_1)|_p/M_n = +\infty \). Then from the dominated convergence theorem, \( \lim_{n \to +\infty} p^nM_{nI2,1} = 0. \)

Thus we get, \( \forall \lambda_1, \lambda_2 \in Q_p, p^nM_{nI2,1} = \delta(\lambda_2 - \lambda_1)f^2_X(\lambda_1) \int_{Q_p} W^2(v)dv + o(1). \)

So, \( p^nM_{nI2} = \delta(\lambda_2 - \lambda_1)f^2_X(\lambda_1) \int_{Q_p} W^2(v)dv + o(1) + O(M_n). \)

With analogous calculations, we get

\[
p^nM_{nI3} = \delta(\lambda_2 + \lambda_1) \int_{Q_p} W(v)^2df^2_X(\lambda_1) + o(1) + O(M_n).
\]

Thus,

\[
p^nM_{ncov}[\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)] = [\delta(\lambda_2 + \lambda_1) + \delta(\lambda_2 - \lambda_1)] f^2_X(\lambda_1) \int_{Q_p} W^2(v)dv + o(1) + O(M_n),
\]

and \( \hat{f}_{X,n} \) is consistent, thus proposition 4.2 is proved.

**Corollary 4.1** Under \( \mathcal{H}_1 \) - \( \mathcal{H}_2 \), we have \( \hat{f}_{X,n}(\lambda) \) converges to \( f_X(\lambda) \) in quadratic mean as \( n \to \infty. \)

**Proof of corollary 4.1.** We know that, the mean square error is:

\[
MSE(n) = \text{bias}^2(\hat{f}_{X,n}) + \text{var}(\hat{f}_{X,n}(\lambda)).
\]

From propositions 4.1 and 4.2, we get \( MSE(n) \to 0 \) as \( n \to +\infty. \)

This implies mean quadratic convergence.

5 Spectral density estimation of a \( p \)-adic stationary process from random sampling

5.1 Preliminaries

Let \( X = \{X(t), t \in Q_p\} \) be a \( p \)-adic stationary second order process, with mean zero, continuous covariance function element of \( L^1(Q_p) \), and with spectral density function \( f_X \).

From [3] and [7], there exists a counting process, denoted by \( \mathcal{N} \), which is associated to a
sequence \((\tau_k)_{k \in \mathbb{Z}}\) of random variables taking their values in \(\mathbb{Q}_p\).

The process \(N\) is defined by

\[
N : B_{\mathbb{Q}_p} \times \Omega \rightarrow N
(A, \omega) \mapsto N(A, \omega) = \sum_{k \in \mathbb{Z}} 1_A(\tau_k(\omega))
\]

and \(N(A, \omega)\) is the number of \(\tau_k\)'s belonging to \(A\).

We suppose that, for every \(A\) element of \(B_{\mathbb{Q}_p}\), the random variable \(N(A)\) has a Poisson distribution \(P(\Lambda(A))\) (such a process exists by [3, 7]), where \(\Lambda(A) = \beta \mu(A)\) and \(\mu\) is the Haar measure on \(\mathbb{Q}_p\). In the sequel, we suppose also that also the mean intensity \(\beta = \mathbb{E}\{U_0\}\) is known.

For every \(A, B\) disjoint in \(\mathbb{Q}_p\), \(N(A)\) and \(N(B)\) are independent.

Thus \(\mathbb{E}[N(A)] = \beta \mu(A) = \beta \int_{\mathbb{Q}_p} 1_A(x) dx\) which implies \(\mathbb{E}[N(dt)] = \beta dt\).

Let \(A, B \in B_{\mathbb{Q}_p}\), we have

\[
\mathbb{E}[N(A)N(B)] = \mathbb{E}[N(A \cap B)^2] + \mathbb{E}[N(A \cap B)N(A \cap B^c)]
+ \mathbb{E}[N(A \cap B)N(A^c \cap B)] + \mathbb{E}[N(A \cap B^c)N(A^c \cap B)],
\]

where \(A^c\) denotes the complement of \(A\) in \(\mathbb{Q}_p\).

Since \(N(A)\) is Poisson for every \(A\) in \(B_{\mathbb{Q}_p}\), we get:

\[
\mathbb{E} \left[ N(A \cap B)^2 \right] = \beta^2 \mu(A \cap B)^2 + \beta \mu(A \cap B).
\]

Thus, since

\[
\int_{\mathbb{Q}_p} f(x, x) dx = \int_{\mathbb{Q}_p} f(x, y) \left[ \int_{\mathbb{Q}_p} d\delta_x(y) \right] dx,
\]

we have

\[
\mathbb{E}[N(A)N(B)] = \beta^2 \mu(A) \mu(B) + \beta \mu(A \cap B)
= \int_{A \times B} \beta^2 dx dy + \int_{A \times B} \beta d\delta_x(y) dx,
\]

Then

\[
\mathbb{E}[N(dt)N(ds)] = \beta^2 dt ds + \beta d\delta(s) dt
\]

and

\[
\mathbb{E}[N(t + dt)N(t + s + ds)] = \beta^2 dt ds + \beta d\delta(s) dt.
\]

Since \(N(A)\) is Poisson, \(\mathbb{E}[N(t + dt)] = \mathbb{E}[N(dt)] = \beta dt\) and

\[
\text{cov}[N(t + dt), N(t + s + ds)] = \beta^2 dt ds + \beta d\delta(s) dt - \beta^2 dt ds
= \beta d\delta(s) dt.
\]

### 5.2 Construction of the spectral density estimator

**Definition 5.1** The sample process \(Z\) is defined by

\[
Z(A) = \int_A X(t) N(dt) = \sum_{k \in \mathbb{Z}} X(\tau_k) 1_A(\tau_k) = \sum_{\tau_k \in A} X(\tau_k), \quad \forall A \in B_{\mathbb{Q}_p}.
\]
This definition may be written, too: \( Z(t + dt) = X(t)N(t + dt) \).

Since \( X \) and \( N \) are independent, we get

\[
\mathbb{E} [Z(dt)] = \mathbb{E} [X(t)N(dt)] = \mathbb{E} [X(t)] \mathbb{E} [N(dt)] = 0
\]

and

\[
\text{cov} [Z(t + dt), Z(t + s + ds)] = \mathbb{E} [X(t)X(t + s)] \mathbb{E} [N(t + dt)N(t + s + ds)] = c_2(s) \beta^2 ds + \beta \delta(s) dt.
\]

Thus the increment process \( Z \) is second order stationary.

Since \( c_2 \in L^1(Q_p) \), we define the measure \( \theta_Z \), for all \( B \) element of \( \mathcal{B}_Q \), by:

\[
\theta_Z(B) = \int_B \theta_Z(dt) = \int_B c_2(u) \left[ \beta^2 dt + \beta \delta(t) \right],
\]

then

\[
\theta_Z(dt) = c_2(t) \left[ \beta^2 dt + \beta \delta(t) \right].
\]

Defining the measure \( \theta_X \), for every \( B \) in \( \mathcal{B}_Q \), by:

\[
\theta_X(B) = \int_B \theta_X(dt) = \int_B \beta^2 du + \beta \delta(u),
\]

(under differential form)

\[
\theta_X(dt) = \beta c_2(0) \delta(B) + \int_B c_2(u) \beta^2 du.
\]

The Haar measure being \( \sigma \)-finite on \( Q_p \), then the measures \( \theta_X \) and \( \theta_Z \) are also \( \sigma \)-finite.

The spectral density \( f_Z \) associated to the process \( Z \) is defined, for \( \lambda \in Q_p \) by:

\[
f_Z(\lambda) = \int_{Q_p} e^{-2i\pi \langle t, \lambda \rangle} \theta_Z(dt).
\]

By definition of \( \theta_Z \), we get

\[
f_Z(\lambda) = \beta \int_{Q_p} e^{-2i\pi \langle \lambda t \rangle} c_2(0) \delta(t) + \beta^2 \int_{Q_p} e^{-2i\pi \langle \lambda t \rangle} c_2(t) dt
\]

\[
= \beta c_2(0) + \beta^2 f_X(\lambda). \tag{15}
\]

In order to estimate \( f_X \), we write from the formula (15): \( f_X(\lambda) = 1/\beta^2 [f_Z(\lambda) - \beta c_2(0)] \).

So we have to estimate \( c_2(0) \) and \( f_Z \).

1. Estimation of \( c_2(0) \).

   We propose the estimator: \( \hat{c}_{z,n}(0) = 1/(\beta p^n) \int_{U_n} X^2(t)N(dt) \).

2. Estimation of \( f_Z(\lambda) \).

   First we introduce a sequence \( (M_n)_{n \in \mathbb{N}} \) of rational numbers, a \( p \)-adic kernel \( W \), like in section 2.1, formula (11); and the same spectral window, i.e. \( W_n(\lambda) = 1/M_n W(M_n \lambda) \).

   Let \( d_{z,n}(\lambda) \) be the finite Fourier transform associated to the observations \( Z(t), t \in U_n \), i.e.

   \[
d_{z,n}(\lambda) = \int_{U_n} e^{-2i\pi \langle \lambda t \rangle} Z(dt) = \int_{U_n} e^{-2i\pi \langle \lambda t \rangle} X(t)N(dt).
\]

   The associated periodogram is:

   \[
   I_{z,n}(\lambda) = \frac{1}{p^n} |d_{z,n}(\lambda)|^2 = \frac{1}{p^n} \left| \int_{U_n} e^{-2i\pi \langle \lambda t \rangle} X(t)N(dt) \right|^2.
\]

   We estimate \( f_Z(\lambda) \) by: \( \hat{f}_{z,n}(\lambda) = \int_{Q_p} W_n(\lambda - u) I_{z,n}(u) du \).
Then we propose the following estimator for $f_X(\lambda)$:

$$\hat{f}_{X,n}(\lambda) = \frac{1}{\beta^2} \left[ \hat{f}_{X,n}(\lambda) - \beta \hat{c}_{2,n}(0) \right].$$

### 5.3 Asymptotic behaviour of the estimators

The asymptotic behaviour of $\hat{f}_{X,n}(\lambda)$ will be established from the properties of the estimators $\hat{f}_{X,n}(\lambda)$ and $\hat{c}_{2,n}(0)$.

Our hypothesis are the following:

- $\mathcal{H}_3$: $c_2 \in L^1(Q_p)$
- $\mathcal{H}_4$: $c_2 \in L^2(Q_p)$
- $\mathcal{H}_5$: $\mathcal{F}W \in L^1(Q_p)$
- $\mathcal{H}_6$: $c_4(u, u, u) \in L^1(Q_p)$
- $\mathcal{H}_7$: $c_4(u, 0, 0) \in L^1(Q_p)$
- $\mathcal{H}_8$: $c_4(u, u, 0) \in L^1(Q_p)$
- $\mathcal{H}_9$: $c_4(u, v, 0) \in L^1(Q_p^2)$
- $\mathcal{H}_{10}$: $c_4(u, u + v, 0) \in L^1(Q_p^3)$
- $\mathcal{H}_{11}$: $c_4(u, u + v, u + v) \in L^1(Q_p^3)$
- $\mathcal{H}_{12}$: $c_4(u, u + v, u + v, 0) \in L^1(Q_p^3)$

**Proposition 5.1** Under $\mathcal{H}_4$ and $\mathcal{H}_8$, $\hat{c}_{2,n}(0)$ is unbiased and consistent.

**Proof of proposition 5.1.**

a. As $X$ and $N$ are independent,

$$\mathbb{E}[\hat{c}_{2,n}(0)] = \frac{1}{\beta p^n} \int_{U_n} \mathbb{E}[X^2(t)N(dt)] = \frac{1}{\beta p^n} c_2(0) \int_{U_n} \beta dt = c_2(0).$$

Then, $\hat{c}_{2,n}(0)$ is unbiased.

b. $\hat{c}_{2,n}(0)$ is consistent. Indeed:

$$p^{2n} \beta^2 \text{var} [\hat{c}_{2,n}(0)] = \int_{U_n} \int_{U_n} \mathbb{E} \left[ X^2(t)X^2(s)N(dt)N(ds) \right] - p^{2n} \beta^2 c_2^2(0)$$

$$= \int_{U} \int_{U} \mathbb{E} \left[ X^2(t)X^2(s) \right] \mathbb{E} \left[ N(dt)N(ds) \right] - p^{2n} \beta^2 c_2^2(0).$$

As $X$ and $N$ are independent, with $s = t + u$, we get

$$p^{2n} \beta^2 \text{var} [\hat{c}_{2,n}(0)] = \int_{U_n} \int_{U_n} \left[ c_4(0, u, u) + 2c_2^2(u) + c_2^2(0) \right] [\beta^2 du + \beta d\delta(u)] dt - p^{2n} \beta^2 c_2^2(0)$$

$$= \beta^2 p^n \int_{U_n} c_4(0, u, u) du + \beta p^n c_4(0, 0, 0) + 2\beta^2 p^n \int_{U_n} c_2^2(u) du + 3\beta p^n c_2^2(0).$$

From $\mathcal{H}_4$ and $\mathcal{H}_8$, we deduce: $\text{var} [\hat{c}_{2,n}(0)] = O(1/p^n)$.

**Proposition 5.2** Under $\mathcal{H}_3 - \mathcal{H}_{12}$, $\hat{f}_{X,n}(\lambda)$ is asymptotically unbiased and consistent.

**Proof of proposition 5.2.**

a. $\hat{f}_{X,n}(\lambda)$ is asymptotically unbiased. With change of variate $v = M_n \lambda - M_n u$, such that: $dv = |M_n|^p du = 1/M_n du$, we get

$$\mathbb{E}[\hat{f}_{X,n}(\lambda)] = \frac{1}{M_n} \mathbb{E} \left[ \int_{Q_p} W(M_n \lambda - M_n u) I_{X,n}(u) du \right]$$

$$= \int_{Q_p} W(v) \mathbb{E} \left[ I_{X,n}(\lambda - \frac{v}{M_n}) \right] dv.$$
As, \( \int_{Q_p} W(v)dv = 1 \), we obtain
\[
\mathbb{E}[f_{z,n}(\lambda)] - f_z(\lambda) = \int_{Q_p} W(v) \mathbb{E} \left[ I_{z,n}(\lambda - \frac{v}{M_n}) - f_z(\lambda) \right] dv.
\]
Let \( g_{n,\lambda}(v) = W(v) \mathbb{E} \left[ I_{z,n}(\lambda - \frac{v}{M_n}) - f_z(\lambda) \right] \). With \( w = u - t \), we can write
\[
\mathbb{E} \left[ I_{z,n}(\lambda - \frac{v}{M_n}) \right] = \frac{1}{p^n} \int_{U_n} \int_{U_n} e^{-2i\pi \langle(\lambda - v/M_n)(u-t)\rangle} \mathbb{E} [Z(dt)Z(du)]
\]
\[
= \frac{\beta^2}{p^n} \int_{U_n} \int_{U_n} e^{-2i\pi \langle(\lambda - v/M_n)(u-t)\rangle} c_2(t - u)dudt
\]
\[
+ \frac{\beta}{p^n} \int_{U_n} \int_{U_n} e^{-2i\pi \langle(\lambda - v/M_n)(u-t)\rangle} c_2(t - u)d\delta_t(u)dt
\]
\[
= \frac{\beta^2}{p^n} \int_{U_n} \int_{U_n} e^{-2i\pi \langle(\lambda - v/M_n)w\rangle} c_2(w)dw \]
\[
+ c_2(0) \frac{\beta}{p^n} \mu \{U_n\}
\]
\[
= \beta^2 \int_{U_n} e^{-2i\pi \langle(\lambda - v/M_n)w\rangle} c_2(w)dw + c_2(0)\beta. \tag{16}
\]
From dominated convergence theorem, and \( \lim_{n \to +\infty} v/M_n = 0 \) in \( Q_p \), we obtain:
\[
\lim_{n \to +\infty} \mathbb{E} \left[ I_{z,n}(\lambda - \frac{v}{M_n}) \right] = \beta c_2(0) + \beta^2 f_x(\lambda) = f_z(\lambda).
\]
Thus \( \lim_{n \to +\infty} g_{n,\lambda}(v) = 0 \). Moreover, from (16), and since \( |f_z(\lambda)| \leq \beta^2|c_2|L_1(Q_p) + |c_2(0)\beta| \), we have
\[
|g_{n,\lambda}(v)| \leq W(v) \left[ |c_2(0)|\beta^2 + \beta|c_2|L_1(Q_p) + \beta^2|c_2|L_1(Q_p) + \frac{|c_2(0)|}{\beta} \right]. \tag{17}
\]
The right hand side of (17) is integrable, since \( c_2 \) and \( W \) are; from dominated convergence theorem, we get the asymptotic unbiasedness of \( \hat{f}_{z,n} \).

b. With the change of variates \( v = \lambda - u \), we get
\[
\hat{f}_{z,n}(\lambda)
\]
\[
= \frac{1}{p^n} \int_{Q_p} \int_{U_n} \int_{U_n} W_n(\lambda - u) e^{-2i\pi (u(t-s))} X(t)X(s)N(t + dt)N(s + ds)du
\]
\[
= \frac{1}{p^n} \int_{U_n} \int_{U_n} \int_{Q_p} W_n(v) e^{2i\pi (v(t-s))} du e^{-2i\pi (\lambda(t-s))} X(t)X(s)N(t + dt)N(s + ds),
\]
and, with \( u = M_n v \), which implies \( dv = M_n du \), we can write
\[
\int_{Q_p} \frac{1}{M_n} W(M_n v) e^{2i\pi (v(t-s))} dv = \int_{Q_p} W(u) e^{2i\pi (u((t-s)/M_n))} = \mathcal{F}W ((t-s)/M_n).
\]
Let \( V_n(u) = \mathcal{F}W (u/M_n) \). We obtain
\[
\hat{f}_{z,n}(\lambda) = 1/p^n \int_{U_n} \int_{U_n} V_n(t - s) e^{-2i\pi (\lambda(t-s))} X(t)X(s)N(t + dt)N(s + ds).
\]
Let \( \lambda_1 \) and \( \lambda_2 \) be elements of \( Q_p \). We have from the independence of \( X \) and \( N \):
\[
p^{2n} \text{cov} \left[ \hat{f}_{z,n}(\lambda_1), \hat{f}_{z,n}(\lambda_2) \right]
\]
\[
= \int \int \int \int_{Q_p^4} V_n(t - s)V_n(u - v) e^{-2i\pi (\lambda_1(t-s) - \lambda_2(u-v))}
\]
\[
\times \left[ \mathbb{E} [X(s)X(t)X(u)X(v)] \mathbb{E} [N(ds)N(dt)N(du)N(dv)]
\]
\[
- \mathbb{E} [X(s)X(t)] \mathbb{E} [N(ds)N(dt)] \mathbb{E} [X(u)X(v)] \mathbb{E} [N(du)N(dv)] \right].
\]
As,
\[
\mathbb{E} [X(s)X(t)X(u)X(v)] = c_4(t-s,u-s,v-u) + c_2(t-s)c_2(u-v)
\]
\[
+ c_2(t-u)c_2(s-v) + c_2(t-v)c_2(s-u),
\]
we have, \(p^{2n}\) \text{cov} \left[ f_{x,n}(\lambda_1), f_{x,n}(\lambda_2) \right] = \sum_{i=1}^{4} J_i, \text{ where }
\]
\[
J_1 = \int \int \int U_{\lambda_1} c_2(t-s)c_2(u-v)V_n(t-s)V_n(u-v) e^{-2i\pi(\lambda_1(t-s)-\lambda_2(u-v))} \times (\mathbb{E}[N(ds)N(dt)N(du)N(dv)] - \mathbb{E}[N(ds)N(dt)] \mathbb{E}[N(du)N(dv)]),
\]
\[
J_2 = \int \int \int U_{\lambda_1} c_2(t-v)c_2(s-u)V_n(t-s)V_n(u-v) \times e^{-2i\pi(\lambda_1(t-s)-\lambda_2(v-u))} \mathbb{E}[N(ds)N(dt)N(du)N(dv)],
\]
\[
J_3 = \int \int \int U_{\lambda_1} c_2(t-u)c_2(s-v)V_n(t-s)V_n(u-v) \times e^{-2i\pi(\lambda_1(t-s)-\lambda_2(u-v))} \mathbb{E}[N(ds)N(dt)N(du)N(dv)],
\]
\[
J_4 = \int \int \int U_{\lambda_1} c_4(t-s,u-s,v-u)V_n(t-s)V_n(u-v) \times e^{-2i\pi(\lambda_1(t-s)-\lambda_2(u-v))} \mathbb{E}[N(ds)N(dt)N(du)N(dv)].
\]

From [10], we get the following formulas:
\[
\mathbb{E}[N(ds)N(dt)N(du)N(dv)]
\]
\[
= \beta d\delta_1(s)d\delta_3(u)d\delta_3(v)dt + \beta^2 d\delta_1(s)d\delta_5(v)dt + \beta^2 d\delta_2(s)d\delta_4(v)dt + \beta^3 d\delta_1(s)d\delta_5(v)dsdt
\]
\[
+ \beta^2 d\delta_1(s)d\delta_3(u)ds + \beta^2 d\delta_2(s)d\delta_4(v)ds + \beta^3 d\delta_1(s)d\delta_5(v)ds
\]
\[
+ \beta^2 d\delta_3(u)dsdt + \beta^3 d\delta_1(u)ds + \beta^4 dsdt
\]
\[
\text{and}
\]
\[
\mathbb{E}[N(ds)N(dt)N(du)N(dv)] - \mathbb{E}[N(ds)N(dt)] \mathbb{E}[N(du)N(dv)]
\]
\[
= \beta d\delta_1(s)d\delta_3(u)d\delta_3(v)dt + \beta^2 d\delta_1(s)d\delta_5(v)dt + \beta^2 d\delta_2(s)d\delta_4(v)dt + \beta^3 d\delta_1(s)d\delta_5(v)dt
\]
\[
+ \beta^2 d\delta_1(s)d\delta_3(u)dt + \beta^2 d\delta_2(s)d\delta_4(v)dt + \beta^3 d\delta_1(s)d\delta_5(v)dt
\]
\[
+ \beta^2 d\delta_3(u)dt + \beta^3 d\delta_1(u)dt + \beta^4 dt
\]

\textbf{Remark 5.1} Without any supplementary condition on \(X\), we get \(J_{1,1} = J_{2,1} = J_{3,1} = J_{4,1} = O(p^n)\).

Calculating \(J_i\) for \(i = 1, \ldots, 4\) we obtain the following lemmas.

\textbf{Lemma 5.1}

i) \textit{Under } \mathcal{H}_3\textit{)
\[
J_{1,j} = O(p^n) \text{ for } j = 2, \cdots, 11,
\]
\[ J_{2,j} = O(p^n) \text{ for } j = 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, \]
\[ J_{3,j} = O(p^n) \text{ for } j = 2, 3, 4, 5, 9, 10, 11, 12, 13. \]

ii) Under \( \mathcal{H}_3 \) and \( \mathcal{H}_4 \),
\[ J_{2,8} = J_{2,15} = J_{3,15} = J_{3,8} = O(p^n/M_n). \]

Proof of lemma 5.1.

i) We give a detailed proof for the result: \( J_{1,3} = O(p^n) \), the other results follow analogously. First, by definition, we get

\[
J_{1,3} = \beta^2 \int_{U_n} \int_{U_n} \int_{U_n} \int_{U_n} c_2(t-s)c_2(u-v)V_n(t-s)V_n(u-v) \times e^{-2\pi i \lambda_1(t-s) - \lambda_2(u-v)} d\delta_u(s) d\delta_v(v) dtdudv
\]

\[ = \beta^2 c_2(0)V_n(0) \int_{U_n} \int_{U_n} c_2(t-u)V_n(t-u) e^{-2\pi i \lambda_1(t-u)} dtdudv. \]

with \( x = (t-u)/M_n \), from Fubini’s theorem and definition of \( V_n \), we get

\[
|J_{1,3}| \leq \beta^2 |c_2(0)| \mathcal{F}W(0) \frac{p^n}{M_n} \int_{Q_p} |c_2(xM_n)| \mathcal{F}W(x)dx.
\]

Let \( y = xM_n \). Then as \( \mathcal{F}W \) is bounded, \( |J_{1,3}| \leq \text{const } \beta^2 |c_2(0)| \mathcal{F}W(0)p^n \int_{Q_p} |c_2(y)| dy. \)

From \( \mathcal{H}_3 \), we get the result.

ii) We are going to prove: \( J_{2,15} = O(p^n/M_n) \).
First, we can write

\[
J_{2,15} = \beta^4 \int_{U_n} \int_{U_n} \int_{U_n} \int_{U_n} c_2(t-v)c_2(s-u)V_n(t-s)V_n(u-v) \times e^{-2\pi i \lambda_1(t-s) - \lambda_2(u-v)} dsdtdudv.
\]

Successively, let \( x = \frac{t-s}{M_n}; y = s-v + xM_n; z = \frac{u-v}{M_n}; \) and \( v = s-u. \)

By Fubini’s theorem and \( V_n \) being bounded, we get

\[
|J_{2,15}| \leq \text{const } \frac{p^n}{M_n} |c_2|_{L^1(Q_p)} |\mathcal{F}W|_{L^1(Q_p)} \int_{Q_p} |c_2(v)| dv.
\]

Thus, \( \mathcal{H}_3 \) and \( \mathcal{H}_4 \) imply: \( J_{2,15} = O\left(\frac{p^n}{M_n}\right) \).

Lemma 5.2

i) Under \( \mathcal{H}_4 \), \( J_{3,7} = J_{3,8} = O(p^n). \) ii) Under \( \mathcal{H}_5 \), \( J_{3,14} = O(p^n/M_n). \)

iii) Under \( \mathcal{H}_6 \), \( J_{4,2} = J_{4,6} = O(p^n). \) iv) Under \( \mathcal{H}_7 \), \( J_{4,3} = J_{4,4} = J_{4,5} = O(p^n). \)

v) Under \( \mathcal{H}_8 \), \( J_{4,7} = J_{4,8} = O(p^n). \) vi) Under \( \mathcal{H}_9 \), \( J_{4,9} = J_{4,10} = O(p^n). \)

vii) Under \( \mathcal{H}_{10} \), \( J_{4,11} = J_{4,13} = O(p^n). \) viii) Under \( \mathcal{H}_{11} \), \( J_{4,12} = O(p^n). \)

ix) Under \( \mathcal{H}_{12} \), \( J_{4,15} = O(p^n). \)

Proof. We only prove i) and iii); the other results are proved analogously.
i) We can write
\[
J_{3,7} = \beta^2 \int_{U_n} \int_{U_n} \int_{U_n} \int_{U_n} c_2(t-u)c_2(s-v)V_n(t-s)V_n(u-v)
\times e^{-2i\pi(\lambda_1(t-s)-\lambda_2(u-v))} d\delta_t(v)d\delta_u(u)dsdt
\]
\[
= \beta^2 \int_{U_n} \int_{U_n} c_2(t-s)^2V_n(t-s)^2 e^{-2i\pi((\lambda_1+\lambda_2)(t-s))} dt ds.
\]
From \( x = (t-s)/M_n, \ y = xM_n \), we get
\[
|J_{3,7}| \leq \beta^2 p^n \int_{Q_p} |c_2(xM_n)^2|FW(x)^2 dx M_n Q_p n~
\]
\[
\leq \beta^2 p^n |FW|_{L^\infty(Q_p)}^2 |c_2|_{L^2(Q_p)}^2,
\]
thus, \( J_{3,7} = O(p^n) \) by \( \mathcal{H}_2 \).

ii) We can write:
\[
J_{4,2} = \beta^2 \int_{U_n} \int_{U_n} \int_{U_n} \int_{U_n} c_4(t-s, u-s, v-s)V_n(t-s)V_n(u-v)
\times e^{-2i\pi(\lambda_1(t-s)-\lambda_2(u-v))} d\delta_t(v)d\delta_u(u)dt ds
\]
\[
= \beta^2 \int_{U_n} \int_{U_n} c_4(t-s, t-s, t-s)V_n(t-s)FW(0) e^{-2i\pi(\lambda_1(t-s))} dt ds,
\]
with \( t-s/M_n = x \) and \( u = xM_n \), we obtain
\[
|J_{4,2}| \leq \text{const} \beta^2 \int_{U_n} \left[ \int_{Q_p} |c_4(xM_n, xM_n, xM_n)|FW(x)dx \right] ds
\]
\[
\leq \text{const} \beta^2 p^n \int_{Q_p} |c_4(u, u, u)| du.
\]
Then, \( J_{4,2} = O(p^n) \) comes from \( \mathcal{H}_6 \).
We come back to the proof of proposition 5.2.
From lemmas 5.1 and 5.2, we get
\[
J = \sum_{i=1}^4 J_i = O(p^n) + O\left(\frac{p^n}{M_n}\right) + O\left(\frac{1}{M_n}\right) = O\left(\frac{p^n}{M_n}\right),
\]
and thus \( p^{2n}\text{cov}\left[\hat{f}_{x,n}(\lambda_1), \hat{f}_{x,n}(\lambda_2)\right] = O\left(\frac{p^n}{M_n}\right)\),
which implies, \( \text{cov}\left[\hat{f}_{x,n}(\lambda_1), \hat{f}_{x,n}(\lambda_2)\right] = O\left(\frac{1}{p^nM_n}\right)\).
So, \( \lim_{n \to +\infty} \text{cov}\left[\hat{f}_{x,n}(\lambda_1), \hat{f}_{x,n}(\lambda_2)\right] = 0 \), for every \( \lambda_1, \lambda_2 \in Q_p \).
This proves that \( \hat{f}_{x,n} \) is consistent.

Our main result in this section is the following:

**Theorem 5.1** Under the hypothesis of propositions 5.1 and 5.2, \( \hat{f}_{x,n}(\lambda) \) is asymptotically unbiased and consistent.
Proof of theorem 5.1.

a) \( \hat{f}_{X,n}(\lambda) \) is asymptotically unbiased.

Indeed, \( \mathbb{E} \left[ \hat{f}_{X,n}(\lambda) \right] = \frac{1}{\beta^2} \mathbb{E} \left[ \hat{f}_{z,n}(\lambda) \right] - \beta c_z(0) \).

Under conditions of lemma 5.1, as \( \hat{f}_{z,n} \) is asymptotically unbiased, then

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \hat{f}_{X,n}(\lambda) \right] = \frac{1}{\beta^2} \mathbb{E} \left[ \hat{f}_{z}(\lambda) - \beta c_z(0) \right] = f_X(\lambda).
\]

b) \( \hat{f}_{X,n}(\lambda) \) is consistent.

Let \( \lambda_1 \) and \( \lambda_2 \) be elements of \( Q_p \). We have

\[
\text{cov} \left[ \hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2) \right] = \frac{1}{\beta^4} \text{cov} \left[ \hat{f}_{z,n}(\lambda_1), \hat{f}_{z,n}(\lambda_2) \right] - \frac{1}{\beta^3} \text{cov} \left[ \hat{f}_{z,n}(\lambda_1), \hat{c}_{z,n}(0) \right] \\
- \frac{1}{\beta^3} \text{cov} \left[ \hat{f}_{z,n}(\lambda_2), \hat{c}_{z,n}(0) \right] + \frac{1}{\beta^2} \text{var} \left[ \hat{c}_{z,n}(0) \right].
\]

For \( i = 1, 2 \), we can write

\[
|\text{cov} \left[ \hat{f}_{z,n}(\lambda_i), \hat{c}_{z,n}(0) \right] | \leq \sqrt{\text{var} \left[ \hat{f}_{z,n}(\lambda_i) \right] \text{var} \left[ \hat{c}_{z,n}(0) \right]}.
\]

The two propositions 5.1, 5.2 imply:

\[
\text{cov} \left[ \hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2) \right] = O(\frac{1}{p^n}) + O\left( \frac{1}{p^n M_n} \right) + O\left( \frac{1}{p^n \sqrt{M_n}} \right)
\]

The theorem is then proved.

Corollary 5.1 Under \( H_3) - H_{12} \), \( \hat{f}_{X,n}(\lambda) \) converges to \( f_X(\lambda) \) in quadratic mean as \( n \to +\infty \).

Proof of corollary 5.1. We know that \( MSE(n, \lambda) = \text{bias}^2(\hat{f}_{X,n}(\lambda)) + \text{var}(\hat{f}_{X,n}(\lambda)) \).

Then, from theorem 5.1, we obtain \( MSE(n) \to 0 \) as \( n \to +\infty \). This implies mean quadratic convergence.

6 Discussion and extensions

This paper has been concerned with the case of real-valued process. Extensions to the complex-valued and \( r \)-vector valued cases are immediate. We think that, it will be very important to treat the case of \( p \)-adic valued process, and afterward, observe the almost sure convergence and give the asymptotic distributions of the estimates.

As the convergence rate of the estimators depends on the sequence \( (M_n)_{n \in \mathbb{N}} \), we think that the choice of this sequence is crucial, and methods like Cross-Validation procedure's (cf. [11] and [5]), will solve this problem.

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References


