ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 27, nº 1 (2020), p. 1-17.

<http://ambp.centre-mersenne.org/item?id=AMBP_2020__27_1_1_0>

Publication éditée par le laboratoire de mathématiques Blaise Pascal de l'université Clermont Auvergne, UMR 6620 du CNRS Clermont-Ferrand — France

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Annales mathématiques Blaise Pascal **27**, 1-17 (2020)

Largeness and equational probability in groups

Khaled Jaber
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Abstract

We define k-genericity and k-largeness for a subset of a group, and determine the value of k for which a k-large subset of G^n is already the whole of G^n, for various equationally defined subsets. We link this with the inner measure of the set of solutions of an equation in a group, leading to new results and/or proofs in equational probabilistic group theory.

1. Introduction

In probabilistic group theory we are interested in what proportion of (tuples of) elements of a group have a particular property; if this property is given by an equation, we talk about *equational probability*. In [9] a notion of *largeness* was introduced for a subset of a group, and it was shown that certain equational properties of a group hold everywhere as soon as they hold largely. In this paper, we shall introduce a quantitative version of largeness, and deduce some results in equational probabilistic group theory.

Throughout this paper, G will be a group and μ a left-invariant probability measure on some algebra of subsets of G.

Example 1.1.

(1) G finite, μ the counting measure.

(2) G_1 a group, μ_1 a left-invariant measure on G_1, and $G = G_1^n$ with the product measure $\mu = \mu_1^n$.

(3) More generally, G_1 a group, $G \le G_1^n$ and μ a left-invariant measure on G.

(4) G arbitrary and the measure algebra reduced to $\{\emptyset, G\}$. While this set-up trivialises the probability statements, the largeness results remain meaningful.

If X is a measurable subset of G we can interpret $\mu(X)$ as the probability that a random element of G lies in X. If H is another group, $f : G \to H$ is a function and $c \in H$ some constant, we put $\mu(f(x) = c) = \mu(\{g \in G : f(g) = c\})$.

The second author was partially supported by the ANR-DFG project AAPG2019 GeoMod.
Keywords: probabilistic group theory, largeness.
2020 *Mathematics Subject Classification*: 20A15, 03C60, 20P99.

Example 1.2. Let G_1 be a group, $G \leq G_1^n$ a subgroup, $\bar{g} \in G_1^m$ constants, and $w(\bar{x}, \bar{y})$ a word in $\bar{x}\bar{y}$ and their inverses, with $|\bar{x}| = n$ and $|\bar{y}| = m$. Then $w(\bar{x}, \bar{g})$ induces a function from G to G_1.

We shall now list some known results, starting with Frobenius in 1895.

Fact 1.3. *Let G be a finite group.*

- Frobenius 1895 [5] *If n divides* $|G|$ *then the number of solutions of* $x^n = 1$ *is a multiple of n. In particular,* $\mu(x^n = 1) \geq \frac{n}{|G|}$.
- Miller 1907 [14] *If G is non-abelian, then* $\mu(x^2 = 1) \leq \frac{3}{4}$.
- Laffey 1976 [11] *If G is a* 3*-group not of exponent* 3 *then* $\mu(x^3 = 1) \leq \frac{7}{9}$.
- Laffey 1976 [12] *If p is prime and divides* $|G|$*, but G is not a p-group, then* $\mu(x^p = 1) \leq \frac{p}{p+1}$.
- Laffey 1979 [13] *If G is not a* 2*-group, then* $\mu(x^4 = 1) \leq \frac{8}{9}$.
- Iiyori, Yamaki 1991 [8] *If n divides* $|G|$ *and* $X = \{g \in G : g^n = 1\}$ *has cardinality n, then X forms a subgroup of G.*
- Erdős, Turan, 1968 [3] *If* $k(G)$ *is the number of conjugacy classes in G, then* $\mu([x, y] = 1) = \frac{k(G)}{|G|}$.
- Joseph 1977 [10], Gustafson 1973 [6] *If G is non-abelian, then* $\mu([x, y] = 1) \leq \frac{5}{8}$.
- Neumann, 1989 [16] *For any real* $r > 0$ *there are* $n_1(r)$ *and* $n_2(r)$ *such that if* $\mu([x, y] = 1) \geq r$ *then G contains normal subgroups* $H \leq K$ *such that* K/H *is abelian,* $|G : K| \leq n_1(r)$ *and* $|H| \leq n_2(r)$.
- Barry, MacHale, Ní Shé, 2006 [1] *If* $\mu([x, y] = 1) > \frac{1}{3}$ *then G is supersoluble.*
- Heffernan, MacHale, Ní Shé, 2014 [7] *If* $\mu([x, y] = 1) > \frac{7}{24}$ *then G is metabelian. If* $\mu([x, y] = 1) > \frac{83}{675}$ *then G is abelian-by-nilpotent.*

In Section 2 we shall introduce largeness and prove the main connection between largeness and measure, Lemma 2.5, which will be used throughout the rest of the paper. Section 3 deals with central elements, or more generally FC and BFC groups. We shall treat equations of the form $x^n = c$ for arbitrary c in Section 4, recovering Miller's result for $n = 2$, and a weaker bound than Laffey for $n = 3$ (namely $\frac{6}{7}$). In Section 5 we shall consider commutator equations; while our methods allow us to deal with more complicated commutators, they are too general to obtain the bounds from Fact 1.3. Section 6 deals with nilpotent groups via linearisation, and the short Section 7 places Sherman's autocommutativity degree in our context.

Notation. We shall write $x^y = y^{-1}xy$, $x^{-y} = (x^{-1})^y = y^{-1}x^{-1}y$ and $[x, y] = x^{-1}y^{-1}xy = y^{-x}y = x^{-1}x^y$.

2. **Largeness and Probability**

The following notion of largeness was introduced in [9].

Definition 2.1. If $X \subseteq G$, we say that X is *k-large* in G if the intersection of any k left translates of X is non-empty, and X is *k-generic* in G if k left translates of X cover G. A subset X is *large* if it is k-large for all k; it is *generic* if it is k-generic for some k.

Of course, analogous notions exist for right and two-sided genericity/largeness. Both genericity and largeness are notions of prominence, increasing with k for largeness and decreasing with k for genericity. Clearly, if $X \subseteq G$ and X is (k-)large/generic, so is any left or right translate or superset of X. Largeness and genericity are co-complementary:

Lemma 2.2. *Let $X \subseteq G$. Then X is 1-large if and only if $X \neq \emptyset$, and X is 1-generic if and only if $X = G$. More generally, X is k-large if and only if $G \setminus X$ is not k-generic. Finally, X is k-generic/large if and only if $X \cap Y \neq \emptyset$ for all k-large/generic $Y \subseteq G$.*

Proof. We only show the last assertion. If X is not k-generic/large, then $Y := G \setminus X$ is k-large/generic, and $X \cap Y = \emptyset$. Conversely, if X is k-generic, say $G = \bigcup_{i<k} g_i X$, and Y is k-large, then

$$\begin{aligned}\emptyset \neq \bigcap_{i<k} g_i Y = G \cap \bigcap_{i<k} g_i Y &= \bigcup_{i<k} g_i X \cap \bigcap_{i<k} g_i Y \\ &= \bigcup_{i<k} \left(g_i X \cap \bigcap_{i<k} g_i Y \right) \subseteq \bigcup_{i<k} (g_i X \cap g_i Y) = \bigcup_{i<k} g_i (X \cap Y).\end{aligned}$$

Thus $X \cap Y \neq \emptyset$. □

Remark 2.3. If $\phi : G \to H$ is an epimorphism and $X \subseteq G$ is (k-)large/generic, so is $\phi(X) \subseteq H$. Conversely, if $Y \subseteq H$ is (k-)large/generic in H, so is $\phi^{-1}[X]$ in G.

In particular, if $X \subseteq G \times H$ is (k-)large/generic, so are the projections to each coordinate. Conversely, if $X \subseteq G$ and $Y \subseteq H$ are (k-)large, so is $X \times Y \subseteq G \times H$; if X is k-generic and Y is ℓ-generic, $X \times Y$ is $k\ell$-generic.

Lemma 2.4. *Suppose X is $k\ell$-large in G and $H \leq G$ is a subgroup of index k. Then $X \cap H$ is ℓ-large in H.*

Proof. Let $(g_i : i < k)$ be coset representatives of H in G, and consider $(h_j : j < \ell)$ in H. By $k\ell$-largeness of X in G there is $x \in \bigcap_{i<k,\ j<\ell} g_i h_j X$. As $\bigcup_{i<k} g_i H = G$, there is $i_0 < k$ with $x \in g_{i_0} H$. But then

$$g_{i_0}^{-1} x \in H \cap \bigcap_{i<k,\ j<\ell} g_{i_0}^{-1} g_i h_j X \subseteq H \cap \bigcap_{j<\ell} h_j X = \bigcap_{j<\ell} h_j (X \cap H),$$

so $X \cap H$ is ℓ-large. □

The link between largeness and probability is given by the following lemma, which will be used throughout the paper. Recall that the *inner measure* of an arbitrary subset X of a measurable group G is

$$\mu_*(X) = \sup\{\mu(Y) : Y \subseteq X \text{ measurable}\},$$

and the *outer measure* is given by

$$\mu^*(X) = \inf\{\mu(Y) : Y \supseteq X \text{ measurable}\}.$$

Clearly the inner measure is superadditive, the outer measure is subadditive, and $\mu_*(X) + \mu^*(G \setminus X) = 1$.

Lemma 2.5. *If X is k-generic in G, then $\mu^*(X) \geq \frac{1}{k}$. If $\mu_*(X) > 1 - \frac{1}{k}$ then X is k-large in G.*

Proof. If X is k-generic there are $g_1, \ldots, g_k$ in G with $G = \bigcup_{i \leq k} g_i X$. Hence

$$1 = \mu^*(G) = \mu^*\left(\bigcup_{i \leq k} g_i X\right) \leq \sum_{i \leq k} \mu^*(g_i X) = k\, \mu^*(X)$$

by left invariance, whence $\mu^*(X) \geq \frac{1}{k}$.

Now if X is not k-large, its complement is k-generic, so $\mu^*(G \setminus X) \geq \frac{1}{k}$. But then $\mu_*(X) \leq 1 - \frac{1}{k}$. □

These bounds are strict, as we can take X a subgroup of index k (resp. its complement).

Remark 2.6. For any group G the set $(G \times \{1\}) \cup (\{1\} \times G)$ is 2-large in G^2; if G is infinite it is of measure 0.

We shall now prove some results about finite groups, which owing to their non-linearity do not generalise easily to the measurable context.

Remark 2.7. Let G be a finite group of order n, and $X \subseteq G$ a non-empty proper subset of size m. Then X is $(n-m+1)$-generic and at most m-large, since we can form the union of X with $n-m$ translates of X to cover all the $n-m$ points of $G \setminus X$, and we can intersect X with m translates of X to remove all m points of X.

Theorem 2.8. *Let G be a finite group of order n, and $X \subseteq G$ a non-empty proper subset of size m. If $m > n - \frac{1}{2} - \sqrt{n - \frac{3}{4}}$, then X is 2-generic. Hence if $m < \frac{1}{2} + \sqrt{n - \frac{3}{4}}$ then X is not 2-large.*

Proof. If $m > n - \frac{1}{2} - \sqrt{n - \frac{3}{4}}$, then

$$n - \frac{3}{4} > (n - m - \frac{1}{2})^2 = (n-m)(n-m-1) + \frac{1}{4}.$$

Put $Z = \{xy^{-1} : x, y \in G \setminus X\}$. Then

$$|Z| \leq (n-m)(n-m-1) + 1 < n,$$

so there is $g \in G \setminus Z$. But if $h \in G \setminus (X \cup gX)$, then $h, g^{-1}h \in G \setminus X$, and $g = h(g^{-1}h)^{-1} \in Z$, a contradiction. Thus $G = X \cup gX$ and X is 2-generic.

The second assertion follows by taking complements. $\square$

Theorem 2.9. *Let G be a finite group of order n. If the exponent of G does not divide ℓ then $\mu(x^\ell = 1) \leq 1 - \frac{1}{\sqrt{2n}}$, where μ is the counting measure.*

Proof. Put $X = \{g \in G : g^\ell = 1\}$, of size $m < n$, and take any $g \in G \setminus X$. Note that $X \cap gX \cap C_G(g)$ is empty, as otherwise there would be $y \in C_G(g)$ with $y^\ell = 1 = (gy)^\ell$, whence $g^\ell = 1$ and $g \in X$.

Thus $|C_G(g)| \leq 2\,|G \setminus X|$. Moreover $g^G \cap X = \emptyset$, and

$$|G|/|C_G(g)| = |g^G| \leq |G \setminus X|.$$

Thus $n = |G| \leq 2\,|G \setminus X|^2$ and $\sqrt{\frac{n}{2}} \leq n - m$, whence

$$\mu(x^\ell = 1) = \frac{m}{n} \leq \frac{n - \sqrt{\frac{n}{2}}}{n} = 1 - \frac{1}{\sqrt{2n}}. \qquad \square$$

Definition 2.10. Let $f : G \to H$ be a function, and $c \in H$. The equation $f(x) = c$ is *k-largely satisfied* in G if $\{g \in G : f(g) = c\}$ is k-large in G. By abuse of notation, if $G = G_1^n$ and $x = (x_1, \ldots, x_n)$, we shall also say that $f(x_1, \ldots, x_n) = c$ is k-largely satisfied in G_1.

3. **FC-Groups**

In this section we shall work in the set-up of Example 1.2: G_1 will be a group, $G \leq G_1^n$, $w(\bar{x}, \bar{y})$ a word in $\bar{x}\bar{y}$ and their inverses with $n = |\bar{x}|$ and $m = |\bar{y}|$, $\bar{g} \in G_1^m$ and $c \in G_1$ constants, and $f(\bar{x}) = w(\bar{x}, \bar{g})$.

Recall that a group is FC if the centraliser of any element has finite index; it is BFC if the index is bounded independently of the element.

We shall first need a preparatory lemma. For two tuples $\bar{g} = (g_i : i < k)$ and $\bar{g}' = (g_i' : i < k)$ in G_1^k we shall put $\bar{g}^{-1} = (g_i^{-1} : i < k)$ and $\bar{g} \cdot \bar{g}' = (g_i g_i' : i < k)$.

Lemma 3.1. *Suppose $\bar{g}, \bar{g}' \in G_1^m$ and $\bar{h}, \bar{h}' \in G_1^n$ are such that all elements from $\bar{g}\bar{h}$ commute with all elements from $\bar{g}'\bar{h}'$. Then*

$$w(\bar{h} \cdot \bar{h}', \bar{g} \cdot \bar{g}') = w(\bar{h}, \bar{g})\, w(\bar{h}', \bar{g}').$$

Proof. Obvious. □

Theorem 3.2. *Let G_1 be an FC-group. If the equation $w(\bar{x}, \bar{g}) = c$ is largely satisfied in G then it is identically satisfied in G.*

Proof. Consider $\bar{h} \in G$, and $C = C_{G_1}(\bar{g}, \bar{h})$, a subgroup of finite index in G_1. Put $H = C^n \cap G$, a subgroup of finite index in G, and $X = \{\bar{h}' \in G : w(\bar{h}', \bar{g}) = c\}$. Then $X \cap \bar{h}^{-1} X \cap H$ is large in H, whence non-empty. So there is $\bar{x} \in H$ with

$$w(\bar{1}, \bar{g})\, w(\bar{x}, \bar{1}) = w(\bar{x}, \bar{g}) = c = w(\bar{h} \cdot \bar{x}, \bar{g}) = w(\bar{h}, \bar{g})\, w(\bar{x}, \bar{1}).$$

Hence $w(\bar{h}, \bar{g}) = w(\bar{1}, \bar{g})$ for all $\bar{h} \in G$, and $w(\bar{1}, \bar{g}) = w(\bar{x}, \bar{g}) = c$. □

For a BFC-group, we can bound the degree of largeness needed:

Theorem 3.3. *Suppose every centraliser of a single element has index at most k in G_1. If the equation $w(\bar{x}, \bar{g}) = c$ is $2k^{n^2+mn}$-largely satisfied in G then it is identically satisfied in G.*

Proof. In the notation of the previous proof, $C = C_{G_1}(\bar{g}, \bar{h})$ has index at most k^{n+m} in G_1, so

$$|G : H| = |G : G \cap C^n| \leq |G_1^n : C^n| = |G_1 : C|^n \leq (k^{n+m})^n = k^{n^2+mn}.$$

Now $2k^{n^2+mn}$-largeness of X in G implies k^{n^2+mn}-largeness of $X \cap \bar{h}^{-1} X$ in G, whence 1-largeness of $X \cap \bar{h}^{-1} X \cap H$ in H. So we can find the $\bar{x}$ required to finish the proof. □

Corollary 3.4. *Suppose every centraliser of a single element has index at most k in G_1. If $w(\bar{x}, \bar{g}) = c$ is not an identity on G, then*

$$\mu_*(w(\bar{x}, \bar{g}) = c) \leq 1 - \frac{1}{2k^{n^2+mn}}.$$

Proof. If $\mu_*(w(\bar{x},\bar{g}) = c) > 1 - \frac{1}{2k^{n^2+mn}}$, then $\{\bar{x} \in G : w(\bar{x},\bar{g}) = c\}$ is $2k^{n^2+mn}$-large in G by Lemma 2.5, and $w(\bar{x},\bar{g}) = c$ is identically satisfied in G by Theorem 3.3. □

Remark 3.5. This holds in particular for the equation $x^\ell = c$, with $n = 1$ and $m = 0$.

If the group is central-by-finite, the largeness needed does not depend on the number of parameters.

Corollary 3.6. *Suppose $Z(G_1)$ has index k in G_1. If the equation $w(\bar{x},\bar{g}) = c$ is $2k^n$-largely satisfied in G then it is identically satisfied in G.*

Proof. $H = G \cap Z(G_1)^n$ has index at most k^n in G. We finish as in Theorem 3.3. □

Corollary 3.7. *If $|G_1 : Z(G_1)| \leq k$ and $w(\bar{x},\bar{g}) = c$ is not an identity in G, then $\mu_*(w(\bar{x},\bar{g}) = 1) \leq 1 - \frac{1}{2k^n}$.*

Of course, for an abelian group G_1 we have $k = 1$ in the above results.

Remark 3.8. If $w(\bar{x},\bar{g}) = c$ is 2-largely satisfied in G^n, then it is identically satisfied in the abelian quotient G/G'. If moreover G is a BFC-group, then G' is finite by B.H. Neumann's Lemma [15], and G^n satisfies a finite disjunction $\bigvee_{c' \in G'} w(\bar{x},\bar{g}) = cc'$.

We can also deduce results for central elements just from 2-largeness (although for infinite index $|G_1 : Z(G_1)|$ there is no reason that if X is large in G the intersection $X \cap Z(G_1)^n$ is still large in $G \cap Z(G_1)^n$).

Theorem 3.9. *If $w(\bar{x},\bar{g}) = c$ is 2-largely satisfied in G, then $w(\bar{x},\bar{1}) = 1$ identically on $G \cap Z(G_1)^n$.*

Proof. Consider $\bar{h} \in G \cap Z(G_1)^n$. Put $X = \{\bar{h}' \in G : w(\bar{h}',\bar{g}) = 1\}$. Then $X \cap \bar{h}^{-1}X$ is non-empty, so there is $\bar{x} \in G$ with

$$w(\bar{x},\bar{g}) = c = w(\bar{h} \cdot \bar{x},\bar{g}) = w(\bar{h},\bar{1})\, w(\bar{x},\bar{g}).$$

Hence $w(\bar{h},\bar{1}) = 1$. □

Corollary 3.10. *If $x_1^{k_1} \dots x_n^{k_n} = c$ is 2-largely satisfied in G^n and $k = \gcd(k_1,\dots,k_n)$, then $x^k = 1$ identically on $Z(G)$.*

Proof. We have $x_1^{k_1} \dots x_n^{k_n} = 1$ on $Z(G)$. Putting $x_i = g \in Z(G)$ and $x_j = 1$ for $j \neq i$ we have $g^{k_i} = 1$ for all $1 \leq i \leq n$. The result follows. □

Corollary 3.11. *If the exponent of $Z(G)$ does not divide $\gcd(k_1,\dots,k_n)$, then*

$$\mu_*(x_1^{k_1} \dots x_n^{k_n} = c) \leq \frac{1}{2}.$$

4. **Burnside and Engel Equations**

In Remark 3.5 we have already seen that if every centraliser of a single element has index at most k in G, then $\mu_*(x^m = c) \leq 1 - \frac{1}{2k}$ unless the exponent of G divides m. In this case necessarily $c = x^m = 1$.

We shall first prove Miller's Theorem mentioned in the introduction.

Theorem 4.1. *Let $c \in G$. If $x^2 = c$ is 4-largely satisfied in G, then G is abelian of exponent 2, and $c = 1$.*

Proof. Fix $g, h \in G$. Then there is x with $c = x^2 = (gx)^2 = (hx)^2 = (ghx)^2$. But this implies $x^{-1}gx = g^{-1}$, $x^{-1}hx = h^{-1}$ and $x^{-1}ghx = (gh)^{-1}$. On the other hand,

$$x^{-1}ghx = x^{-1}gx\, x^{-1}hx = g^{-1}h^{-1} = (hg)^{-1}.$$

Hence $gh = hg$ and G is abelian. But now $c = x^2 = (gx)^2 = g^2x^2 = g^2c$, whence $g^2 = 1$. □

If G satisfies 4-largely $xax = b$ for some $a, b \in G$, then it satisfies 4-largely $(ax)^2 = ab$, whence $x^2 = ab$. Hence G is abelian of exponent 2, and $a = b$.

Corollary 4.2. *If G is not of exponent 2 or $a \neq b$, then $\mu_*(xax = b) \leq \frac{3}{4}$.*

Recall that the n^{th} *Engel condition* is the condition $[x,_n y] = 1$, where $[x,_1 y] = [x, y]$ and $[x,_{n+1} y] = [[x,_n y], y]$. Note that

$$[x, y, y] = [y^{-x}y, y] = y^{-1}y^xy^{-1}y^{-x}yy = [y^{-x}, y]^y.$$

Thus the 2-Engel condition $[x, y, y] = 1$ is equivalent to $[y^{-x}, y] = 1$, that is all conjugacy classes being commutative.

Theorem 4.3. *If G satisfies 7-largely $x^3 = 1$ then G is 2-Engel.*

Proof. Put $X = \{g \in G : g^3 = 1\}$. For $g, h \in G$ consider

$$x \in X \cap g^{-1}X \cap h^{-1}X \cap gX \cap (gh)^{-1}X \cap gh^{-1}X \cap gh^{-1}g^{-1}X.$$

Then $(yx)^3 = 1$ for $y \in \{1, g, h, g^{-1}, gh, hg^{-1}, ghg^{-1}\}$, which means that $xyx = y^{-1}x^{-1}y^{-1}$. We calculate the product xhx^2gx in two ways:

$$\begin{aligned} xhx^2gx &= (xhx)(xgx) = h^{-1}(x^{-1}h^{-1}g^{-1}x^{-1})g^{-1} \\ &= h^{-1}ghxghg^{-1} \end{aligned}$$

and

$$\begin{aligned} xhx^2gx &= xh(g^{-1}x)^{-1}x = xh(g^{-1}x)^2x = (xhg^{-1}x)g^{-1}x^2 \\ &= gh^{-1}(x^{-1}gh^{-1}g^{-1}x^{-1}) = gh^{-1}ghg^{-1}xghg^{-1}. \end{aligned}$$

Thus $h^{-1}gh = gh^{-1}ghg^{-1}$ and $g^h g = gg^h$. As $h \in G$ was arbitrary, the conjugacy class of g is commutative; as g was arbitrary, all conjugacy classes are commutative. □

Theorem 4.4. *Let G be 2-Engel. If G satisfies 2-largely $x^3 = 1$ then G has exponent 3.*

Proof. For any $g \in G$ there is $x \in G$ with $x^3 = (gx)^3 = 1$. As x^G is commutative,

$$g^x g^{-1} g^{x^{-1}} = x^{-1}gxg^{-1}xgx^{-1} = gx^{-g}xx^g x^{-1} = gx^{-g}x^g xx^{-1} = g.$$

Since g^G is commutative, we have

$$g^3 = g^2 g^x g^{-1} g^{x^{-1}} = g^2 g^{-1} g^{x^{-1}} g^x = (gx)^3 = 1.$$ □

Corollary 4.5. *If G satisfies 7-largely $x^3 = 1$, then G has exponent 3. If G is not of exponent 3 then $\mu_*(x^3 = 1) \leq \frac{6}{7}$. If moreover G is 2-Engel, then $\mu_*(x^3 = 1) \leq \frac{1}{2}$.*

Note that the bound $\frac{6}{7}$ is not as good as Laffey's bound $\frac{7}{9}$ cited in the introduction.

Problem 4.6. A group which satisfies 5-largely $x^3 = 1$, is it 2-Engel? This would improve our bound to $\frac{4}{5}$.

Corollary 4.7. *If $|G : Z(G)| \leq 7$ and G satisfies 7-largely $x^3 = c$ for some $c \in G$, then $c = 1$ and G has exponent 3.*

Proof. $\{x \in G : x^3 = c\} \cap Z(G)$ is 1-large, whence non-empty, and contains an element z. But now there is $x \in G$ with $x^3 = 1 = (zx)^3 = z^3x^3 = cx^3$, whence $c = 1$. We finish by Corollary 4.5. □

If $|G : Z(G)|$ is prime, then G is abelian, and 2-largeness is sufficient by Corollary 3.10.

5. Commutator Equations

Consider the equation $[x, g] = c$ for some $c, g \in G$. Since $\{x \in G : [x, g] = c\}$ is a coset of $C_G(g)$ or empty, and a coset of a proper subgroup cannot be 2-large, it follows that if G satisfies 2-largely $[x, g] = c$ then $g \in Z(G)$ and $c = 1$. The following argument generalises this result.

Theorem 5.1. *Suppose $f : G \to H$ satisfies $f(xx') = f(x)^h f(x')$ for some $h \in H$ which depends on $x, x' \in G$. If G_0 and G_1 are groups, $f_0 : G_0 \to H$ and $f_1 : G_1 \to H$ are functions such that $G_0 \times G \times G_1$ satisfies k-largely $f_0(x_0)\, f(x)\, f_1(x_1) = c$ for some $k \geq 2$, then $f(G) = 1$ and $G_0 \times G_1$ satisfies k-largely $f_0(x_0)\, f_1(x_1) = c$.*

Proof. Fix $g \in G$. By 2-largeness there is $(x_0, x, x_1) \in G_0 \times G \times G_1$ such that

$$f_0(x_0)\, f(x)\, f(x_1) = c = f_0(x_0)\, f(gx)\, f(x_1).$$

Thus $f(x) = f(gx) = f(g)^h f(x)$ and $f(g) = 1$. It follows that $f_0(x_0) f(x) f_1(x_1) = f_0(x_0) f_1(x_1)$ on $G_0 \times G \times G_1$. The result follows. □

Corollary 5.2. *If G satisfies 2-largely $\prod_{i<n}[x_i, g_i] = c$ for some $g_i \in G$, then $g_i \in Z(G)$ for all $i < n$ and $c = 1$. If not all g_i are central or $c \neq 1$ then $\mu_*(\prod_{i<n}[x_i, g_i] = c) \leq \frac{1}{2}$.*

Proof. We have $[xx', y] = [x, y]^{x'}[x', y]$. Now use Theorem 5.1. □

Remark 5.3. Theorem 5.1 also holds if $f(xx') = f(x')f(x)^h$, with almost the same proof. Hence Corollary 5.2 also holds if some factors are of the form $[g_i, x_i]$.

Gustafson [6] has shown that $\mu_2([x, y] = 1) \leq \frac{1}{2}(1 + \mu(Z(G)) \leq \frac{5}{8}$ for a non-abelian compact topological group G, where μ is the Haar measure on G and μ_2 the product measure on G^2. Pournaki and Sobhani [17] have generalised this to calculate that $\mu([x, y] = g) < \frac{1}{2}$ for any $g \neq 1$ in a finite group, using Rusin's classification [18] of all finite groups with $\mu([x, y] = 1) > \frac{11}{32}$ (see also [4]). We have only been able to establish results using 4-largeness, giving the bound of $\frac{3}{4}$ in Corollary 5.7, so the following two problems remain open:

Problem 5.4.

(1) If G satisfies 2-largely $[x, y] = 1$, is $G' = C_2$ and $G/Z(G)$ of exponent 2, or $G' = C_3$ and $G/Z(G) = S_3$?

(2) If G satisfies 2-largely $[x, y] = c$ for some $c \in G$, is $c = 1$?

Theorem 5.5. *If $w(\bar{x}, \bar{g})[x, y] = c$ is satisfied 4-largely in G^{n+1}, where $x \in \bar{x}$ and $y \notin \bar{x}$, then G is abelian and $w(\bar{x}, \bar{g}) = c$.*

Proof. For any $h \in G$ the set

$$\{(\bar{x}, x, y) : w(\bar{x}, \bar{g})[x, y] = c = w(\bar{x}, \bar{g})[x, hy]\}$$

is 2-large in G^{n+1}. Hence $\{(x, y) \in G^2 : [x, y] = [x, hy]\}$ is 2-large in G^2. Now $[x, hy] = [x, y][x, h]^y$, so $[x, h] = 1$ is satisfied 2-largely in G, whence $h \in Z(G)$. It follows that G is abelian. But then $w(\bar{x}, \bar{g}) = c$ is satisfied 4-largely in G^n, and must be an identity in G by commutativity and Corollary 3.6. □

Corollary 5.6. *If G is a group with $\mu_*(w(\bar{x}, \bar{g})[x, y] = c) > \frac{3}{4}$, then G is abelian satisfying $w(\bar{x}, \bar{g}) = c$.*

Corollary 5.7. *If G satisfies 4-largely $[x, y] = c$, then G is abelian and $c = 1$. If G is not abelian or $c \neq 1$, then $\mu_*([x, y] = c) \leq \frac{3}{4}$.*

Remark 5.8. The same holds for the equation $xcy = yc'x$ with $c \neq c'$: putting $x' = xc$ and $y' = yc'$, this is equivalent to $[x', y'] = c^{-1}c'$.

Theorem 5.9. *Let $g, h \in G$ and $k = \min\{|G : C_G(g)|, |G : C_G(h)|\}$. If G satisfies k-largely $[g, h^x] = 1$, then g^G and h^G commute.*

Proof. If $k = |G : C_G(h)|$, then $\{x \in G : [g, h^x] = 1\} \cap C_G(h)$ is 1-large, whence non-empty, and $[g, h] = 1$. Now note that for any $a \in G$ also $|G : C_G(h^a)| = k$ and $[g, h^{ax}] = 1$ is satisfied k-largely, whence $[g, h^a] = 1$ and $[g, h^G] = 1$.

If $k = |G : C_G(g)|$, then $\{x \in G : [g^{x^{-1}}, h] = 1\} \cap C_G(g)$ is 1-large (still on the left) and non-empty, whence $[g, h] = 1$ and we finish as above. □

Corollary 5.10. *If $[g^G, h^G]$ is non-trivial for some $g, h \in G$, then $\mu_*([g, h^x] = 1) \leq 1 - \frac{1}{k}$, where $k = \min\{|G : C_G(g)|, |G : C_G(h)|\}$.*

Theorem 5.11. *If $g, h, c \in G$ and $[x, g, h] = c$ is $2k$-largely satisfied, where $k = |G : C_G(h)|$, then $[G, g, h] = 1$. Similarly, if $[g, x, h] = c$ is $2k$-largely satisfied for some $c \in Z(G)$, then $[g, G, h] = 1$.*

Proof. Choose $a \in G$. Then the set $X = \{x \in G : [x, g, h] = c = [ax, g, h]\}$ is k-large, and for $x \in X$ we have

$$[x, g, h] = c = [ax, g, h] = [[a, g]^x[x, g], h] = [[a, g]^x, h]^{[x,g]}[x, g, h],$$

whence $[[a, g]^x, h] = 1$. By Theorem 5.9 we have $[a, g, h] = 1$.

If $[g, x, h] = c$ is $2k$-largely satisfied with $c \in Z(G)$, then for $a \in G$ we obtain a k-large $X \subseteq G$ such that for $x \in X$ we have

$$[g, x, h] = c = [g, ax, h] = [[g, x][g, a]^x, h] = [g, x, h]^{[g,a]^x}[[g, a]^x, h],$$

whence $[[g, a]^x, h] = 1$, and $[g, a, h] = 1$ by Theorem 5.9. □

Corollary 5.12. *If $g, h \in G$ and $k = |G : C_G(h)|$, then $[G, g, h] \neq 1$ implies $\mu_*([x, g, h] = c) \leq 1 - \frac{1}{2k}$ for any $c \in G$, and $[g, G, h] \neq c$ implies $\mu_*([g, x, h] = c) \leq 1 - \frac{1}{2k}$ for any $c \in Z(G)$.*

We shall now generalise Corollary 5.7 to higher nilpotency classes. However, the proof requires an additional assumption.

Theorem 5.13. *Suppose $s < \omega$ is such that for all $i < k$ there is a set A_i of size at most s such that $Z(G/Z_i(G)) = C_{G/Z_i(G)}(A_i)$. If G satisfies $2(s + 1)^k$-largely $[x_0, x_1, \ldots, x_k] = c$, then $c = 1$ and G is nilpotent of class at most k.*

Proof. We use induction on k. For $k = 1$ note that $s \geq 1$ (otherwise G is abelian and we are done), so the result follows from Corollary 5.7.

Now suppose the assertion is true for k, and

$$X = \{\bar{x} \in G^{k+2} : [x_0, x_1, \dots, x_{k+1}] = c\}$$

is $2(s+1)^{k+1}$-large in G^{k+2}. If $A_0 = \{a_i : i < s\}$ consider the projection Y of $X \cap \bigcap_{i<s}(1, \dots, 1, a_i^{-1})X$ to the first $k+1$ coordinates, and note that it is $2(s+1)^k$-large. Then for all $(x_0, \dots, x_k) \in Y$ there is $y \in G$ such that

$$[x_0, \dots, x_k, y] = c = [x_0, \dots, x_k, a_i y] = [x_0, \dots, x_k, y]\,[x_0, \dots, x_k, a_i]^y$$

for all $i < s$, whence $[x_0, \dots, x_k] \in Z(G)$. By inductive assumption $G/Z(G)$ is nilpotent of class at most k, and we are done. □

Corollary 5.14. *Let s be as above. If G is not nilpotent of class at most k or $c \neq 1$, then* $\mu_*([x_0, x_1, \dots, x_k] = c) \leq 1 - \frac{1}{2}(s+1)^{-k}$.

Remark 5.15. Recall that an $\mathfrak{M}$c-group is a group G such that for every subset A there is a finite subset $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$. Equivalently, G satisfies the ascending (or the descending) chain condition on centralisers. Roger Bryant [2] has shown that in an $\mathfrak{M}$c-group, for every iterated centre $Z_i(G)$ there is a finite set A_i such that $Z(G/Z_i(G)) = C_{G/Z_i(G)}(A_i)$. So in an $\mathfrak{M}$c-group we can find some s as needed for Theorem 5.13 and Corollary 5.14.

Problem 5.16. To what extent do we need the $\mathfrak{M}$c-condition (or similar) in Theorem 5.13 and Corollary 5.13? It is not needed for nilpotency class 1 (Corollary 5.7). In general, assuming just 2^{k+1}-largeness of $[x_0, \dots, x_k] = c$, we obtain that $\{\bar{x} \in G^k : [x_0, \dots, x_{k-1}] \in C_G(g)\}$ is 2^k-large in G^k for any $g \in G$. Does this imply $\gamma_k(G) \leq C_G(g)$, or even $\gamma_k(G) \leq Z(G)$?

6. Nilpotent groups

We shall first introduce the notion of a supercommutator from [9].

Definition 6.1. Any variable and any constant from G is a *supercommutator*; if v and w are supercommutators, then v^{-1} and $[v, w]$ are supercommutators.

Alternatively, we could have said that x, x^{-1} and g are supercommutators for any variable x and any $g \in G$, and that if v and w are supercommutators, so is $[v, w]$.

Definition 6.2. The set $\mathrm{Var}(v)$ of variables of a supercommutator v is defined by $\mathrm{Var}(x) = \{x\}$, $\mathrm{Var}(g) = \emptyset$, $\mathrm{Var}(v^{-1}) = \mathrm{Var}(v)$, and $\mathrm{Var}([v, w] = \mathrm{Var}(v) \cup \mathrm{Var}(w)$. We put $\mathrm{var}(v) = |\mathrm{Var}(v)|$, the *variable number* of v. If $\bar{x}$ is a tuple of variables, we put $\mathrm{Var}_{\bar{x}} = \mathrm{Var}(v) \cap \bar{x}$, $\mathrm{Var}'_{\bar{x}}(v) = \mathrm{Var}(v) \setminus \bar{x}$, $\mathrm{var}_{\bar{x}}(v) = |\mathrm{Var}_{\bar{x}}(v)|$ and $\mathrm{var}'_{\bar{x}}(v) = |\mathrm{Var}'_{\bar{x}}(v)|$.

Clearly $\mathrm{var}([v,v']) \geq \max\{\mathrm{var}(v), \mathrm{var}(v')\}$, and similarly for $\mathrm{var}_{\bar{x}}$ and $\mathrm{var}'_{\bar{x}}$.

Lemma 6.3. *Let $H \trianglelefteq G$ and $v(\bar{x}, \bar{z})$ a supercommutator.*

(1) *v defines a function from $H^{|\bar{x}\bar{z}|}$ to $\gamma_{\mathrm{var}(v)}(H)$.*

(2) *If $\mathrm{var}_{\bar{x}}(v) > 0$ and $\bar{x}$, $\bar{y}$ and $\bar{z}$ are pairwise disjoint, then*

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z})\, v(\bar{y}, \bar{z})\, \Phi(\bar{x}, \bar{y}, \bar{z}),$$

where Φ is a product of supercommutators whose factors w satisfy

(†) $\mathrm{Var}_{\bar{z}}(w) = \mathrm{Var}_{\bar{z}}(v)$, *and if $x_i \in \mathrm{Var}_{\bar{x}}(v)$ then $x_i \in \mathrm{Var}(w)$ or $y_i \in \mathrm{Var}(w)$, and both possibilities occur for at least one i.*

(3) *If $v(\bar{x}, \bar{z})$ is a product of supercommutators whose factors w satisfy $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}}(w) \geq n$, then*

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z})\, v(\bar{y}, \bar{z})\, \Phi(\bar{x}, \bar{y}, \bar{z}),$$

where Φ is a product of supercommutators whose factors w satisfy $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}}(w) > n$.

Proof. (1) is proved as in [9, Lemme 6 (1)] by induction, using that $\gamma_n(H)$ is characteristic in H, whence normal in G, and $[\gamma_n(H), \gamma_m(H)] \leq \gamma_{n+m}(H)$. We shall show (2) by induction on the construction of v.

If $v = x \in \bar{x}$ we have $v(yx) = yx = xy[y,x] = v(x)v(y)[y,x]$; if $v = x^{-1}$ we have $v(yx) = x^{-1}y^{-1} = v(x)v(y)$. This leaves the case $v = [v_1, v_2]$ for two supercommutators v_1 and v_2. We shall assume $\mathrm{var}_{\bar{x}}(v_1) > 0$ and $\mathrm{var}_{\bar{x}}(v_2) > 0$ (the case $\mathrm{var}_{\bar{x}}(v_1)\,\mathrm{var}_{\bar{x}}(v_2) = 0$ is analogous, but simpler).

By inductive hypothesis, there are Φ_i for $i = 1, 2$, products of supercommutators satisfying (†) relative to v_i, such that

$$v_i(\bar{y} \cdot \bar{x}, \bar{z}) = v_i(\bar{x}, \bar{z})\, v_i(\bar{y}, \bar{z})\, \Phi_i.$$

Then

$$\begin{aligned} v(\bar{y} \cdot \bar{x}, \bar{z}) &= [v_1(\bar{y} \cdot \bar{x}, \bar{z}), v_2(\bar{y} \cdot \bar{x}, \bar{z})] \\ &= [v_1(\bar{x}, \bar{z})\, v_1(\bar{y}, \bar{z})\, \Phi_1, v_2(\bar{x}, \bar{z})\, v_2(\bar{y}, \bar{z})\, \Phi_2] \\ &= [v_1(\bar{x}, \bar{z}), v_2(\bar{x}, \bar{z})]\, [v_1(\bar{y}, \bar{z}), v_2(\bar{y}, \bar{z})]\, \Phi = v(\bar{x}, \bar{z})\, v(\bar{y}, \bar{z})\, \Phi, \end{aligned}$$

where Φ is a product of supercommutators $[w, w']$

(i) where $w \in \Phi_1 \cup \{v_1(\bar{x}, \bar{z}), v_1(\bar{y}, \bar{z})\}$ and $w' \in \Phi_2 \cup \{v_2(\bar{x}, \bar{z}), v_2(\bar{y}, \bar{z})\}$, except for $[v_1(\bar{x}, \bar{z}), v_2(\bar{x}, \bar{z})]$ and $[v_1(\bar{y}, \bar{z}), v_2(\bar{y}, \bar{z})]$; it is clear that these must satisfy (†).

(ii) where one of w, w' is from (i), so $[w, w']$ satisfies (†).

(iii) where one of w, w' is equal to $v(\bar{x}, \bar{z})$ and the other contains at least one y_i, or one is equal to $v(\bar{y}, \bar{z})$ and the other contains at least one x_i; again $[w, w']$ satisfies (†).

(iv) which are obtained iteratively from supercommutators from (ii) and (iii) by commutation with other supercommutators, thus satisfying (†).

Here (i) takes care of the commutators of various factors of the two products, while (ii)–(iv) takes care of the correct order. Note that the only factor without a variable y_i is $v(\bar{x}, \bar{z})$ and the only factor without a variable x_j is $v(\bar{y}, \bar{z})$.

To show (3) note first that for a single supercommutator v the factorisation given in (2) satisfies the requirement. So for a product of supercommutators, we apply (2) to every factor, and then use commutators to get them into the right order. Note that we never have to commute a $w(\bar{x}, \bar{z})$ with a $w'(\bar{x}, \bar{z})$, or a $w(\bar{y}, \bar{z})$ with a $w'(\bar{y}, \bar{z})$, as they already appear in the correct order with respect to one another. It follows that all new commutators satisfy (†), whence $\mathrm{var}'_{\bar{x}} > n$. □

Theorem 6.4. *If G is nilpotent of class k and v is a product of supercommutators w with $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}}(w) \geq n$ such that G satisfies $\max\{2^{k-n}, 1\}$-largely $v(\bar{x}, \bar{g}) = c$, then $c = 1$.*

Proof. This is true for $n \geq k$, as then $\mathrm{var}(w) = \mathrm{var}_{\bar{x}}(w) + \mathrm{var}'_{\bar{x}}(w) \geq 1 + n$, and

$$c = w(\bar{x}, \bar{g}) \in \gamma_{\mathrm{var}(w)} G \leq \gamma_{n+1} G = \{1\}$$

for some $\bar{x} \in G$.

Now suppose it is true for $n + 1 \leq k$, and let $v(\bar{x}, \bar{z})$ be a product of supercommutators w with $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}} \geq n$, such that H satisfies 2^{k-n}-largely $v(\bar{x}, \bar{g}) = c$. By Lemma 6.3 there is Φ, a product of supercommutators whose factors w satisfy $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}}(w) > n$, such that

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z})\, v(\bar{y}, \bar{z})\, \Phi(\bar{x}, \bar{y}, \bar{z}).$$

Choose $\bar{h} \in G$ with $v(\bar{h}, \bar{g}) = c$. If $X = \{\bar{x} \in G : v(\bar{x}, \bar{g}) = c\}$, then X is 2^{k-n}-large, and $Y = X \cap \bar{h}^{-1} X$ is 2^{k-n-1}-large. Moreover, for $\bar{x} \in Y$ we have

$$\Phi(\bar{x}, \bar{h}, \bar{g}) = v(\bar{h}, \bar{g})^{-1} v(\bar{x}, \bar{g})^{-1} v(\bar{h} \cdot \bar{x}, \bar{g}) = c^{-1} c^{-1} c = c^{-1}.$$

By hypothesis $c^{-1} = 1$ and we are done. □

Theorem 6.5. *If G is nilpotent of class k and satisfies 2^k-largely an equation $v(\bar{x}, \bar{g}) = c$, then it satisfies $v(\bar{x}, \bar{g}) = c$.*

Proof. Bringing all the constants to the right-hand side, we may assume that $v(\bar{x}, \bar{z})$ is a product of supercommutators w with $\mathrm{var}_{\bar{x}}(w) > 0$. By Lemma 6.3 there is Φ, a product of supercommutators whose factors w satisfy $\mathrm{var}_{\bar{x}}(w) > 0$ and $\mathrm{var}'_{\bar{x}}(w) > 0$, such that

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z})\, v(\bar{y}, \bar{z})\, \Phi(\bar{x}, \bar{y}, \bar{z}).$$

Fix $\bar{h} \in G$. Then

$$\Phi(\bar{x}, \bar{h}, \bar{g}) = v(\bar{h}, \bar{g})^{-1} c^{-1} c = v(\bar{h}, \bar{g})^{-1}$$

2^{k-1}-largely on G. By Theorem 6.4 we have $v(\bar{h}, \bar{g}) = 1$. So $v(\bar{x}, \bar{g})$ is constant. □

Corollary 6.6. *If G is nilpotent of class k and $x^n = c$ is true 2^k-largely, then $c = 1$ and the exponent of G divides n.*

Proof. Immediate from Theorem 6.5. □

Corollary 6.7. *If G is nilpotent of class k and $\mu_*(x^n = c) > 1 - 2^{-k}$, then $c = 1$ and the exponent of G divides n.*

7. Autocommutativity

The notion of autocommutativity has been introduced by Sherman in 1975 [19].

Definition 7.1. Let G be a finite group, Σ a group of automorphisms of G, and H a subgroup of G. The *degree of autocommutativity relative to* $(H; \Sigma)$ is given by

$$\mathrm{ac}(H; \Sigma) = \frac{|\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}|}{|\Sigma| \cdot |H|}.$$

It gives the probability that a random element of H is fixed by a random automorphism in Σ.

Note that $\mathrm{ac}(H; \Sigma) = \mu(\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\})$, where μ is the counting measure on $\Sigma \times H$.

Theorem 7.2. *Let $H \leq G$ be finite groups, Σ a group of automorphisms of G, and suppose that $\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}$ is 4-large in $\Sigma \times H$. Then $H \leq \mathrm{Fix}(\Sigma)$.*

Proof. Given $\sigma \in \Sigma$ and $g \in H$, by 4-largeness there are $x \in H$ and $\tau \in \Sigma$ with

$$\tau(x) = x, \quad (\sigma \circ \tau)(x) = x, \quad \tau(gx) = gx \quad \text{and} \quad (\sigma \circ \tau)(gx) = gx.$$

Then

$$gx = \sigma(\tau(gx)) = \sigma(gx) = \sigma(g)\sigma(x) = \sigma(g)\sigma(\tau(x)) = \sigma(g)x,$$

whence $g = \sigma(g)$. □

Corollary 7.3. *If $H \leq G$ are finite groups and Σ is a group of automorphisms of G with $H \not\leq \mathrm{Fix}(\Sigma)$, then* $\mathrm{ac}(H;\Sigma) \leq \frac{3}{4}$.

Proof. If $\mathrm{ac}(H;\Sigma) > \frac{3}{4}$ then $\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}$ is 4-large in $\Sigma \times H$ by Lemma 2.5. Hence $H \leq \mathrm{Fix}(\Sigma)$ by Theorem 7.2. □

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