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Optimal Hardy–Littlewood inequalities uniformly bounded by a universal constant

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Abstract

The m-linear version of the Hardy–Littlewood inequality for m-linear forms on ℓ_p spaces and $m < p < 2m$, recently proved by Dimant and Sevilla-Peris, asserts that

$$\left(\sum_{\substack{j_i=1 \\ 1\le i\le m}}^{\infty} \left| T\left(e_{j_1}, \dots, e_{j_m}\right) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \le 2^{\frac{m-1}{2}} \sup_{\substack{\|x_i\|\le 1 \\ 1\le i\le m}} |T(x_1, \dots, x_m)|$$

for all continuous m-linear forms $T : \ell_p \times \cdots \times \ell_p \to \mathbb{R}$ or $\mathbb{C}$. We prove a technical lemma, of independent interest, that pushes further some techniques that go back to the seminal ideas of Hardy and Littlewood. As a consequence, we show that the inequality above is still valid with $2^{(m-1)/2}$ replaced by $2^{(m-1)(p-m)/p}$. In particular, we conclude that for $m < p \le m+1$ the optimal constants of the above inequality are uniformly bounded by 2; also, when $m = 2$, we improve the estimates of the original inequality of Hardy and Littlewood.

1. Introduction

Littlewood's famous 4/3 inequality [19], proved in 1930, asserts that

$$\left(\sum_{j,k=1}^{\infty} \left| T(e_j, e_k) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \le \sqrt{2} \sup_{\|x_1\|, \|x_2\|\le 1} |T(x_1, x_2)|,$$

for all continuous bilinear forms $T\colon c_0 \times c_0 \to \mathbb{C}$, and the exponent 4/3 cannot be improved. In some sense this result is the starting point of the theory of multiple summing operators (for recent results on summing operators we refer to [1, 8, 11, 15] and the references therein). From now on, for all Banach spaces $E_1, \dots, E_m, F$ and all m-linear

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maps $T\colon E_1 \times \cdots \times E_m \to F$, we denote

$$\|T\| := \sup_{\|x_1\|,\ldots,\|x_m\|\leq 1} \|T(x_1,\ldots,x_m)\|.$$

Besides its own beauty, Littlewood's insights motivated further important works of Bohnenblust and Hille (1931) and Hardy and Littlewood (1934). The Bohnenblust–Hille inequality [10] assures the existence of a constant $B_m \geq 1$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq B_m \|T\|,$$

for all continuous m–linear forms $T\colon c_0 \times \cdots \times c_0 \to \mathbb{C}$. The case $m = 2$ recovers Littlewood's $4/3$ inequality. Three years later, using quite delicate estimates, Hardy and Littlewood [18] extended Littlewood's $4/3$ inequality to bilinear maps defined on $\ell_p \times \ell_q$. In 1981, Praciano-Pereira [25] extended the Hardy–Littlewood inequalities to m-linear forms on ℓ_p spaces for $p \geq 2m$ and, quite recently, Dimant and Sevilla-Peris [17] generalized the estimates for the case $m < p < 2m$. These results were extensively investigated in various directions in the recent years [2, 3, 5, 6, 7, 13, 14, 17, 21, 24]. As a matter of fact, all results hold for both real and complex scalars with eventually different constants.

From now on we denote $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and, for any function f, whenever it makes sense we formally define $f(\infty) = \lim_{p\to\infty} f(p)$; moreover, we adopt $\frac{a}{0} = \infty$ for all $a > 0$.

In general terms we have the following m-linear inequalities:

- If $2m \leq p \leq \infty$, then there are constants $B^{\mathbb{K}}_{m,p} \geq 1$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq B^{\mathbb{K}}_{m,p} \|T\|$$

  for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and for all positive integers n.

- If $m < p \leq 2m$, then there are constants $B^{\mathbb{K}}_{m,p} \geq 1$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq B^{\mathbb{K}}_{m,p} \|T\|$$

  for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and for all positive integers n.

The exponents of all above inequalities are optimal: if replaced by smaller exponents the constants will depend on n. However, looking at the above inequalities by an anisotropic viewpoint a much richer complexity arise (see, for instance, [2, 3, 5, 6, 13, 23]).

The investigation of the sharp constants in the above inequalities is more than a puzzling mathematical challenge; for applications in physics we refer to [20]. The first estimates for $B_{m,p}^{\mathbb{K}}$ had exponential growth; more precisely,

$$B_{m,p}^{\mathbb{K}} \leq \left(\sqrt{2}\right)^{m-1},$$

for any $m \geq 1$. It was just quite recently that the estimates for $B_{m,p}^{\mathbb{K}}$ were refined, see for instance [5, 6, 9] and references therein. It was proved in [9] that

$$\begin{aligned} B_{m,\infty}^{\mathbb{R}} &< \kappa_1 \cdot m^{\frac{2-\log 2-\gamma}{2}} \approx \kappa_1 \cdot m^{0.36482}, \\ B_{m,\infty}^{\mathbb{C}} &< \kappa_2 \cdot m^{\frac{1-\gamma}{2}} \approx \kappa_2 \cdot m^{0.21139}, \end{aligned}$$

for certain constants $\kappa_1, \kappa_2 > 0$, where γ is the Euler–Mascheroni constant, but, as conjectured in [24], these estimates seem to be suboptimal. For $2m(m-1)^2 < p < \infty$, among other results it was shown in [5] that we also have

$$\begin{aligned} B_{m,p}^{\mathbb{R}} &< \kappa_1 \cdot m^{\frac{2-\log 2-\gamma}{2}} \approx \kappa_1 \cdot m^{0.36482}, \\ B_{m,p}^{\mathbb{C}} &< \kappa_2 \cdot m^{\frac{1-\gamma}{2}} \approx \kappa_2 \cdot m^{0.21139}. \end{aligned}$$

The best known estimates of $B_{m,p}^{\mathbb{K}}$ for the case $m < p \leq 2m$ are $(\sqrt{2})^{m-1}$ (see [3, 17]). Recently, Cavalcante [12] has shown that these estimates are a straightforward consequence of a kind of regularity principle proved in [4], and also investigated monotonicity properties of the optimal constants. However, the search of the optimal constants in this setting ($m < p \leq 2m$) is still quite intriguing. For instance, if $p = m$ it is easy to show that the only Hardy–Littlewood type inequality

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^s \right)^{\frac{1}{s}} \leq B_{m,m}^{\mathbb{K}} \, \|T\|$$

happens for $s = \infty$ (of course, here we consider the sup norm in the left-hand-side of the inequality) and in this case it is obvious that the optimal constants are $B_{m,m}^{\mathbb{K}} = 1$. So, we have optimal constants equal to 1 for $p = m$ and the best known constants equal to $(\sqrt{2})^{m-1}$ for p close to m. In this paper, among other results, we show that in fact the estimates $(\sqrt{2})^{m-1}$ are far from being optimal: we prove that

$$B_{m,p}^{\mathbb{K}} \leq 2^{\frac{(m-1)(p-m)}{p}},$$

and now, since $2^{\frac{(m+1)(p-m)}{p}} \longrightarrow 1$, we have a smooth connection between the estimates for $p = m$ and $p > m$. From now on, for $\mathbf{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$ and $1 \leq k \leq m$, let

$$\left|\frac{1}{\mathbf{p}}\right|_k := \frac{1}{p_1} + \cdots + \frac{1}{p_k} \qquad \text{and} \qquad \left|\frac{1}{\mathbf{p}}\right| := \left|\frac{1}{\mathbf{p}}\right|_m.$$

We present below the estimate obtained by Dimant and Sevilla-Peris [17] for further reference:

Theorem 1.1 (Dimant and Sevilla-Peris)**.** *Let $m \geq 2$ be a positive integer and $\mathbf{p} = (p_1, \ldots, p_m) \in (1, \infty]^m$ with*

$$\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1.$$

Then

$$\left(\sum_{j_1, \ldots, j_m = 1}^{n} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{\frac{1}{1 - \left|\frac{1}{\mathbf{p}}\right|}} \right)^{1 - \left|\frac{1}{\mathbf{p}}\right|} \leq \left(\sqrt{2}\right)^{m-1} \|T\|,$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n. In particular, if $m < p \leq 2m$, then

$$\left(\sum_{j_1, \ldots, j_m = 1}^{n} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq \left(\sqrt{2}\right)^{m-1} \|T\|$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n.

The exponent $\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}$ is optimal, but if one works in the anisotropic setting the result is not optimal (see, for instance, [7, 23]). The main results of the present paper are the forthcoming Theorems 3.3, 3.4 and 3.5 which also improve the original constants of the bilinear Hardy–Littlewood inequalities. As a consequence of these results, when $m < p \leq m + 1$, we shall prove that the optimal constants of the Hardy–Littlewood inequality are uniformly bounded by 2.

2. **A multipurpose lemma**

Let $m \geq 2$ be a positive integer, F be a Banach space, $A \subset I_m := \{1, \ldots, m\}$, $\mathbf{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$, $s, \alpha \geq 1$ and

$$B_{\mathbf{p},s,\alpha}^{A,F}(n) := \inf \left\{ C(n) \geq 0 \ : \ \begin{array}{r} \left(\displaystyle\sum_{j_i = 1}^{n} \left(\sum_{\widehat{j_i} = 1}^{n} \left\| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right\|^{s} \right)^{\frac{1}{s}\alpha} \right)^{\frac{1}{\alpha}} \\ \leq C(n), \ \text{for all } i \in A \end{array} \right\},$$

in which $\widehat{j_i}$ means that the sum runs over all indexes but j_i, and the infimum is taken over all norm-one m-linear maps $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to F$. The following lemma, fundamental in the proof of our main results, is based on ideas dating back to Hardy and Littlewood (see [2, 16, 17, 18, 25]). It is our belief that it is a result of independent interest, with potential applications in the theory of multiple summing operators:

Lemma 2.1. *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *and* $\mathbf{q} = (q_1, \ldots, q_m)$ *be such that* $1 \leq p_k < q_k \leq \infty$, $k = 1, \ldots, m$, *and also let* $\lambda_0, s \geq 1$.

(1) *If*

$$s \geq \eta_1 := \left[\frac{1}{\lambda_0} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right|\right]^{-1} > 0, \tag{2.1}$$

then

$$B^{I_m,F}_{\mathbf{p},s,\eta_1}(n) \leq B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n).$$

(2) *If*

$$s \geq \eta_2 := \left[\frac{1}{\lambda_0} - \left|\frac{1}{\mathbf{p}}\right|_{m-1} + \left|\frac{1}{\mathbf{q}}\right|_{m-1}\right]^{-1} > 0,$$

then

$$B^{\{m\},F}_{\mathbf{p},s,\eta_2}(n) \leq B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n).$$

Proof. To prove (1), let s, λ_0 be such that (2.1) is fulfilled. Let us define

$$\lambda_j := \left[\frac{1}{\lambda_0} - \left|\frac{1}{\mathbf{p}}\right|_j + \left|\frac{1}{\mathbf{q}}\right|_j\right]^{-1}, \quad j = 1, \ldots, m.$$

Notice that $\lambda_m = \eta_1$. Moreover, for all $j = 1, \ldots, m$, we have $\lambda_{j-1} < \lambda_j$ and

$$\left[\frac{q_j p_j}{\lambda_{j-1}(q_j - p_j)}\right]^* = \frac{\lambda_j}{\lambda_{j-1}}, \tag{2.2}$$

where the notation above denotes the conjugate number, i.e, if $a \geq 1$, then $1/a + 1/a^* = 1$.

Let us suppose that, for $k \in \{1, \ldots, m\}$, the inequality

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left\|T(e_{j_1}, \ldots, e_{j_m})\right\|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \leq B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|$$

is true for all m–linear maps $T : \ell_{p_1}^n \times \cdots \times \ell_{p_{k-1}}^n \times \ell_{q_k}^n \times \cdots \times \ell_{q_m}^n \to F$ and for all $i = 1, \ldots, m$. Let us prove that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left\|T(e_{j_1}, \ldots, e_{j_m})\right\|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}} \leq B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\| \tag{2.3}$$

for all m–linear maps $T:\ell_{p_1}^n\times\cdots\times\ell_{p_k}^n\times\ell_{q_{k+1}}^n\times\cdots\times\ell_{q_m}^n\to F$ and for all $i=1,\ldots,m$.

Consider

$$T:\ell_{p_1}^n\times\cdots\times\ell_{p_k}^n\times\ell_{q_{k+1}}^n\times\cdots\times\ell_{q_m}^n\to F,$$

a m-linear map and, for each $x\in B_{\ell^n_{\frac{q_kp_k}{q_k-p_k}}}$, define $T^{(x)}:\ell_{p_1}^n\times\cdots\times\ell_{p_{k-1}}^n\times\ell_{q_k}^n\times\cdots\times\ell_{q_m}^n\to F$ by

$$T^{(x)}\left(z^{(1)},\ldots,z^{(m)}\right)=T\left(z^{(1)},\ldots,z^{(k-1)},xz^{(k)},z^{(k+1)},\ldots,z^{(m)}\right),$$

where $xz^{(k)}=\left(x_jz_j^{(k)}\right)_{j=1}^n\in\ell_{p_k}^n$. Observe that

$$\|T\|\geq\sup\left\{\left\|T^{(x)}\right\|:x\in B_{\ell^n_{\frac{q_kp_k}{q_k-p_k}}}\right\}.$$

By applying the induction hypothesis to $T^{(x)}$, we get, for all $i=1,\ldots,m$,

$$\begin{aligned}
&\left(\sum_{j_i=1}^n\left(\sum_{\widehat{j_i}=1}^n\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^s\left|x_{j_k}\right|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}\\
&\qquad\qquad=\left(\sum_{j_i=1}^n\left(\sum_{\widehat{j_i}=1}^n\left\|T\left(e_{j_1},\ldots,e_{j_{k-1}},xe_{j_k},e_{j_{k+1}},\ldots,e_{j_m}\right)\right\|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}\\
&\qquad\qquad=\left(\sum_{j_i=1}^n\left(\sum_{\widehat{j_i}=1}^n\left\|T^{(x)}\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}\\
&\qquad\qquad\leq B_{\mathbf{q},s,\lambda_0}^{I_m,F}(n)\left\|T^{(x)}\right\|\\
&\qquad\qquad\leq B_{\mathbf{q},s,\lambda_0}^{I_m,F}(n)\|T\|.
\end{aligned}$$

By (2.2), using the characterization of the dual of ℓ_p-type spaces, we have

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}}\\
&\qquad=\left\|\left(\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)_{j_k=1}^{n}\right\|_{\left[\frac{q_k p_k}{\lambda_{k-1}(q_k-p_k)}\right]^*}^{\frac{1}{\lambda_{k-1}}}\\
&\qquad=\left(\sup_{y\in B_{\ell^n_{\frac{q_k p_k}{\lambda_{k-1}(q_k-p_k)}}}}\sum_{j_k=1}^{n}|y_{j_k}|\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}.
\end{aligned}$$

We continue the proof by making some other changes on the arguments borrowed from [17, 18, 25] to encompass our relaxed hypotheses. The main difference is that we now work in a broader scenario with $q_j \leq \infty$ for all j, and for this task some technical modifications are in order. Since for all positive integers N and all scalars $w_1,\ldots,w_N$, we have

$$\sup_{y\in B_{\ell^N_{\frac{p}{\nu}}}}\sum_{i=1}^{N}|w_i||y_i| = \sup_{x\in B_{\ell^N_p}}\sum_{i=1}^{N}|w_i||x_i|^{\nu}, \tag{2.4}$$

for all $1\leq \nu\leq p<\infty$, we get

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}}\\
&\qquad=\left(\sup_{y\in B_{\ell^n_{\frac{q_k p_k}{\lambda_{k-1}(q_k-p_k)}}}}\sum_{j_k=1}^{n}|y_{j_k}|\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}\\
&\qquad=\left(\sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}}\sum_{j_k=1}^{n}|x_{j_k}|^{\lambda_{k-1}}\left(\sum_{\widehat{j_k}=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{x \in B_{\ell^n_{\frac{q_k p_k}{q_k - p_k}}}} \left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} \left\| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right\|^s \left| x_{j_k} \right|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&\leq B^{I_m, F}_{\mathbf{q}, s, \lambda_0}(n) \|T\|, \qquad (2.5)
\end{aligned}
$$

and this concludes the proof of (2.3) for $i = k$. To prove (2.3) for $i \neq k$ let us initially consider $k \neq m$ or $s > \lambda_m = \eta_1$. For each $i = 1, \ldots, m$, to simplify our notation, let us denote

$$
S_i := \left(\sum_{\widehat{j_i}=1}^{n} \|T(e_{j_1}, \ldots, e_{j_m})\|^s \right)^{\frac{1}{s}}.
$$

Note that

$$
\begin{aligned}
&\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} \|T(e_{j_1}, \ldots, e_{j_m})\|^s \right)^{\frac{1}{s}\lambda_k} \\
&\qquad = \sum_{j_i=1}^{n} S_i^{\lambda_k} = \sum_{j_i=1}^{n} S_i^{\lambda_k - s} S_i^s = \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1}, \ldots, e_{j_m})\|^s}{S_i^{s-\lambda_k}} \\
&\qquad = \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{\|T(e_{j_1}, \ldots, e_{j_m})\|^s}{S_i^{s-\lambda_k}} \\
&\qquad = \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{\|T(e_{j_1}, \ldots, e_{j_m})\|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}} \|T(e_{j_1}, \ldots, e_{j_m})\|^{\frac{s(\lambda_k - \lambda_{k-1})}{s-\lambda_{k-1}}}.
\end{aligned}
$$

From Hölder's inequality with exponents

$$
r = \frac{s - \lambda_{k-1}}{s - \lambda_k} \text{ and } r^* = \frac{s - \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}
$$

we have

$$
\begin{aligned}
&\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} \|T(e_{j_1}, \ldots, e_{j_m})\|^s \right)^{\frac{1}{s}\lambda_k} \\
&\quad = \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{\|T(e_{j_1}, \ldots, e_{j_m})\|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}} \|T(e_{j_1}, \ldots, e_{j_m})\|^{\frac{s(\lambda_k - \lambda_{k-1})}{s-\lambda_{k-1}}}
\end{aligned}
$$

$$\leq \sum_{j_k=1}^{n}\left[\left(\sum_{\widehat{j_k}=1}^{n}\frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{s-\lambda_k}{s-\lambda_{k-1}}}\left(\sum_{\widehat{j_k}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{\lambda_k-\lambda_{k-1}}{s-\lambda_{k-1}}}\right].$$

Now, by the Hölder inequality with exponents

$$r=\frac{\lambda_k\,(s-\lambda_{k-1})}{\lambda_{k-1}\,(s-\lambda_k)}\quad\text{and}\quad r^*=\frac{\lambda_k\,(s-\lambda_{k-1})}{s\,(\lambda_k-\lambda_{k-1})}$$

we have

$$\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\lambda_k}\\
&\qquad\leq\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{\lambda_k}{\lambda_{k-1}}}\right)^{\frac{\lambda_{k-1}}{\lambda_k}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}}\\
&\qquad\qquad\times\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \qquad (2.6)
\end{aligned}$$

Let us estimate separately the two factors of this product. It follows from the case $i=k$ that

$$\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}\leq\left(B_{\mathbf{q},s,\lambda_0}^{I_m,F}(n)\|T\|\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \qquad (2.7)$$

In order to estimate the first factor of the product in (2.6), we first observe that, by (2.4), we obtain

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{\lambda_k}{\lambda_{k-1}}}\right)^{\frac{\lambda_{k-1}}{\lambda_k}}\\
&\qquad=\sup_{y\in B_{\ell^n_{\frac{q_kp_k}{\lambda_{k-1}(q_k-p_k)}}}}\sum_{j_k=1}^{n}|y_{j_k}|\sum_{\widehat{j_k}=1}^{n}\frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}}.
\end{aligned}$$

Straightforward computations give us

$$\begin{aligned}
&\sup_{y\in B_{\ell^n_{\frac{q_k p_k}{\lambda_{k-1}(q_k-p_k)}}}} \sum_{j_k=1}^{n} |y_{j_k}| \sum_{\widehat{j_k}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}} \\
&\quad = \sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}} \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}} |x_{j_k}|^{\lambda_{k-1}} \\
&\quad = \sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}} \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}} |x_{j_k}|^{\lambda_{k-1}} \\
&\quad = \sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}} \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^{s-\lambda_{k-1}}}{S_i^{s-\lambda_{k-1}}} \|T(e_{j_1},\ldots,e_{j_m})\|^{\lambda_{k-1}} |x_{j_k}|^{\lambda_{k-1}}.
\end{aligned}$$

By the Hölder inequality with exponents

$$r = \frac{s}{s-\lambda_{k-1}} \qquad \text{and} \qquad r^* = \frac{s}{\lambda_{k-1}},$$

we have

$$\begin{aligned}
&\sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}} \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^{s-\lambda_{k-1}}}{S_i^{s-\lambda_{k-1}}} \|T(e_{j_1},\ldots,e_{j_m})\|^{\lambda_{k-1}} |x_{j_k}|^{\lambda_{k-1}} \\
&\qquad \leq \sup_{x\in B_{\ell^n_{\frac{q_k p_k}{q_k-p_k}}}} \sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^s} \right)^{\frac{s-\lambda_{k-1}}{s}} \\
&\qquad\qquad \times \left(\sum_{\widehat{j_i}=1}^{n} \|T(e_{j_1},\ldots,e_{j_m})\|^s |x_{j_k}|^s \right)^{\frac{1}{s}\lambda_{k-1}}.
\end{aligned}$$

Since

$$\sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^s} = \sum_{\widehat{j_i}=1}^{n} \frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{\sum_{\widehat{j_i}=1}^{n} \|T(e_{j_1},\ldots,e_{j_m})\|^s} = 1,$$

we finally conclude that

$$\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\frac{\|T(e_{j_1},\ldots,e_{j_m})\|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{\lambda_k}{\lambda_{k-1}}}\right)^{\frac{\lambda_{k-1}}{\lambda_k}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}}$$

$$\leq \sup_{x\in B_{\ell^n_{\frac{q_kp_k}{q_k-p_k}}}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s|x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1}}$$

$$= \sup_{x\in B_{\ell^n_{\frac{q_kp_k}{q_k-p_k}}}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left\|T^{(x)}\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^s\right)^{\frac{1}{s}\lambda_{k-1}}$$

$$\leq \left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|\right)^{\lambda_{k-1}}. \tag{2.8}$$

Plugging (2.7) and (2.8) in (2.6), we obtain

$$\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\lambda_k}$$

$$\begin{aligned}
&\leq \left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|\right)^{\lambda_{k-1}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}}\cdot\left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}\\
&= \left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\right)^{\lambda_{k-1}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}}\cdot\left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}\\
&\qquad\times\|T\|^{\lambda_{k-1}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}}\cdot\|T\|^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}\\
&= \left(B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\right)^{\lambda_k}\|T\|^{\lambda_k},
\end{aligned}$$

that is,

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left\|T(e_{j_1},\ldots,e_{j_m})\right\|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}}\leq B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|.$$

It remains to consider $k = m$ and $s = \lambda_m = \eta_1$. Fortunately, this case is simpler than the previous and we have

$$\begin{aligned}\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\eta_1}\right)^{\frac{1}{\eta_1}} &= \left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}\|T(e_{j_1},\ldots,e_{j_m})\|^s\right)^{\frac{1}{s}\eta_1}\right)^{\frac{1}{\eta_1}} \\ &\le B^{I_m,F}_{\mathbf{q},s,\lambda_0}(n)\|T\|,\end{aligned}$$

where the inequality is due to the case $i = k$.

The proof of (2) is similar, except for the last step (case $k = m$) where one may use an argument as (2.5), but it is somewhat predictable and we omit the proof. □

Remark 2.2. The case $q_k = \infty$ for all $k = 1, \ldots, m$ in (1) is known and follows the ideas from [18, 25]; our approach follows the lines of the modern presentation of [17].

3. Main results

We begin this section by recalling a particular instance of the Contraction Principle of [16, Theorem 12.2]. From now on $r_i(t)$ are the Rademacher functions.

Lemma 3.1 (Corollary of the Contraction Principle)**.** *Let N be a positive integer, and A and B be subsets of $\{1,\ldots,N\}$ such that $A \subset B$. Regardless of the choice of scalars $a_1, \ldots, a_N$,*

$$\int_0^1 \left|\sum_{i\in A} a_i r_i(t)\right| \mathrm{d}t \le \int_0^1 \left|\sum_{i\in B} a_i r_i(t)\right| \mathrm{d}t.$$

The next lemma seems to be a by now standard consequence of the Contraction Principle (it was recently proved, for instance, in [22, Lemma 1]) but we present a proof here for the sake of completeness.

Lemma 3.2. *Regardless of the choice of the positive integers m, N and the scalars $a_{i_1,\ldots,i_m}$, $i_1, \ldots, i_m = 1, \ldots, N$, we have*

$$\max_{\substack{i_k=1,\ldots,N\\ k=1,\ldots,m}} \left|a_{i_1,\ldots,i_m}\right| \le \int_{[0,1]^m} \left|\sum_{i_1,\ldots,i_m=1}^{N} r_{i_1}(t_1)\ldots r_{i_m}(t_m) a_{i_1,\ldots,i_m}\right| \mathrm{d}t_1 \ldots \mathrm{d}t_m.$$

Proof. We will proceed by induction over m. For the case $m = 1$ consider $l \in B := \{1, \ldots, N\}$ and $A := \{l\}$. Thus, from Lemma 3.1,

$$\int_0^1 \left|\sum_{i\in\{l\}} a_i r_i(t)\right| \mathrm{d}t \le \int_0^1 \left|\sum_{i=1}^{N} a_i r_i(t)\right| \mathrm{d}t$$

and this implies

$$|a_l| \leq \int_0^1 \left| \sum_{i=1}^{N} a_i r_i(t) \right| \mathrm{d}t$$

for all $l \in \{1, \ldots, N\}$. Hence,

$$\max_{i=1,\ldots,N} |a_i| \leq \int_0^1 \left| \sum_{i=1}^{N} a_i r_i(t) \right| \mathrm{d}t.$$

Let us suppose, as the induction step, that the result is valid for $m-1$. For all positive integers $i_1, \ldots, i_m$,

$$\begin{aligned}
&\int_{[0,1]^m} \left| \sum_{i_1,\ldots,i_m=1}^{N} r_{i_1}(t_1) \ldots r_{i_m}(t_m) a_{i_1,\ldots,i_m} \right| \mathrm{d}t_1 \ldots \mathrm{d}t_m \\
&= \int_{[0,1]^{m-1}} \left[\int_0^1 \left| \sum_{i_1=1}^{N} r_{i_1}(t_1) \times \left(\sum_{i_2,\ldots,i_m=1}^{N} r_{i_2}(t_2) \ldots r_{i_m}(t_m) a_{i_1,\ldots,i_m} \right) \right| \mathrm{d}t_1 \right] \mathrm{d}t_2 \ldots \mathrm{d}t_m \\
&\geq \int_{[0,1]^{m-1}} \left| \sum_{i_2,\ldots,i_m=1}^{N} r_{i_2}(t_2) \ldots r_{i_m}(t_m) a_{i_1,\ldots,i_m} \right| \mathrm{d}t_2 \ldots \mathrm{d}t_m \\
&\geq \left| a_{i_1,\ldots,i_m} \right|,
\end{aligned}$$

where we have used the case $m = 1$ and the induction hypothesis on the first and second inequalities, respectively. This concludes the proof. □

Now we are able to prove our first main result, providing better constants for Theorem 1.1. The main difference between the proof of our next result and the original proof of [17] is that here we use Lemma 3.2 in order to get better estimates.

Theorem 3.3. *Let $m \geq 2$ be a positive integer and $\mathbf{p} = (p_1, \ldots, p_m) \in (1, \infty]$ with $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$. Then, for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n,*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)} \|T\| .$$

In particular, if $m < p \leq 2m$, then, for all continuous m-linear forms $T : \ell_p \times \cdots \times \ell_p \to \mathbb{K}$

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m)}{p}} \|T\| .$$

Proof. Let $S : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ be an m-linear form (note that S has a different domain from the domain of T). Consider $s = \left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}$. Since $s \geq 2$, from Lemma 3.2, Hölder's inequality and Khinchin's inequality for multiple sums we have

$$\begin{aligned}
&\sum_{j_1=1}^{n}\left(\sum_{\widehat{j_1}=1}^{n}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^s\right)^{\frac{1}{s}} \\
&\qquad\leq \sum_{j_1=1}^{n}\left(\left(\left(\sum_{\widehat{j_1}=1}^{n}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{s}}\left(\max_{\widehat{j_1}}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|\right)^{1-\frac{2}{s}}\right) \\
&\qquad\leq \sum_{j_1=1}^{n}\left(\left(\left(\sqrt{2}\right)^{m-1}R_n\right)^{\frac{2}{s}}R_n^{1-\frac{2}{s}}\right),
\end{aligned}$$

where

$$R_n := \int_I\left|\sum_{\widehat{j_1}=1}^{n} r_{j_2}(t_2)\ldots r_{j_m}(t_m)S\left(e_{j_1},\ldots,e_{j_m}\right)\right|\mathrm{d}t_2\ldots\mathrm{d}t_m$$

and $I := [0,1]^{m-1}$. Since

$$\begin{aligned}
&\sum_{j_1=1}^{n}\left(\left(\left(\sqrt{2}\right)^{m-1}R_n\right)^{\frac{2}{s}}R_n^{1-\frac{2}{s}}\right) \\
&= 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\times\sum_{j_1=1}^{n}\int_I\left|\sum_{\widehat{j_1}=1}^{n} r_{j_2}(t_2)\ldots r_{j_m}(t_m)S\left(e_{j_1},\ldots,e_{j_m}\right)\right|\mathrm{d}t_2\ldots\mathrm{d}t_m \\
&= 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\times\int_I\sum_{j_1=1}^{n}\left|S\left(e_{j_1},\sum_{j_2=1}^{n} r_{j_2}(t_2)e_{j_2},\ldots,\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right|\mathrm{d}t_2\ldots\mathrm{d}t_m \\
&\leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\times\sup_{t_2,\ldots,t_m\in[0,1]}\sum_{j_1=1}^{n}\left|S\left(e_{j_1},\sum_{j_2=1}^{n} r_{j_2}(t_2)e_{j_2},\ldots,\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right| \\
&\leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)} \\
&\times\sup_{t_2,\ldots,t_m\in[0,1]}\left\|S\left(\,\cdot\,,\sum_{j_2=1}^{n} r_{j_2}(t_2)e_{j_2},\ldots,\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right\| \\
&\leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\left\|S\right\|,
\end{aligned}$$

we conclude that

$$\sum_{j_1=1}^{n}\left(\sum_{\widehat{j_1}=1}^{n}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}} \leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\|S\|.$$

Repeating the same procedure for other indexes we have

$$\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}} \leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\|S\|,$$

for all $i = 1,\ldots,m$. Hence, from Lemma 2.1 (1), we conclude that, for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}} \leq 2^{(m-1)\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)}\|T\|. \qquad \square$$

The above result shows that for $m < p \leq m + c$, with a fixed constant c, we have a kind of uniform Hardy–Littlewood inequality, in the sense that there exists a universal constant, independent of m, satisfying the respective inequalities. For instance, choosing $c = 1$, we have

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq 2^{\frac{m-1}{m+1}}\|T\| < 2\|T\|,$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ with $m < p \leq m+1$, and all positive integers n.

The next result shows that when $p \leq 2m - 2$ it is possible to improve both exponents and constants:

Theorem 3.4. *Let $m \geq 2$ be a positive integer and $m < p \leq 2m-2$. Then, for all positive integers n, all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all $i = 1,\ldots,m$, we have*

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{\frac{p}{p-(m-1)}}\right)^{\frac{p-(m-1)}{p}\cdot\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}}\|T\|.$$

Proof. Consider $s = \frac{p}{p-(m-1)}$. Since $p \leq 2m - 2$ we have $s \geq 2$. From Lemma 3.2, Hölder's inequality and Khinchin's inequality for multiple sums we have, as in the proof of Theorem 3.3, for all $i = 1, \dots, m$,

$$\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|T\left(e_{j_1},\dots,e_{j_m}\right)\right|^{\frac{p}{p-(m-1)}}\right)^{\frac{p-(m-1)}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}}\|T\|$$

for all m-linear forms $T \in \mathcal{L}\left({}^m\ell_\infty^n;\mathbb{K}\right)$ and all positive integers n. Note that

$$\left[\frac{1}{\lambda_0}-\sum_{j=1}^{m-1}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right]^{-1} = \frac{1}{1-\frac{m-1}{p}} = \frac{p}{p-(m-1)} = s.$$

From Lemma 2.1 (2), we conclude that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|T\left(e_{j_1},\dots,e_{j_m}\right)\right|^{\frac{p}{p-(m-1)}}\right)^{\frac{p-(m-1)}{p}\cdot\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}}\|T\|$$

for all continuous m-linear forms $T : \ell_p \times \cdots \times \ell_p \to \mathbb{K}$. □

In our final result we will show that if we have the additional hypothesis

$$\frac{1}{2} \leq \frac{1}{p_1}+\frac{1}{p_2},$$

then the optimal constants of the respective Hardy–Littlewood inequalities are always bounded by $2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}$.

Theorem 3.5. *Let $m \geq 3$ and $\mathbf{p} = (p_1, \dots, p_m) \in (1,\infty]^m$ be such that*

$$\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right|_2 \quad \text{and} \quad \left|\frac{1}{\mathbf{p}}\right| < 1.$$

Then

$$\left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_{m-1}}}\right)^{\frac{1-\left|\frac{1}{\mathbf{p}}\right|_{m-1}}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|T\|,$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n.

Proof. Since $\left|\frac{1}{\mathbf{p}}\right|_2 \geq \frac{1}{2}$, by Theorem 3.3 we have

$$\left(\sum_{i,j=1}^{n}|T_2(e_i,e_j)|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2} \leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|T_2\|$$

for all bilinear forms $T_2 : \ell_{p_1}^n \times \ell_{p_2}^n \to \mathbb{K}$ and all positive integers n. By the Khinchin inequality we conclude that

$$\left(\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\left|T_3\left(e_i,e_j,e_k\right)\right|^2\right)^{\frac{1}{2}\cdot\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2} \leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|T_3\|$$

for all 3-linear forms $T_3 : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_\infty^n \to \mathbb{K}$ and all positive integers n. In fact, for all positive integers n we have

$$\begin{aligned}
&\left(\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\left|T_3\left(e_i,e_j,e_k\right)\right|^2\right)^{\frac{1}{2}\cdot\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\\
&\qquad\leq A^{-1}_{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\left(\sum_{i,j=1}^{n}\int_0^1\left|\sum_{k=1}^{n}r_k(t)T_3\left(e_i,e_j,e_k\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\mathrm{d}t\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\\
&\qquad=\left(\int_0^1\sum_{i,j=1}^{n}\left|T_3\left(e_i,e_j,\sum_{k=1}^{n}r_k(t)e_k\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\mathrm{d}t\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\\
&\qquad\leq\sup_{t\in[0,1]}\left(\sum_{i,j=1}^{n}\left|T_3\left(e_i,e_j,\sum_{k=1}^{n}r_k(t)e_k\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\\
&\qquad\leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|T_3\|,
\end{aligned}$$

where $A_{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}$ is the constant of the Khinchin inequality, and in our case this constant is 1. Thus,

$$\left(\sum_{i,j,k=1}^{n}\left|T_3\left(e_i,e_j,e_k\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|_2} \leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|T_3\|,$$

for all 3-linear forms $T_3 : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_\infty^n \to \mathbb{K}$ and all positive integers n. This means that for any Banach spaces E_1, E_2, E_3, every continuous 3-linear form $R : E_1 \times E_2 \times E_3 \to \mathbb{K}$ is multiple $\left(\left(1-\left|\frac{1}{\mathbf{p}}\right|_2\right)^{-1}; p_1^*, p_2^*, 1\right)$-summing and the associated constant is $2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}$ (see [17]). Since

$$\frac{1}{1}-\frac{1}{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}=\frac{1}{p_3^*}-\frac{1}{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}},$$

using the inclusion theorem for absolutely summing linear operators, we conclude that:

Every continuous 3*-linear form* $R : E_1 \times E_2 \times E_3 \to \mathbb{K}$

$$\textit{is multiple } \left(\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}, \frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}, \frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}; p_1^*, p_2^*, p_3^*\right)\textit{-summing with the same constant.}$$

(This is an anisotropic notion of multiple summing operators; we refer to [4] for further details.) Using the canonical isometric isomorphisms for the spaces of weakly summable sequences (see [16, Proposition 2.2]) we know that this is equivalent to assert that

$$\left(\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n}\left|S\left(e_i, e_j, e_k\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|_2}}\right)^{\frac{1-\left|\frac{1}{\mathbf{p}}\right|_2}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq 2^{1-\left|\frac{1}{\mathbf{p}}\right|_2}\|S\|,$$

for all 3-linear forms $S : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_{p_3}^n \to \mathbb{K}$ and all positive integers n. The proof is completed by a standard induction argument. $\square$

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