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A Simple Proof of Berry–Esséen Bounds for the Quadratic Variation of the Subfractional Brownian Motion

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Abstract

We give a simple technic to derive the Berry–Esséen bounds for the quadratic variation of the subfractional Brownian motion (subfBm). Our approach has two main ingredients: (i) bounding from above the covariance of quadratic variation of subfBm by the covariance of the quadratic variation of fractional Brownian motion (fBm); and (ii) using the existing results on fBm in [1, 2, 4]. As a result, we obtain simple and direct proof to derive the rate of convergence of quadratic variation of subfBm. In addition, we also improve this rate of convergence to meet the one of fractional Brownian motion in [2].

Résumé

Nous donnons une technique simple pour calculer les limites Berry–Esséen pour la variation quadratique du mouvement Brownien subfractional (subfBm). Notre approche a deux ingrédients principaux : (i) majorer la covariance des variations quadratiques de subfBm par la covariance de la variation quadratique du mouvement Brownien fractionnaire (FBM) ; et (ii) utiliser les résultats existants sur fBm dans [1, 2, 4]. En conséquence, nous obtenons une simple et directe preuve pour calculer le taux de convergence des variations quadratiques de subfBm. En outre, nous améliorons aussi ce taux de convergence pour obtenir ceux du mouvement Brownien fractionnaire dans [2].

1. Introduction and preliminaries

The subfractional Brownian motion (subfBm in short) $S = (S_t,\, t \geq 0)$ with parameters $H \in (0,1)$, is defined on some probability space $(\Omega, \mathcal{F}, P)$ (Here, and everywhere else, we do assume that $\mathcal{F}$ is the sigma-field generated by S). This means that S is a centered Gaussian process with

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covariance

$$\mathbb{E}[S_s S_t] = R_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right], \; s,t \geq 0 \quad (1.1)$$

The following result, proved in [6], states the convergence of quadratic variation of subfractional Brownian motion to a centered reduced normal variable, also provides its rate of convergence. Hereafter, we denote

$$Z_n = \sum_{k=0}^{n-1} n^{2H}\left[(S_{(k+1)/n} - S_{k/n})^2 - Var\left(S_{(k+1)/n} - S_{k/n}\right)\right], \quad n \geq 1.$$

Theorem 1.1 ([6]). *Let N be a standard Gaussian random variable ($N \sim N(0,1)$) and suppose that $H \in (0, \frac{3}{4}]$. Then $\frac{Z_n}{Var(Z_n)}$ converges in distribution to N and the following Berry–Esséen bounds hold for every $n \geq 1$,*

$$d_{Kol}\left(\frac{Z_n}{\sqrt{Var(Z_n)}}, N\right) \leq c_H \times \begin{cases} n^{-\frac{1}{2}}, & H \in \left(0, \frac{1}{2}\right), \\ n^{2H-\frac{3}{2}}, & H \in \left[\frac{1}{2}, \frac{3}{4}\right), \\ \dfrac{1}{\sqrt{\log n}}, & H = \frac{3}{4}, \end{cases}$$

where c_H is a constant depending only on H.

In [6], the proof uses Stein method and Malliavin calculus, based on the idea developed in [1, 4] for the case of fractional Brownian motion (fBm in short), which leads to the same rate of convergence. Recently, [2] used the convolution product of two sequences which improve clearly the rate of convergence of the fBm. The natural question imposes itself, it is possible to obtain a rate of convergence of subfBm similar to the one proved by [2] for the fBm?

The goal of this paper, is to improve the rate of convergence of the subfBm so that we have at least the same one as the fBm. To perform our calculation, we will mainly follow the idea taken from [2]. With the proof of [4] and [2] in hand, we will show how we can retrieve the result of [6], and how we can improve this result to reach the one of fBm in [2].

For the case of Hurst parameter $H > 3/4$, we think that it deserves an entire work, the quadratic variation will converge to a Hermite random variable similarly to the fractional Brownian Motion [3].

We claim the main result of this paper:

Theorem 1.2. *Let $N \sim \mathcal{N}(0,1)$, there exist a constant c_H depending only on H, such that for every $n \geq 1$,*

$$d_{Kol}\left(\frac{Z_n}{\sqrt{Var(Z_n)}}, N\right) \leq c_H \times \begin{cases} \dfrac{1}{\sqrt{n}}, & H \in \left(0, \frac{5}{8}\right), \\ \dfrac{(\log n)^{3/2}}{\sqrt{n}}, & H = \frac{5}{8}, \\ n^{4H-3}, & H \in \left(\frac{5}{8}, \frac{3}{4}\right), \\ \dfrac{1}{\log n}, & H = \frac{3}{4}. \end{cases}$$

We recall briefly some important tools of Malliavin calculus used throughout this paper. We mean by $\mathfrak{H}$ a real separable Hilbert space defined as follows:

(i) denote by $\mathcal{E}$ the set of all $\mathbb{R}$-valued functions on $[0,\infty)$,

(ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathfrak{H}} = R_H(s,t).$$

For every $q \geq 1$, let $\mathcal{H}_q$ be the q^{th} Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q\left(X\left(h\right)\right), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the q^{th} Hermite polynomial defined as $H_q(x) = (-1)^q\, e^{\frac{x^2}{2}}\, \dfrac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$. The mapping $I_q(h^{\otimes q}) = q! H_q\left(X\left(h\right)\right)$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathfrak{H}^{\odot q}} = \sqrt{q!}\, \|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f,g \in \mathfrak{H}^{\odot q}$ and $q \geq 1$, one has

$$\mathbb{E}\big[I_q(f) I_q(g)\big] = q!\, \langle f, g\rangle_{\mathfrak{H}^{\otimes q}}. \tag{1.2}$$

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the r^{th} contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1=1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}.$$

In particular, note that $f \otimes_0 g = f \otimes g$ and when $p = q$, that $f \otimes_p g = \langle f, g \rangle_{\mathfrak{H}^{\otimes p}}$. Since, in general, the contraction $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. The following formula is useful to compute the product of such multiple integrals: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f)\ I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{1.3}$$

We will use the notation $\delta_{k/n} = 1_{[k/n,(k+1)/n]}$, and we send the reader to [5] for more details on Malliavin calculus.
Now, by self-similarity property of S and (1.1) we deduce for $k \leq l$

$$\begin{aligned}
n^{2H} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} &= n^{2H} \mathbb{E}\left(\left(S_{(k+1)/n} - S_{k/n} \right) \left(S_{(l+1)/n} - S_{l/n} \right) \right) \\
&= \mathbb{E}\left((S_{k+1} - S_k)(S_{l+1} - S_l) \right) \\
&= (k+l+1)^{2H} - \frac{1}{2}(k+l+2)^{2H} - \frac{1}{2}(k+l)^{2H} \\
&\quad - |l-k|^{2H} + \frac{1}{2}|l-1-k|^{2H} + \frac{1}{2}|l+1-k|^{2H} \\
&= \frac{1}{2}\rho(l-k) - \frac{1}{2}\rho(l+k+1),
\end{aligned}$$

where $\rho(r) = |r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H},\ r \in \mathbb{Z}$.

On the other hand, we have the relation, for any $k, l \in \mathbb{N}$

$$\begin{aligned}
\left| n^{2H} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \right| &= \frac{1}{2} |\rho(l-k) - \rho(l+k+1)| \\
&\leq |\rho(l-k)|.
\end{aligned} \tag{1.4}$$

Indeed, we shall prove that the application $r \geq 1 \mapsto |\rho(r)|$ is nonincreasing. We can write the function ρ as

$$\rho(r) = f(r) - f(r-1),$$

where $f(r) := (r+1)^{2H} - r^{2H}$. Hence, we have two cases:

Case 1. $H > \frac{1}{2}$:

$$f'(r) = 2H\left((r+1)^{2H-1} - r^{2H-1}\right) > 0 \Longrightarrow f \nearrow \Longrightarrow \rho(r) > 0, \forall r \geq 1;$$
$$f''(r) = 2H(2H-1)\left((r+1)^{2H-2} - r^{2H-2}\right) < 0 \Longrightarrow f' \searrow \Longrightarrow \rho' < 0$$
$$\Longrightarrow \rho \searrow \text{ on } [1, +\infty[.$$

Hence, $|\rho|$ is nonincreasing on $[1, +\infty[$.

Case 2. $H < \frac{1}{2}$:

$$f'(r) = 2H\left((r+1)^{2H-1} - r^{2H-1}\right) < 0 \Longrightarrow f \searrow \Longrightarrow \rho(r) < 0, \forall r \geq 1;$$
$$f''(r) = 2H(2H-1)\left((r+1)^{2H-2} - r^{2H-2}\right) > 0 \Longrightarrow f' \nearrow \Longrightarrow \rho' > 0$$
$$\Longrightarrow \rho \nearrow \text{ on } [1, +\infty[.$$

Hence, $|\rho|$ is nonincreasing on $[1, +\infty[$.

Moreover, $|\rho(0)| \geq |\rho(1)|$ for any $H \in [0, 1]$. As consequence, $|\rho|$ is nonincreasing on $[0, +\infty[$, combining this with the fact that ρ is symmetric, i.e. $\rho(r) = \rho(-r)$, we have for any $l, k \in \mathbb{N}$,

$$\begin{aligned} |\rho(l-k) - \rho(l+k+1)| &\leq |\rho(l-k)| + |\rho(l+k+1)| \\ &\leq 2|\rho(l-k)|. \end{aligned} \tag{1.5}$$

With inequality (1.4) in hand, it is now straightforward to obtain Theorem 1.2. Hence, we can write the quadratic variation of S, with respect to a subdivision $\pi_n = \{0 < \frac{1}{n} < \frac{2}{n} < \ldots < 1\}$ of $[0, 1]$, as follows

$$\begin{aligned} Z_n &= \sum_{k=0}^{n-1} \left[n^{2H} \left(S_{(k+1)/n} - S_{k/n} \right)^2 - 1 + \frac{1}{2}\rho(2k+1) \right] \\ &= \sum_{k=0}^{n-1} \left[n^{2H} \left(I_1(\delta_{k/n}) \right)^2 - 1 + \frac{1}{2}\rho(2k+1) \right] \\ &= I_2 \left(\underbrace{\sum_{k=0}^{n-1} n^{2H} \delta_{k/n}^{\otimes 2}}_{g_n} \right). \end{aligned} \tag{1.6}$$

Thus, we can write the correct renormalization of Z_n as follows,

$$V_n = \frac{Z_n}{\sqrt{Var(Z_n)}} = \frac{I_2(g_n)}{\sqrt{Var(Z_n)}}. \tag{1.7}$$

2. Proof of Theorem 1.2

In the first step, let study the convergence of $\frac{Var(Z_n)}{n}$ and $\frac{Var(Z_n)}{n\log n}$. Therefore, we have

$$\begin{aligned}
\frac{Var(Z_n)}{n} &= n^{-1}\mathbb{E}[I_2^2(g_n)] = 2\|g_n\|^2_{\mathfrak{H}^{\otimes 2}} \\
&= 2n^{4H-1}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}^{\otimes 2}, \delta_{l/n}^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}} = 2n^{4H-1}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}, \delta_{l/n}\rangle^2_{\mathfrak{H}} \\
&= \frac{1}{2n}\sum_{k,l=0}^{n-1}|\rho(l-k)-\rho(l+k+1)|^2 \\
&= \frac{1}{2n}\sum_{k,l=0}^{n-1}\rho^2(l-k) + \frac{1}{2n}\sum_{k,l=0}^{n-1}\rho^2(l+k+1) \\
&\quad - \frac{1}{n}\sum_{k,l=0}^{n-1}\rho(l-k)\,\rho(l+k+1).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{k,l=0}^{n-1}\rho^2(l+k+1) &= \sum_{l=0}^{n-1}\sum_{j=l+1}^{n+l}\rho^2(j) \\
&= \sum_{j=1}^{2n-1}\sum_{l=0\vee(j-n)}^{(j-1)\wedge(n-1)}\rho^2(j) \\
&= \sum_{j=1}^{n}\rho^2(j)\sum_{l=0}^{j-1} + \sum_{j=n+1}^{2n-1}\rho^2(j)\sum_{l=j-n}^{n-1} \\
&= \sum_{j=1}^{n} j\rho^2(j) + \sum_{j=n+1}^{2n-1}(2n-j)\rho^2(j) \\
&= A_{n,1} + A_{n,2}.
\end{aligned} \tag{2.1}$$

Assume now that $H < \frac{3}{4}$.

Thus, since $\sum_{r\in\mathbb{Z}} \rho^2(r) < \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{n} A_{n,1} = 0.$$

and

$$\frac{1}{n} A_{n,2} \leq \sum_{j=n+1}^{2n-1} \rho^2(j) \longrightarrow 0, \text{ as } n \to \infty.$$

Which leads to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k,l=0}^{n-1} \rho^2(l+k+1) = 0. \tag{2.2}$$

According to the proof of [2, Theorem 5.6], we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k,l=0}^{n-1} \rho^2(l-k) = \sum_{r\in\mathbb{Z}} \rho^2(r). \tag{2.3}$$

Finally, by (2.3) and (2.2), together with Cauchy Schwartz inequality

$$\begin{aligned} \frac{1}{n} \sum_{k,l=0}^{n-1} |\rho(l-k)|\ |\rho(l+k+1)| &\leq \left(\frac{1}{n} \sum_{k,l=0}^{n-1} \rho^2(l-k) \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{1}{n} \sum_{k,l=0}^{n-1} \rho^2(l+k+1) \right)^{\frac{1}{2}} \\ &\longrightarrow 0, \quad \text{ as } n \to 0. \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we conclude that

$$\lim_{n\to\infty} \frac{Var(Z_n)}{n} = \frac{1}{2} \sum_{r\in\mathbb{Z}} \rho^2(r). \tag{2.5}$$

Assume now $H = \frac{3}{4}$. Following similar argument as above we have

$$\begin{aligned} \frac{Var(Z_n)}{n\log(n)} &= \frac{1}{n\log(n)} \sum_{k,l=0}^{n-1} \rho^2(l-k) + \frac{1}{n\log(n)} \sum_{k,l=0}^{n-1} \rho^2(l+k+1) \\ &\quad - \frac{2}{n\log(n)} \sum_{k,l=0}^{n-1} \rho(l-k)\ \rho(l+k+1). \end{aligned}$$

Again from the proof of [2, Theorem 5.6], we have

$$\lim_{n\to\infty} \frac{1}{n\log(n)} \sum_{k,l=0}^{n-1} \rho^2(l-k) = \frac{9}{32}. \tag{2.6}$$

From other side, we have $\rho^2(r) \sim \frac{9}{64|r|}$ as $|r| \to \infty$. Implying in turn from (2.1) that

$$\begin{aligned}
\sum_{k,l=0}^{n-1} \rho^2(l+k+1) &= \sum_{j=1}^{n} j\rho^2(j) + \sum_{j=n+1}^{2n-1} (2n-j)\rho^2(j) \\
&\leq \frac{9}{64}\left(\sum_{j=1}^{n} 1 + \sum_{j=n+1}^{2n} (2n-j)\frac{1}{j}\right) \\
&\leq \frac{9}{64} n \left(1 + \sum_{j=n+1}^{2n} \frac{1}{j}\right) \\
&\leq \frac{9}{64} n \left(1 + \frac{n}{n+1}\right)
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{n\log(n)} \sum_{k,l=0}^{n-1} \rho^2(l+k+1) = 0. \tag{2.7}$$

Similarly to (2.4), we obtain by (2.6), (2.7) and Cauchy–Schwartz inequality

$$\lim_{n\to\infty} \frac{1}{n\log(n)} \sum_{k,l=0}^{n-1} \rho(l-k)\ \rho(l+k+1) = 0. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8) we deduce that

$$\frac{Var(Z_n)}{n\log(n)} = \frac{9}{64}. \tag{2.9}$$

Let us now derive the explicit bounds. From (1.6), multiplication formula (1.3) and the fact that $\mathbb{E}\|DZ_n\|^2_{\mathfrak{H}} = 2Var(Z_n)$, we obtain

$$\frac{1}{2}\|DV_n\|^2_{\mathfrak{H}} - 1 = \frac{2n^{4H}}{Var(Z_n)} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n}\widetilde{\otimes}\delta_{l/n})\ \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathfrak{H}}.$$

It follows by (1.4) that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{2}\,\|DV_n\|^2_{\mathfrak{H}} - 1\right)^2\right] \\
&= \frac{4n^{8H}}{Var^2(Z_n)}\mathbb{E}\left[\left(\sum_{k,l=0}^{n-1} I_2(\delta_{k/n}\widetilde{\otimes}\delta_{l/n})\ \langle\delta_{k/n},\delta_{l/n}\rangle_{\mathfrak{H}}\right)^2\right] \\
&= \frac{8n^{8H}}{Var^2(Z_n)}\sum_{i,j,k,l=0}^{n-1}\langle\delta_{i/n},\delta_{j/n}\rangle_{\mathfrak{H}}\ \langle\delta_{k/n},\delta_{l/n}\rangle_{\mathfrak{H}}\ \langle\delta_{i/n}\widetilde{\otimes}\delta_{j/n},\delta_{k/n}\widetilde{\otimes}\delta_{l/n}\rangle_{\mathfrak{H}^{\otimes 2}} \\
&= \frac{4n^{8H}}{Var^2(Z_n)}\sum_{i,j,k,l=0}^{n-1}\langle\delta_{i/n},\delta_{j/n}\rangle_{\mathfrak{H}}\ \langle\delta_{k/n},\delta_{l/n}\rangle_{\mathfrak{H}}\ \Big(\langle\delta_{i/n},\delta_{k/n}\rangle_{\mathfrak{H}}\ \langle\delta_{j/n},\delta_{l/n}\rangle_{\mathfrak{H}} \\
&\qquad\qquad + \langle\delta_{i/n},\delta_{l/n}\rangle_{\mathfrak{H}}\langle\delta_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}\Big) \\
&= \frac{8n^{8H}}{Var^2(Z_n)}\sum_{i,j,k,l=0}^{n-1}\langle\delta_{i/n},\delta_{j/n}\rangle_{\mathfrak{H}}\ \langle\delta_{i/n},\delta_{k/n}\rangle_{\mathfrak{H}}\ \langle\delta_{k/n},\delta_{l/n}\rangle_{\mathfrak{H}}\ \langle\delta_{j/n},\delta_{l/n}\rangle_{\mathfrak{H}} \\
&\le \frac{8n^2}{Var^2(Z_n)}\ \frac{1}{n^2}\sum_{i,j,k,l=0}^{n-1}|\rho(i-j)|\cdot|\rho(i-k)|\cdot|\rho(k-l)|\cdot|\rho(j-l)|.
\end{aligned} \tag{2.10}$$

Then, combining the convergence (2.5) and (2.9) together with inequality (2.10), the rest of the proof is now similar to the one of Theorem 5.6 in [2]. □

Remark 2.1. To retrieve the result of Tudor [6], we start from equality (2.9) and we follow the same steps as in the proof of Theorem 4.1 in [4].

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