ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 17, nº 1 (2010), p. 1-16.

<http://ambp.cedram.org/item?id=AMBP_2010__17_1_1_0>

Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **17**, 1-16 (2010)

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Abstract

In this paper we give a geometric cobordism description of differential integral cohomology. The main motivation to consider this model (for other models see [5, 6, 7, 8]) is that it allows for simple descriptions of both the cup product and the integration. In particular it is very easy to verify the compatibilty of these structures. We proceed in a similar way in the case of differential cobordism as constructed in [4]. There the starting point was Quillen's cobordism description of singular cobordism groups for a differential manifold X. Here we use instead the similar description of integral cohomology from [11]. This cohomology theory is denoted by $SH^*(X)$. In this description smooth manifolds in Quillen's description are replaced by so-called stratifolds, which are certain stratified spaces. The cohomology theory $SH^*(X)$ is naturally isomorphic to ordinary integral cohomology $H^*(X)$, thus we obtain a cobordism type definition of the differential extension of ordinary integral cohomology.

Une description géométrique de la cohomologie différentielle

Résumé

Nous donnons une définition géométrique de la cohomologie intégrale différentielle. Nous utilisons des cycles de cobordisme avec singularités, et des formes différentielles distributionnelles. Avec cette description, la construction de la multiplication et de l'intégration avec toutes les proprietés désirées est particulièrement simple.

1. Axioms of differential cohomology theories

To begin, let us recall what is meant by a *differential extension of the functor H^**, ordinary integral cohomology. Compare [2, Definition 1.1] for

Keywords: differential cohomology, smooth cohomology, geometric cycles, cobordism.
Math. classification: 55N20, 57R19.

the formal definition and the fundamental paper [8] for a general construction of such theories. We denote the differential k-forms by $\Omega^k(X)$, the subspace of closed forms by $\Omega^k_{cl}(X)$ and the forms with compact support by $\Omega^k_c(X)$. The map to de Rham cohomology, which we identify with real singular cohomology via the de Rham isomorphism, is denoted $Rham\colon \Omega^k_{cl}(X) \to H^k(X;\mathbb{R})$.

Definition 1.1. A *differential extension* of H^* is a functor $X \mapsto \hat{H}^*(X)$ from the category of smooth manifolds to $\mathbb{Z}$-graded groups together with natural transformations

(1) $R\colon \hat{H}^*(X) \to \Omega^*_{cl}(X)$ (curvature)

(2) $I\colon \hat{H}^*(X) \to H^*(X)$ (forget differential data)

(3) $a\colon \Omega^{*-1}(X)/\mathrm{im}(d) \to \hat{H}^*(X)$ (action of forms).

These transformations have to satisfy the following axioms.

(1) $R \circ a = d\colon \Omega^{*-1}(X) \to \Omega^*_{cl}(X)$.

(2) The following diagram commutes:

$$\begin{array}{ccc} \hat{H}^*(X) & \xrightarrow{I} & H^*(X) \\ \downarrow{\scriptstyle R} & & \downarrow \\ \Omega^*_{cl}(X) & \xrightarrow{Rham} & H^*(X;\mathbb{R}) \end{array} .$$

(3) For every smooth manifold X the sequence

$$H^{*-1}(X) \to \Omega^{*-1}(X)/\mathrm{im}(d) \xrightarrow{a} \hat{H}^*(X) \xrightarrow{I} H^*(X) \to 0$$

is exact, where the first map is the composition

$$H^{*-1}(X) \longrightarrow \frac{ker(d\colon \Omega^{*-1}(X) \to \Omega^*(X))}{\mathrm{im}(d)} \subseteq \frac{\Omega^{*-1}(X)}{\mathrm{im}(d)} .$$

To *have a compatible ring structure* means that $\hat{H}^*$ actually takes values in graded commutative rings (we denote the product by $\cup$), that R, I are ring maps, and that for all $x \in \hat{H}^*(X)$ and $\omega \in \Omega(X)^{*-1}/\mathrm{im}(d)$ we have

$$a(\omega) \cup x = a(\omega \wedge R(x)).$$

In this case we call the differential extension a *multiplicative differential extension.*

We will use a construction of $H^*(X)$ in terms of cobordism classes similar to Quillen's description of singular cobordism. The difference is that we replace manifolds by manifolds with singularities called stratifolds. In the next section we briefly introduce stratifolds and prove some basic properties needed for our construction. To distinguish the cohomology theory constructed from stratifolds from singular cohomology we denote it by $SH^*(X)$. The use of bordism with singularities to describe ordinary homology goes back to Sullivan, compare [1]. However, our singularities are quite different from those employed in that approach.

2. Stratifolds

Now we give a short introduction to stratifolds, where we are rather sketchy and refer the reader to [11] for details. A stratifold is a topological space $\mathbf{S}$ together with a subsheaf $\mathcal{C}$ of the sheaf of continuous functions, which in case of a smooth manifold plays the role of the sheaf of smooth functions. The space $\mathbf{S}$ and the sheaf $\mathcal{C}$ have to fulfill certain natural axioms, which in particular give a decomposition of $\mathbf{S}$ into smooth manifolds, the *strata* of $\mathbf{S}$. The top-dimensional stratum $\mathbf{S}^m$ is also called the regular part $\mathbf{S}^{reg}$. There is an obvious definition of *morphisms* between stratifolds, which are continuous maps $f\colon \mathbf{S} \to \mathbf{S}'$, which pull elements in $\mathcal{C}'$ back to elements in $\mathcal{C}$. A basic property which relates the strata $\mathbf{S}^k$ to the stratifold $\mathbf{S}$ is that for each $x \in \mathbf{S}^k$ there is an open neighborhood $U \subseteq \mathbf{S}$ of x and a retract $r\colon U \to U \cap \mathbf{S}^k$ which is a morphism. These are called local retracts. It is useful to note that a map is a morphism if and only if it is smooth on all strata and commutes with appropriate local retracts. We will only consider *regular stratifolds* which means that locally near each x in a stratum $\mathbf{S}^k$ the stratifold looks like $V \times F$ for some stratifold F and open subset V in $\mathbf{S}^k$. A stratifold is called *orientable* if the codimension 1 stratum is empty and the top stratum is orientable. Once an orientation on the top stratum is fixed we call such a stratifold an *oriented stratifold.* More generally, if $\mathbf{S}$ is a stratifold and X a smooth manifold, a continuous map $f\colon \mathbf{S} \to X$ is called orientable if the codimension 1 stratum of $\mathbf{S}$ is empty and if $f|^*_{\mathbf{S}^{reg}}\Lambda_X^{\mathbb{Z}/2}$ is isomorphic to $\Lambda_{\mathbf{S}^{reg}}^{\mathbb{Z}/2}$. Here, $\Lambda_X^{\mathbb{Z}/2}$ is the orientation covering of X. An *orientation of f* is then the choice of such an isomorphism. Note that it also gives an isomorphism between $\Lambda_{\mathbf{S}^{reg}}$ and $f|_{\mathbf{S}^{ref}}^{*}\Lambda_X$, where Λ_X is the real orientation bundle of X.

We also consider stratifolds with boundary. This is a pair of spaces $(\mathcal{T}, \partial\mathcal{T})$ together with stratifold structures on $\mathcal{T} - \partial\mathcal{T}$ and on $\partial\mathcal{T}$ and a germ of collars $c\colon \partial\mathcal{T} \times [0,1) \to \mathcal{T}$. Many basic properties of smooth manifolds generalize to stratifolds, like tangent spaces (the vector space of derivations of the germ of morphisms to $\mathbb{R}$), the differential of a morphism, differential forms (see below), Sard's theorem, approximation of continuous maps from a stratifold to a smooth manifold by morphisms and the transversality theorem for a map from a stratifold $\mathbf{S}$ to a smooth manifold X and a smooth map from a manifold $Y \to X$. For all this see [11].

Since we will use differential forms intensively we define them on stratifolds. A k-form ω on a stratifold $\mathbf{S}$ is a prescription which to each $x \in \mathbf{S}$ assigns an alternating k-form on $T_x\mathbf{S}$, which fulfills the following property:

(1) The restriction to each stratum is a differential k-form.

(2) For each $x \in \mathbf{S}^r$, the r-stratum, there is an open neighborhood $U \subseteq \mathbf{S}$ of x in $\mathbf{S}$ and a local retract
$$r\colon U \to U \cap \mathbf{S}^r,$$
such that
$$\omega|_U = r^*(\omega|_{U\cap\mathbf{S}^r}).$$

Here the pull back of a differential form under a morphism from a stratifold to a smooth manifold is defined as for smooth manifolds using the differential.

Lemma 2.1. *Let $\mathbf{S}$ be an m-dimensional stratifold and $f\colon \mathbf{S} \to X$ be a proper morphism to an n-dimensional smooth manifold. Then there is an open neighborhood V of the singular part such that for all $\omega \in \Omega_c^m(X;E)$ the pullback $f^*(\omega)$ vanishes on V. Here, $E \to X$ is any coefficient bundle. In other words, there is a fixed compact subset $K = \mathbf{S} \setminus V$ of the regular part such that supp$(f^*\omega) \subset K$.*

Proof. Let $x \in \mathbf{S}^r$ for $r < m$ be a point in the singular part of $\mathbf{S}$. Then there is an open neighborhood of x in $\mathbf{S}$ and a local retract
$$r\colon U \to U \cap \mathbf{S}^r,$$

such that the restriction of f to U factors over r and hence

$$f^*(\omega|_U) = r^*(f^*(\omega)|_{U\cap \mathbf{S}^r}).$$

Note that U and r are determined by f and can be chosen independent of ω.

Since $f^*(\omega)|_{U\cap\mathbf{S}^r} = 0$ for dimensional reasons we conclude that there is an open neighborhood V of the singular part $\mathbf{S} - \mathbf{S}^m$ such that $(f|_{\mathbf{S}^m})^*(\omega)$ vanishes on $\mathbf{S}^m \cap V$.

Now, let K be a compact set in X such that ω vanishes outside of K. Then, since f is proper, $f^{-1}(K)$ is a compact subset of $\mathbf{S}$. Since $\mathbf{S}^m - V$ is a closed subset of $\mathbf{S}^m$, the set $f^{-1}(K) \cap (\mathbf{S}^m - V)$ is compact, and $f^*(\omega)$ vanishes outside this set. □

As a consequence we can define the following integral. Let ω be a form with compact support on X with coefficients in Λ_X and $f\colon \mathbf{S} \to X$ a proper oriented morphism from an m-dimensional stratifold $\mathbf{S}$ to X. Then we define

$$\int_{\mathbf{S}} f^*(\omega) := \int_{\mathbf{S}^m} (f|_{\mathbf{S}^m})^*(\omega),$$

using the identification of $f|_{\mathbf{S}^m}^*\Lambda_X$ with $\Lambda_{\mathbf{S}^m}$ from the orientation of f.

Similarly, for a proper oriented morphism $F\colon \mathcal{T} \to X$ from an m-dimensional stratifold with boundary we define

$$\int_{\mathcal{T}} F^*(\omega) := \int_{\mathcal{T}^m} (F|_{\mathcal{T}^m})^*(\omega).$$

Stokes' Theorem applied to the top stratum gives us **Stokes' Theorem**: If $\omega \in \Omega_c^{m-1}(X;\Lambda_X)$, then

$$\int_{\mathcal{T}} d(F^*(\omega)) = \int_{\partial\mathcal{T}} F|_{\partial\mathcal{T}}^*(\omega). \tag{2.1}$$

Remark 2.2. Throughout this article, we will work with oriented morphisms and maps $f\colon \mathbf{S} \to X$ and with differential forms with values in the orientation bundle of X. In the special case that X is an oriented manifold, these are ordinary forms, and an orientation of f is precisely an orientation of $\mathbf{S}$. A reader not used to the more general setting might just assume the orientability and choice of orientations throughout. The passage to the general case is a mere technical point.

3. Differential cohomology via stratifolds

Now we define the cycles of our differential cohomology following the recipe as for for singular cobordism [4]. The starting point is the description of k-th ordinary integral cohomology of X as bordism classes of continuous oriented proper maps from oriented regular stratifolds $\mathbf{S}$ of dimension $(n-k)$ to X [11, Chapter 12]. Actually, compared to [11] one has to make the (obvious) modifications of passing from oriented manifolds to arbitrary manifolds by working with oriented maps. We call such oriented proper maps *k-cycles*. We denote this cohomology group $SH^k(X)$, the *stratifold cohomology* of X. Since every proper map is homotopic via proper maps to a morphism [11] we will always assume that f is a morphism. Note that a proper homotopy can be considered as a special case of a bordism.

Let $f\colon \mathbf{S} \to X$ be a k-cycle, i.e. a proper oriented morphism from a regular $(n-k)$-dimensional stratifold to X. Then we construct a current $T(\mathbf{S}, f)$, i.e. an element in the topological dual space $\Omega_c^{n-k}(X;\Lambda_X)^*$ of continuous linear maps from $\Omega_c^{n-k}(X;\Lambda_X)$ to $\mathbb{R}$ as follows:

$$\omega \mapsto \int_{\mathbf{S}} f^*(\omega).$$

We have an injective map

$$j\colon \Omega^k(X) \to \Omega_c^{n-k}(X;\Lambda_X)^*$$

given by

$$j(\alpha) := \{\omega \mapsto \int_X \alpha \wedge \omega\} \ .$$

After these preparations we define a cycle for k-th differential cohomology:

Definition 3.1. A **differential cycle of degree** k is a triple

$$(\mathbf{S}, f, \alpha)$$

where $(\mathbf{S}, f)$ is as above given by a proper oriented morphism $f\colon \mathbf{S} \to X$, with $\mathbf{S}$ an $(n-k)$-dimensional oriented regular stratifold, and $\alpha \in \Omega_c^{n-(k+1)}(X;\Lambda_X)^*/\mathrm{im}(d^*)$, such that $T(\mathbf{S}, f) - d^*(\alpha)$ is in the image of j. The sum of two differential cycles is defined by disjoint union. The negative of a cycle $(\mathbf{S}, f, \alpha)$ is $(\mathbf{S}, f^-, -\alpha)$ where f^- is f with the reverse orientation.

We define cobordisms of cycles for differential cohomology as follows: if $\mathcal{T}$ is a stratifold with boundary $\partial\mathcal{T} = \mathbf{S}$ and $F\colon \mathcal{T} \to X$ is a proper oriented morphism, we say that $(\mathcal{T}, F)$ is a zero bordism of

$$(\partial W, F|_{\partial\mathcal{T}}, T(\mathcal{T}, \mathcal{F})),$$

and a bordism between two cycles is a zero bordism of the difference. The only thing which is a bit special is that for the map F on a bordism $\mathcal{T}$ to X we require that F commutes with the collar $c\colon \partial\mathcal{T} \times [0, \epsilon) \to \mathcal{T}$ for some appropriate $\epsilon > 0$. This allows to glue bordisms (using the collars) in a compatible way. Note also that $dT(\mathcal{T}, \mathcal{F}) - T(\partial\mathcal{T}, F|_{\partial\mathcal{T}}) = 0$ (in particular is in the image of j) by (2.1):

$$d^*T(\mathcal{T}, \mathcal{F})(\omega) = \int_{\mathcal{T}} F^*(d\omega) = \int_{\mathcal{T}} dF^*\omega = \int_{\partial\mathcal{T}} F|_{\partial\mathcal{T}}^*\omega = T(\partial\mathcal{T}, F|_{\partial\mathcal{T}})(\omega) \ .$$

We denote the corresponding bordism group by

$$\hat{SH}^k(X).$$

We call this group the *differential stratifold cohomology of X*.

As in [4, Definition 4.9] we define the maps

$$\begin{aligned} R\colon \hat{SH}^*(X) \to \Omega^*(X); &\quad [\mathbf{S}, f, \alpha] \mapsto j^{-1}(T(\mathbf{S}, f) - d^*(\alpha)), \\ a\colon \Omega^{*-1}(X)/\mathrm{im}(d) \to \hat{SH}^*(X); &\quad \alpha \mapsto [\emptyset, -j(\alpha)], \\ I\colon \hat{SH}^*(X) \to SH^*(X); &\quad [\mathbf{S}, f, \alpha] \mapsto [\mathbf{S}, f]. \end{aligned}$$

The proof that these maps are well defined is literally the same as in the case, where we have smooth manifolds instead of stratifolds [4, Lemma 4.10], since the basic ingredient, Stokes' Theorem, is available.

The next aim is to construct induced maps for a smooth map $g\colon Y \to X$. The basic idea is that, if $(\mathbf{S}, f, \alpha)$ is a differential cycle in X then we can after a proper homotopy of f (which we can consider as a special case of a bordism) assume that g is transversal to f. Then, as in [11], one can consider the pull back of $\mathbf{S}$ giving a cycle $(g^*(\mathbf{S}), F)$ in Y. We denote the canonical map $g^*(\mathbf{S}) \to \mathbf{S}$ by G. As in [4, Section 4.2.6], the orientation of f induces an orientation of $F = g^*f$.

To extend this pull back to a differential cycle by pulling back α, one has the same situation as in [4], i.e. one has to pull back α along g. Recall that this is only possible if $WF(\alpha) \cap N(g) = \emptyset$. Here $WF(\alpha) \subseteq T^*X$

denotes the wave front set of the distributional form α, and $N(f) \subseteq T^*X$ is the normal set to f. The wave front set of a distributional form α on X is a conical subset of T^*X which measures the locus and the directions of the singularities of α. For a precise definition and for the properties of distributions using the wave front set needed we refer to [9, Section 8]. Compare [4, Section 4.2.6] for the notation and more details. In terms of normal sets transversality of f and g can be expressed as $N(f)\cap N(g) = \emptyset$ (where $N(f)$ is the normal set of the restriction of f to the top stratum). Hence $g^*\alpha$ is defined if $WF(\alpha) \subseteq N(f)$. In order to match this condition we use the freedom to change α by an element in the image of d^*.

We observe that by definition and by Lemma 2.1 $T(\mathbf{S}, f) = (f|_{\mathbf{S}}^{reg})_!(\rho)$ where $\rho\colon \mathbf{S}^{reg} \to \mathbb{R}$ is a smooth compactly supported cutoff function which is zero in a neighborhood of the singular set of $\mathbf{S}$, and which is identically 1 on the support of any $f^*\omega$ for $\omega \in \Omega_c^{n-k}(X)$.

Consequently, the construction of T is described entirely in the context of smooth manifolds, smooth maps and smooth forms; as in the context of [4]. Now, the arguments there, in particular [4, Lemma 4.12] literally apply in our situation to show that we can change α to a representative α' such that the wave front set of α' satisfies $WF(\alpha') \subseteq N(f)$. Then $g^*(\alpha')$ is a well defined distribution and we can make the following definition.

Definition 3.2. We set $g^*[\mathbf{S}, f, \alpha] := [g^*(\mathbf{S}), g^*f, g^*(\alpha)]$, where we choose a representative such that f is transversal to g and $WF(\alpha) \subseteq N(f)$.

The proof that this induced map is well defined and functorial is the same as in the case where $\mathbf{S}$ is a smooth manifold. Naturality of the transformations R, I and a is checked in a straigtforward way.

4. Ring structure on differential stratifold cohomology

Definition 4.1. We define the $\times$*-product* of classes $[\mathbf{S}, f, \alpha] \in \hat{SH}^k(X)$ and $[\mathbf{S}', f', \alpha'] \in \hat{SH}^r(X')$ with values in $\hat{SH}^{k+r}(X \times X')$ as

$$[\mathbf{S}, f, \alpha] \times [\mathbf{S}', f', \alpha'] := \\ [(-1)^{kr}\mathbf{S} \times \mathbf{S}', f \times f', (-1)^k R([\mathbf{S}, f, \alpha]) \times \alpha' + \alpha \times T(\mathbf{S}', f')].$$

The sign $(-1)^{kr}$ comes from the fact that in contrast to [4] we work with orientations of the tangent bundle, whereas there normal orientations are used. This orientation convention is in agreement with that in [11].

The proof of the following fundamental properties is the same as in [4], except that for the difference of signs one has to use the arguments in [11].

Proposition 4.2. *The product is well defined, associative, graded commutative, and natural.*

Using the map induced by the diagonal $\Delta\colon X \to X \times X$ we define the cup product.

Definition 4.3. For $a \in \hat{SH}^k(X)$ and $b \in \hat{SH}^r(X)$ we define

$$a \cup b := \Delta^*(a \times b).$$

By a straightforward calculation we see

Proposition 4.4. *The maps R and I are multiplicative and*

$$a(\beta) \cup [\mathbf{S}, f, \alpha] = a(\beta \wedge R([\mathbf{S}, f, \alpha])).$$

5. Differential stratifold cohomology as differential extension of ordinary cohomology

We have constructed a differential extension of the SH^*-homology theory as developped in [11]. However, we argue that $\hat{SH}^*$ is a differential extension of ordinary integral cohomology. For this, we have to observe that the corresponding stratifold cohomology $SH^*(X)$ is naturally isomorphic to ordinary integral cohomology $H^*(X)$. The reason is that this functor fulfills the homotopy axiom (obvious) and that one has a natural Mayer-Vietoris sequence. This was proven in [11] for the case where X is oriented, but the same proof works in the non-oriented case. Now we apply [12] or [3, Section 7]. There, it is proven that a cohomology theory on smooth manifolds is naturally isomorphic to ordinary integral cohomology if it satisfies the homotopy axiom, the Mayer-Vietoris sequence and if the cohomology groups of a 0-dimensional manifold X are the direct product of the cohomology groups of the points in X. Moreover, it was shown in [12] that the natural isomorphism can be chosen to preserve the ring structure. Thus we can identify the multiplicative cohomology theories stratifold cohomology and ordinary integral cohomology.

We now formulate our main theorem.

Theorem 5.1. *Our construction $\hat{SH}^*$ defines a multiplicative differential extension of ordinary cohomology with integer coefficients in the sense of*

Definition 1.1. By [13, Theorem 1.1] *or* [3] *it follows that our theory is uniquely naturally equivalent to any other of the many models for this extension, in particular to Cheeger-Simons differential characters of* [5], *as described in* [13].

This is actually even true as a multiplicative extension: by [13, Theorem 1.2] *or* [3], *there is only one multiplicative differential extension of integral cohomology.*

Proof. Our setup is not quite identical to the one of [13], as there it is required that the kernel of $R\colon \hat{HS}^*(M) \to \Omega^*(M)$ is naturally identified with $H^{*-1}(M;\mathbb{R}/\mathbb{Z})$.

As this is not given, we use instead the method of proof of [3]. There, a natural transformation Φ between any two differential extensions $\hat{H}$ and $\hat{H}'$ of integral cohomology is constructed by making a universal choice. It is shown that Φ is additive and unique in even degrees because $H^k(pt) = 0$ for k odd. The same method implies immediately that the transformation is additive and unique in all degrees except for $* = 1$, as $H^{*-1}(pt) = 0$ except if $* = 1$.

Next, the method shows that for the transformation Φ there is

$$c \in \mathbb{R}/\mathbb{Z} = H^0(S^1 \times S^1; \mathbb{R}/\mathbb{Z}) = H^0(K(\mathbb{Z},1) \times K(\mathbb{Z},1); \mathbb{R}/\mathbb{Z})$$

such that for two classes x, y of degree 1 we have

$$\Phi(x+y) = \Phi(x) + \Phi(y) + a(c).$$

However, we can modify Φ by setting $\Phi'(x) := \Phi(x) - a(c)$ if x is of degree 1 and $\Phi'(x) = \Phi(x)$ otherwise. Then we conclude that Φ' is the unique additive transformation between the two differential extensions of integral cohomology satisfying our axioms.

The methods of [3] finally show that there is at most one ring structure on a differential extension of integral cohomology. Again, this follows because of the vanishing of $H^*(pt;\mathbb{R}/\mathbb{Z})$ if $* \neq 0$, together with the consideration of distributivity for products of classes of degree zero and degree one. □

Proposition 5.2. *The flat theory corresponding to* $\hat{SH}^*$, *i.e. the functor* $U^*(X) := \ker(R\colon \hat{SH}^*(X) \to \Omega^*(X))$ *is naturally isomorphic to the functor* $H^{*-1}(X;\mathbb{R}/\mathbb{Z})$.

In particular, $\hat{SH}^*$ *satisfies the setup of* [13].

Proof. This is a special case of [3, Theorem 7.12]. □

6. Integration along the fiber

Let $p\colon X \to B$ be the projection map of a locally trivial fiber bundle. To define "integration along the fibers" of p for a cohomology theory E, one has to choose an E-orientation for p (which might not exist).

For a general cohomology theory E and a differential extension $\hat{E}$, usually one has to choose further data in addition to an ordinary E-orientation to prescribe an $\hat{E}$-orientation, compare e.g. [2, Section 3.1] or [4, Section 4.3.7].

An exception is ordinary integral cohomology H, as already observed in [5, 6, 10]. Here, an ordinary orientation determines canonically a differential orientation and a differential integration map. In our model of differential cohomology using stratifolds, the definition of the integration map as well as the proof of its main properties is particularly simple. More precisely, we will prove the following theorem.

Theorem 6.1. *Given a locally trivial smooth fiber bundle with closed d-dimensional fibers which is oriented for ordinary integral cohomology, there is a canonical integration for differential stratifold cohomology*

$$\hat{p}_!\colon S\hat{H}^*(E) \to S\hat{H}^{*-d}(B).$$

This has the following properties

(1) *The differential integration is compatible with integration of forms and of ordinary integral cohomology classes, i.e. the following diagrams commute:*

$$\begin{array}{ccccc}
\Omega^{*-1}(E) & \xrightarrow{\alpha} & S\hat{H}^*(E) & \xrightarrow{I} & H^*(E) \\
\big\downarrow \int_{E/B} & & \big\downarrow \hat{p}_! & & \big\downarrow p_! \\
\Omega^{*-1-d}(B) & \xrightarrow{\alpha} & S\hat{H}^{*-d}(B) & \xrightarrow{I} & H^{*-d}(B).
\end{array}$$

$$\begin{array}{ccc}
S\hat{H}^*(E) & \xrightarrow{R} & \Omega^*_{cl}(E) \\
\big\downarrow \hat{p}_! & & \big\downarrow \int_{E/B} \\
S\hat{H}^{*-d}(B) & \xrightarrow{R} & \Omega^{*-d}_{cl}(B).
\end{array}$$

(2) *Naturality: If*

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ \downarrow q & & \downarrow p \\ C & \xrightarrow{u} & B \end{array}$$

is a cartesian diagram, then

$$\begin{array}{ccc} \hat{SH}^*(E) & \xrightarrow{\hat{v}^*} & \hat{SH}^*(F) \\ \downarrow \hat{p}_! & & \downarrow \hat{q}_! \\ \hat{SH}^{*-d}(B) & \xrightarrow{\hat{u}^*} & \hat{SH}^{*-d}(C) \end{array}$$

commutes, where we use on q the pullback of the orientation on p.

(3) *Functoriality: if $r\colon D \xrightarrow{q} E \xrightarrow{p} B$ is a composition of two smooth oriented fiber bundles (with composed orientation), then*

$$\hat{r}_! = \hat{p}_! \circ \hat{q}_!.$$

(4) *Projection formula: if $x \in \hat{SH}^k(B)$, $y \in \hat{SH}^m(E)$ then*

$$\hat{p}_!(\hat{p}^*(x) \cup y) = (-1)^{kd} x \cup \hat{p}_!(y).$$

(5) *On a cycle $x = [\mathbf{S}, f, \alpha]$,*

$$\hat{p}_!(x) = [\mathbf{S}, p \circ f, \int_{E/B} \alpha := p_* \alpha],$$

where by definition $\int_{E/B} \alpha(\omega) = \alpha(p^*\omega)$ and where we equip $p \circ f$ with the composed orientation.

In the remainder of this section we prove Theorem 6.1. Note that (5) actually is a definition of $\hat{p}_!$ which by construction is compatible with the addition in $\hat{SH}^*$. However, we have to check that it is well defined. By compatibility with addition, for a cycle $(\mathbf{S}, f, \alpha)$ representing zero, i.e. $\mathbf{S} = \partial \mathcal{T}$, $f = (F\colon \mathcal{T} \to X)|_{\mathbf{S}}$, $\alpha = T(\mathcal{T}, F)$, we have to check that this is mapped to zero under $\hat{p}_!$. However, $[\mathbf{S}, p \circ f, p_* \alpha]$ is precisely the boundary of $[\mathcal{T}, p \circ F, T(\mathcal{T}, F)]$, as by definition T is the pushdown of the fundamental class, and this is natural for composition, so 5 indeed defines $\hat{p}_!$.

Next we prove the compatibilities of (1). Here, we have for a class of cycles $x = [\mathbf{S}, f, \alpha]$ and a form $\omega \in \Omega^*(E)$:

$$p_! I(x) = p_! [f\colon \mathbf{S} \to E] = [p \circ f\colon \mathbf{S} \to B] = I(\hat{p}_!(x)),$$

$$\int_{E/B} R(x) = \int_{E/B} (T(\mathbf{S}, f) - d^*\alpha) = T(\mathbf{S}, p \circ f) - d^* \int_{E/B} \alpha = R(\hat{p}_!(x)),$$

$$\hat{p}_! \alpha(\omega) = \hat{p}_! [\emptyset, -\omega] = [\emptyset, -\int_{E/B} \omega] = \alpha(\int_{E/B} \omega).$$

To prove naturality (2) with respect to pullback in a diagram

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{u} & B \end{array}$$

choose (without loss of generality) the cycle $x = [\mathbf{S}, f, \alpha]$ such that f is transversal to v and $WF(\alpha) \subseteq N(f)$. Since p and q are submersions, the composition $p \circ f$ is transversal to u. Moreover (with the notation $WF_y(\beta) = WF(\beta) \cap T^*_y B$ and similar for normal sets)

$$\begin{aligned} WF_y(\int_{E/B} \alpha) \subseteq \bigcup_{x \in p^{-1}(y)} (dp^*_x)^{-1}(WF_x(\alpha)) \\ \subseteq \bigcup_{x \in p^{-1}(y)} (dp^*_x)^{-1} N_x(f) \subseteq N_y(p \circ f), \end{aligned}$$

so that $u^*(\hat{p}_!(x))$ is defined using the cycle $(\mathbf{S}, p \circ f, \int_{E/B} \alpha)$.

Then,

$$\begin{aligned} \hat{q}_!(v^*(x)) = \hat{q}_!(v^*\mathbf{S}, v^* f, v^*\alpha) = (v^*\mathbf{S}, q \circ v^* f, \int_{F/C} v^*\alpha) \\ = (u^*\mathbf{S}, u^*(p \circ f), u^* \int_{E/B} \alpha) = u^*(\hat{p}_!(x)). \end{aligned}$$

Here we use that pullback and pushdown of distributional forms in a cartesion square are compatible (which follows from the corresponding statement for differential forms by continuity, as the pullback is extended from differential to distributional forms by continuity).

It remains to prove the projection formula (4). This we do in two steps. First we consider the projection $\mathrm{id} \times p\colon B \times E \to B \times B$. If $x = [\mathbf{S}, f, \alpha] \in$

$\hat{SH}^k(B)$ and $y = [\tilde{\mathbf{S}}, \tilde{f}, \tilde{\alpha}] \in \hat{SH}^m(E)$ then

$$\widehat{\mathrm{id} \times p_!}(x \times y) =$$
$$(-1)^{km}[\mathbf{S} \times \tilde{\mathbf{S}}, f \times (p \circ \tilde{f}), (-1)^k R(S, f, \alpha) \times \int_{E/B} \tilde{\alpha} + \alpha \times T(\tilde{S}, p \circ \tilde{f})]$$
$$= (-1)^{kd} x \times \hat{p}_!(y) \ .$$

Secondly, using the diagonal inclusion $B \to B \times B$ we pull back the whole situation to $p\colon E \to B$ and use the naturality of the differential integration with respect to pullback. Observe that the natural map $E \xrightarrow{p \times \mathrm{id}} B \times E$ which lifts the diagonal map $B \to B \times B$ factors as $E \xrightarrow{diag} E \times E \xrightarrow{p \times \mathrm{id}} B \times E$. Recall finally that the cup product in differential cohomology is defined as the pullback of the exterior product with respect to the diagonal map. We obtain

$$\hat{p}_!(p^* x \times y) = (-1)^{kd} x \cup \hat{p}_!(y).$$

7. Transformations between differential cohomology

The construction of differential cohomology via stratifolds, i.e. generalized oriented manifolds, immediately allows to define a lift of the orientation transformation from a bordism theory which is naturally equipped with an H-orientation to the corresponding differential extensions of the present article and of [4] (provided the characters are chosen appropriately).

As an example, take the canonical orientation from complex bordism to integral homology. As character on complex bordism, use this map composed with the natural map from integral cohomology to cohomology with real coefficients:

$$MU^*(X) \xrightarrow{ori} H^*(X; \mathbb{Z}) \xrightarrow{i_*} H^*(X; \mathbb{R}).$$

In the stratifold description of integral cohomology and for X an oriented manifold, the transformation sends $[E \to X] \in MU^*(X)$ to $[E \to X]$, where the complex oriented manifold E with proper map to X is interpreted as a stratifold with proper morphism to X. A bordism of manifolds over X is also a bordism of stratifolds, so this map is well defined; it is an easy exercise that this indeed describes the natural transformation dual to taking the fundamental class of a stable almost complex manifold.

We immediately get a differential lift

$$\hat{MU}^*(X) \to \hat{HS}^*(X)$$

by mapping the $\hat{MU}^*(X)$-class $[E, f, \alpha]$ to the $\hat{HS}^*(X)$-class $[E, f, \alpha]$. Obviously this is compatible with the curvature homomorphisms as well as with the passage to the underlying homology theories and the transformation *ori*, and with the action of differential forms.

Acknolodgements: Thomas Schick was partially supported by the Courant Research Center "Higher order structures in Mathematics" within the German initiave of excellence.

References

[1] N. A. Baas – "On formal groups and singularities in complex cobordism theory", *Math. Scand.* **33** (1973), p. 303–313 (1974).

[2] U. Bunke & T. Schick – "Smooth K-theory", arXiv:0707.0046, to appear in *From Probability to Geometry.* Volume dedicated to J.-M. Bismut for his 60th birthday (X. Ma, editor), Asterisque 327 & 328, 2009.

[3] ———, "Uniqueness of smooth extensions of generalized cohomology theories", arXiv.org:0901.4423, to appear in Journal of Topology, 2010.

[4] U. Bunke, T. Schick, I. Schröder & M. Wiethaup – "Landweber exact formal group laws and smooth cohomology theories", *Algebr. Geom. Topol.* **9** (2009), no. 3, p. 1751–1790.

[5] J. Cheeger & J. Simons – "Differential characters and geometric invariants", in *Geometry and topology (College Park, Md., 1983/84)*, Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, p. 50–80.

[6] J. L. Dupont & R. Ljungmann – "Integration of simplicial forms and Deligne cohomology", *Math. Scand.* **97** (2005), no. 1, p. 11–39.

[7] R. Harvey & B. Lawson – "From sparks to grundles—differential characters", *Comm. Anal. Geom.* **14** (2006), no. 1, p. 25–58.

[8] M. J. Hopkins & I. M. Singer – "Quadratic functions in geometry, topology, and M-theory", *J. Differential Geom.* **70** (2005), no. 3, p. 329–452.

[9] L. Hörmander – *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition.

[10] M. Köhler – "Integration in glatter Kohomologie", Tech. report, Georg-August-Universität Göttingen, 2007, Diplomarbeit.

[11] M. Kreck – "Differential algebraic topology", Preprint, available at http://www.hausdorff-research-institute.uni-bonn.de/kreck, 2007.

[12] M. Kreck & W. Singhoff – "Homology and cohomology theories on manifolds", to appear in Münster Journal of Mathematics, 2010.

[13] J. Simons & D. Sullivan – "Axiomatic characterization of ordinary differential cohomology", *J. Topol.* **1** (2008), no. 1, p. 45–56.

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