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Abstract

In the first part of the paper, we define an approximated Brunn-Minkowski inequality which generalizes the classical one for metric measure spaces. Our new definition, based only on properties of the distance, allows also us to deal with discrete metric measure spaces. Then we show the stability of our new inequality under convergence of metric measure spaces. This result gives as corollary the stability of the classical Brunn-Minkowski inequality for geodesic spaces. The proof of this stability was done for related inequalities (curvature-dimension inequality, metric contraction property) but not for the Brunn-Minkowski one, as far as we know.
In the second part of the paper, we show that every metric measure space satisfying the classical Brunn-Minkowski inequality can be approximated by discrete metric spaces with some approximated Brunn-Minkowski inequalities.

Une version discrète de l'inégalité de Brunn-Minkowski et sa stabilité

Résumé

Dans une première partie, nous définissons une inégalité de Brunn-Minkowski approchée qui généralise l'inégalité de Brunn-Minkowski classique aux cas des espaces métriques mesurés. Cette nouvelle définition s'applique aussi aux espaces métriques mesurés discrets. Nous montrons alors la stabilité de cette nouvelle inégalité sous la convergence d'espaces métriques mesurés. Nous obtenons en corollaire la stabilité de l'inégalité de Brunn-Minkowski classique pour les espaces géodesiques. Cette stabilité a déjà été montrée pour d'autres inégalités (inégalité de courbure-dimension, propriété de contraction de la mesure) mais, autant que nous sachions, pas pour celle de Brunn-Minkowski.
Dans la seconde partie de l'article, nous montrons que tout espace métrique mesuré satisfaisant l'inégalité de Brunn-Minkowski classique peut être approché par des espaces métriques discrets vérifiant certaines inégalités de Brunn-Minkowski approchées.

Keywords: Brunn-Minkowski inequality, metric measure spaces, $\mathbb{D}$-convergence, Ricci curvature, discretization.
Math. classification: 52A40,53C21.

1. Introduction

Let us recall some facts about the Brunn-Minkowski inequality. It was first established in $\mathbb{R}^n$ for convex bodies by Brunn and Minkowski in 1887 (for more details about the inequality and its birth, one can refer to the great surveys [1, 5] and the references therein). It can be read as follows: if K and L are convex bodies (compact convex sets with non-empty interior) of $\mathbb{R}^n$ and $0 \leq t \leq 1$ then

$$V_n((1-t)K + tL)^{1/n} \geq (1-t)V_n(K)^{1/n} + tV_n(L)^{1/n} \tag{1.1}$$

where V_n is the Lebesgue measure on $\mathbb{R}^n$ and $+$ the Minkowski sum which is given by

$$A + B = \{a + b; a \in A, b \in B\}$$

for A and B two sets of $\mathbb{R}^n$. Equality holds if and only if K and L are equal up to translation and dilation.

The Brunn-Minkowski inequality is a very powerful inequality with a lot of applications. For example it implies very quickly the isoperimetric inequality for convex bodies in $\mathbb{R}^n$ which reads

$$\left(\frac{V_n(K)}{V_n(B_n)}\right)^{1/n} \leq \left(\frac{s_n(K)}{s_n(B_n)}\right)^{1/(n-1)} \tag{1.2}$$

where K is a convex body of $\mathbb{R}^n$, B_n the unit ball of $\mathbb{R}^n$ and s_n the surface area measure; with equality if and only if K is a ball.

The Brunn-Minkowski inequality is valid not only for convex bodies but also for all non-empty compact sets and even for all non-empty Borel sets of $\mathbb{R}^n$. One way to prove it is to establish the Prekopa-Leindler functional inequality (see [1]) which applied to characteristic functions of sets gives the multiplicative Brunn-Minkowski inequality

$$V_n((1-t)K + tL) \geq V_n(K)^{1-t}V_n(L)^t \tag{1.3}$$

where K and L are two Borel sets of $\mathbb{R}^n$. By homogenity of the Lebesgue measure V_n, it can be shown that this a priori weaker inequality is in fact equivalent to the n-dimensional one (1.1).

The Brunn-Minkowski inequality has a very strong geometric content and it is natural to ask on which more general spaces than $\mathbb{R}^n$ the inequality can be extended.

A first answer would be to change the measure: a log-concave measure on $\mathbb{R}^n$ satisfies the multiplicative Brunn Minkowski inequality.

But to be able to get away from $\mathbb{R}^n$, we have to generalize the Minkowski sum. This can be done on geodesic spaces with the notion of s-intermediate sets. This notion is very useful in optimal transportation on geodesic spaces (see [3] for length and geodesic spaces and [11] for optimal transportation) and then appeared in some studies of optimal transportation on Riemmannian manifolds (see [4]). Following an idea of this last paper, for two subsets C_0 and C_1 of a metric space X and $s \in [0,1]$ we define the s-intermediate set between C_0 and C_1 by

$$C_s = \left\{ x \in X; \exists (c_0, c_1) \in C_0 \times C_1, \begin{array}{rcl} d(c_0, x) & = & sd(c_0, c_1) \\ d(x, c_1) & = & (1-s)d(c_0, c_1) \end{array} \right\}. \tag{1.4}$$

This is the set spanned by all geodesics going from a point in C_0 to a point in C_1. On $\mathbb{R}^n$, this set is exactly the Minkowski sum of $(1-s)C_0$ and sC_1. The authors in [4] use it only for a Riemannian manifold but it makes sense for all metric spaces even if it is interesting only for geodesic spaces. In this context we will say a metric measure space (X, d, m) satisfies the N-dimensionnal Brunn-Minkowski inequality if

$$m^{1/N}(C_s) \geq (1-s)\, m^{1/N}(C_0) + s\, m^{1/N}(C_1) \tag{1.5}$$

for all $0 \leq s \leq 1$ and all C_0, C_1 non-empty compact sets of X. We will refer in the sequel at (1.5) as the "classical" N-dimensionnal Brunn-Minkowski inequality. It is proven in [4] that a Riemannian manifold M of dimension n whose Ricci curvature is always non negative satisfies (1.5) with dimension $N = n$ and with its canonical volume measure, i.e.

$$vol(C_s)^{1/n} \geq (1-s) vol(C_0)^{1/n} + s\, vol(C_1)^{1/n} \tag{1.6}$$

for all non-empty compact sets C_0 and C_1 of M and with vol the canonical volume measure of the Riemannian manifold.

Recently, there have been a lot of works on the geometry of metric measure spaces. Lott-Villani and Sturm have given independently a synthetic treatment of metric spaces having Ricci curvature bounded below (see [7, 9, 10]). All these works were motivated by the result of precompactness of Gromov: the class of Riemannian manifolds of dimension n and Ricci curvature bounded below by some constant k is precompact in the Gromov-Hausdorff metric. The notion of lower bound on Ricci curvature they develop for metric spaces generalizes the Riemannian one and is stable under Gromov-Hausdorff convergence. Their definition relies on convexity properties of relative entropy on the Wasserstein space of probability measures and is linked with optimal transportation.

Sturm ([10]) in this context defines a Brunn-Minkowski inequality with curvature bounded below by k. The meaning of this inequality may not be totally satisfactory. Indeed the inequality involves a parameter Θ which is $\inf_{c_0 \in C_0, c_1 \in C_1} d(c_0, c_1)$ or $\sup_{c_0 \in C_0, c_1 \in C_1} d(c_0, c_1)$ whether the curvature is non-negative or negative. It corresponds to the minimal or maximal length of geodesics between the two compact sets C_0 and C_1. But this Brunn-Minkowski inequality is a direct consequence of the dimension-curvature condition $CD(k, N)$ defined in this paper and it gives all the geometric consequences of the theory like a Bishop-Gromov theorem on the growth of balls.

There is another weak concept of lower bound of Ricci curvature which is known as metric contraction property (see [8, 10, 6]) and which is implied by this last Brunn-Minkowski inequality at least in the case of curvature bounded below by 0 and the $m \otimes m$ a.s. existence and uniqueness of geodesics between two points of the metric measure space (X, m, d).

As far as we know the stability of Brunn-Minkowski inequality (1.5) was not proven yet. This is the main result of our paper (corollary 2.4).

In section 2, we prove the result for the classical Brunn-Minkowski inequality (1.5) (which corresponds to the inequality defined by Sturm with curvature bounded below by 0) and then explain in a remark how we can extend our result to the general case with curvature bounded below by k. We introduce an approximated Brunn-Minkowski inequality which will be essential in the core of the proof of our main result and allows us to deal with discrete metric spaces.

In section 3, we show that every metric measure space satisfying a classical Brunn-Minkowski inequality can be approximated by discrete metric spaces with some approximated Brunn-Minkowski inequalities.

To avoid issues with sets of zero measure we will work only with metric spaces (X, d, m) where (X, d) is Polish and m a Borel measure on (X, d) with full support, i.e. that charges every open ball of X.

2. Stability of Brunn-Minkowski inequality

Definition 2.1. Given $h \geq 0$ and $N \in \mathbb{N}, N \geq 1$, we say that a metric measure space (X, d, μ) satisfies the h-Brunn-Minkowski inequality of dimension N denoted by $BM(N, h)$ if for each pair (C_0, C_1) of non-empty compact subsets of X, we have:

$$\forall s \in [0,1],\ \mu^{1/N}({C_s}^h) \geq (1-s)\,\mu^{1/N}(C_0) + s\,\mu^{1/N}(C_1) \tag{2.1}$$

where C_s^h is defined by

$$\left\{ x \in X; \exists (x_0, x_1) \in C_0 \times C_1,\ \begin{array}{ccc} |d(x_0,x) - sd(x_0,x_1)| & \leq & h \\ |d(x,x_1) - (1-s)d(x_0,x_1)| & \leq & h \end{array} \right\}.$$

We call the set C_s^h the set of h(-approximated) s-intermediate points between C_0 and C_1. The idea of introducing this set is due to [2]. If X is a geodesic space and $h = 0$, this inequality is just the classical Brunn-Minkowski inequality. We shall often note $BM(N)$ instead of $BM(N, 0)$. Remark that this definition is meaningful for discrete metric spaces.
Observe also that if X satisfies $BM(N, h)$ it also satisfies $BM(N, h')$ for all $h' \geq h$.
In this work we use the following distance $\mathbb{D}$ between abstract metric measure spaces. We refer to [9] for its properties.

Definition 2.2. Let (M, d, m) and (M', d', m') be two metric measure spaces, their $\mathbb{D}$ distance is given by

$$\mathbb{D}((M,d,m),(M',d',m')) = \inf_{\hat{d},q} \left(\int_{M \times M'} \hat{d}^2(x,y)dq(x,y) \right)^{1/2}$$

where $\hat{d}$ is a pseudo-metric on the disjoint union $M \sqcup M'$ which coincides with d on M and with d' on M' and q a coupling of the measures m and m'.

A pseudo-metric $\hat{d}$ on $M \sqcup M'$ is a metric which coincides with d on M and with d' on M' but for which the property $\hat{d}(x, y) = 0$ does not necessarily imply that x and y are equal. A coupling q of the measures m and m' is a measure on $M \times M'$ whose marginals are m and m'.

Theorem 2.3. *Let (X_n, d_n, m_n) be a sequence of compact metric measure spaces which converges with respect to the distance $\mathbb{D}$ to another compact metric measure space (X, d, m). If (X_n, d_n, m_n) satisfies $BM(N, h_n)$ and if $h_n \to h$ when n goes to infinity, then (X, d, m) satisfies $BM(N, h)$.*

Before proving this theorem, observe that a direct consequence is the stability of the classical Brunn-Minkowski inequality for compact geodesic spaces:

Corollary 2.4. *Let (X_n, d_n, m_n) be a sequence of compact geodesic spaces which converges with respect to the distance $\mathbb{D}$ to another compact metric measure space (X, d, m), then X is also a geodesic space. If (X_n, d_n, m_n) satisfies $BM(N)$ then (X, d, m) satisfies also $BM(N)$.*

The fact that the limit set X is a geodesic space is well known (see [9]).

We will prove theorem 2.3 only for compact sets of positive measure. The remarks following the proof will extend the inequality to all non-empty Borel sets.

The idea of the proof is quite simple. Take two non-empty compact subsets of the limit set X. Choose and fix a nearly optimal coupling of X_n and X. Then one construct two compact sets of X_n by dilating the first two sets of X in $X_n \sqcup X$ with respect to the pseudo-distance $\hat{d}_n$ of the coupling and taking the restriction to X_n.

The fact which makes things work is that the sets built in X_n have nearly the same mass as the initial ones in X. Now we can define an approximated s-intermediate set in X_n and apply the Brunn-Minkowski inequality in X_n. By the same construction as before, we construct a set in the limit set X from the s-intermediate set in X_n without losing too much measure. To conclude we have to study the link between this last set and the set of approximate s-intermediate points between initial compact subsets of X.

Proof of Theorem 2.3. Let C_0, C_1 be two compact sets of X of positive measure and let $s \in [0,1]$. Extracting a subsequence, we may assume that $\mathbb{D}(X_n, X) \leq \frac{1}{2n}$. By definition of $\mathbb{D}$, there exists $\hat{d}_n$ a pseudo-metric on $X_n \sqcup X$ and q_n a coupling of m_n and m such that

$$\left(\int_{X_n \times X} \hat{d}_n^2(x,y) dq_n(x,y) \right)^{1/2} \leq \delta_n = \frac{1}{n}$$

Let $\varepsilon_n = \frac{1}{\sqrt{n}}$. For $i = 0, 1$, define $A_{n,i} = \{x \in X_n; \hat{d}_n(x, C_i) \leq \varepsilon_n\}$, these are compact sets of X_n. Let us first prove that they are non empty for n

large enough. Indeed, for $i = 0, 1$,

$$\begin{aligned} m(C_i) &= q_n(X_n \times C_i) \\ &= q_n(A_{n,i} \times C_i) + q_n(\{X_n \setminus A_{n,i}\} \times C_i) \end{aligned}$$

But if $(x, y) \in \{X_n \setminus A_{n,i}\} \times C_i$, then $\hat{d}_n^2(x, y) \geq \varepsilon_n^2$, so

$$\begin{aligned} q_n(\{X_n \setminus A_{n,i}\} \times C_i) &\leq \int_{\{X_n \setminus A_{n,i}\} \times C_i} \frac{\hat{d}_n^2(x, y)}{{\varepsilon_n}^2} dq_n(x, y) \\ &\leq \frac{\delta_n^2}{{\varepsilon_n}^2} = \frac{1}{n}. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} m_n(A_{n,i}) &= q_n(A_{n,i} \times X) \\ &\geq q_n(A_{n,i} \times C_i) \end{aligned}$$

Therefore, for $i = 0, 1$,

$$m_n(A_{n,i}) \geq m(C_i) - \frac{1}{n}. \tag{2.2}$$

So the sets $A_{n,i}$ are non empty for n large enough.

Now since X_n satisfies $BM(N, h_n)$, introduce the set $\hat{A}_{n,s} \subset X_n$ defined as in definition 2.1 by

$$\Big\{x \in X_n\,;\, \exists (x_{n,0}, x_{n,1}) \in A_{n,0} \times A_{n,1},\ |d(x_{n,0}, x) - s d(x_{n,0}, x_{n,1})| \leq h_n, \\ |d(x, x_{n,1}) - (1-s) d(x_{n,0}, x_{n,1})| \leq h_n\Big\}.$$

This is the set of all the h_n s-intermediate points between $A_{n,0}$ and $A_{n,1}$. By definition of $BM(N, h_n)$, we have

$$m_n^{1/N}(\hat{A}_{n,s}) \geq (1-s) m_n^{1/N}(A_{n,0}) + s\, m_n^{1/N}(A_{n,1}) \tag{2.3}$$

We can now define $C_{n,s} \subset X$ by

$$C_{n,s} = \{y \in X; \exists x \in \hat{A}_{n,s}, \hat{d}_n(x, y) \leq \varepsilon_n\}$$

Similary to (2.2) we have

$$m(C_{n,s}) \geq m_n(\hat{A}_{n,s}) - \frac{1}{n} \tag{2.4}$$

Now since $(x-\frac{1}{n})_+^{1/N} \geq x^{1/N} - (\frac{1}{n})^{1/N}$ for all $x \geq 0$, combining the inequalities (2.2), (2.3) and (2.4) give us:

$$\begin{aligned} m^{1/N}(C_{n,s}) &\geq m_n^{1/N}(\hat{A}_{n,s}) - (\frac{1}{n})^{1/N} \\ &\geq (1-s)\, m_n^{1/N}(A_{n,0}) + s\, m_n^{1/N}(A_{n,1}) - (\frac{1}{n})^{1/N} \\ &\geq (1-s)\, m^{1/N}(C_0) + s\, m^{1/N}(C_1) - 2(\frac{1}{n})^{1/N}. \end{aligned}$$

Now let us prove that the set $C_{n,s}$ of X is included in the set $C_s^{h_n+4\varepsilon_n}$ of all the $h_n + 4\varepsilon_n$ s-intermediate points between C_0 and C_1 defined as in defintion 2.1 by

$$\Big\{x \in X\,;\, \exists (x_0,x_1) \in C_0 \times C_1,\, |d(x_0,x) - sd(x_0,x_1)| \leq h_n + 4\varepsilon_n, \\ |d(x,x_1) - (1-s)d(x_0,x_1)| \leq h_n + 4\varepsilon_n\Big\}.$$

Indeed, let $y \in C_{n,s}$, by definition of this set, there exists $x \in \hat{A}_{n,s}$ so that $\hat{d}_n(x,y) \leq \varepsilon_n$. By definition of $\hat{A}_{n,s}$, it follows that there exists $(x_{n,0},x_{n,1}) \in A_{n,0} \times A_{n,1}$ such that

$$\begin{aligned} |d_n(x,x_{n,0}) - s\, d_n(x_{n,0},x_{n,1})| &\leq h_n \\ |d_n(x,x_{n,1}) - (1-s)\, d_n(x_{n,0},x_{n,1})| &\leq h_n. \end{aligned}$$

By definition of $A_{n,i}$ for $i = 0,1$, there exists $(y_0,y_1) \in C_0 \times C_1$ with $\hat{d}_n(x_{n,0},y_0) \leq \varepsilon_n$ and $\hat{d}_n(x_{n,1},y_1) \leq \varepsilon_n$. It follows:

$$\begin{aligned} |\hat{d}(y,y_0) - s\,\hat{d}(y_0,y_1)| \leq{}& |\hat{d}(y,y_0) - \hat{d}(x,x_{n,0})| \\ &+ |\hat{d}(x,x_{n,0}) - s\,\hat{d}(x_{n,0},x_{n,1})| \\ &+ s\,|\hat{d}(y_0,y_1) - \hat{d}(x_{n,0},x_{n,1})| \\ \leq{}& h_n + 4\varepsilon_n \end{aligned}$$

and

$$|\hat{d}(y,y_1) - (1-s)\,\hat{d}(y_0,y_1)| \leq h_n + 4\varepsilon_n.$$

The sequence $(h_n + 4\varepsilon_n)_n$ is converging to h. Extracting a subsequence, we may assume this sequence is monotone. There are two cases. The first one is when the extracting subsequence is non-decreasing. Then we have

$C_s^{h_n+4\varepsilon_n} \subset C_s^h$. So, for all n,

$$m^{1/N}(C_s^h) \geq m^{1/N}(C_s^{h_n+4\varepsilon_n}) \geq (1-s)\,m^{1/N}(C_0) + s\,m^{1/N}(C_1) - 2(\frac{1}{n})^{1/N}.$$

Letting n go to infinity gives the conclusion.
The second one, more interesting, is when the extracted subsequence is non-increasing. Then we have

$$C_s^h = \bigcap_n C_s^{h_n+4\varepsilon_n}.$$

Indeed if $y \in \bigcap_n C_s^{h_n+4\varepsilon_n}$, for all $n \in \mathbb{N}$, $\exists (y_{n,0}, y_{n,1}) \in C_0 \times C_1$ so that

$$\begin{array}{lcl} |d(y, y_{n,0}) - s\,d(y_{n,0}, y_{n,1})| & \leq & h_n + 4\varepsilon_n \\ |d(y, y_{n,1}) - (1-s)\,d(y_{n,0}, y_{n,1})| & \leq & h_n + 4\varepsilon_n. \end{array}$$

By compactness of C_0 and C_1 we can extract another subsequence so that $y_{n,0} \to y_0 \in C_0$ and $y_{n,1} \to y_1 \in C_1$ and we have

$$\begin{array}{lcl} |d(y, y_0) - s\,d(y_0, y_1)| & \leq & h \\ |d(y, y_1) - (1-s)\,d(y_0, y_1)| & \leq & h \end{array}.$$

The other inclusion is immediate. This intersection is non-increasing so

$$m^{1/N}(C_s^h) = \lim_{n\to\infty} m^{1/N}(C_s^{h_n+4\varepsilon_n})$$

which gives the conclusion

$$m^{1/N}(C_s^h) \geq (1-s)\,m^{1/N}(C_0) + s\,m^{1/N}(C_1).$$

□

Remark 2.5. $BM(N)$ is directly implied by the conditions $CD(O,N)$ of Sturm or Lott and Villani for all the compact sets with positive measure (in fact for all the Borel sets with positive measure) (see [10]). But if the measure m charges all the balls of the space then if the space satisfies $BM(N)$ for all its compact subsets with positive measure it satisfies also it for all its non-empty compact subsets.

Indeed let (X, d, m) be a metric measure space where the measure m charges all the open balls. Assume X satifies $BM(N)$ for all its compact subsets with positive measure. Let C_0, C_1 be non-empty compact subsets with $m(C_0) = 0$ and $m(C_1) > 0$ (the case $m(C_0) = m(C_1) = 0$ is trivial) and $s \in [0,1]$. Define $H_0^\varepsilon = \{y \in X; \exists x \in C_0, d(x,y) \leq \varepsilon\}$, then $m(H_0^\varepsilon) >$

0. Define H_s^ε the set of all the s-intermediate points between H_0^ε and C_1. By Brunn-Minkowski inequality we have:

$$m^{1/N}(H_s^\varepsilon) \geq (1-s)\, m^{1/N}(H_0^\varepsilon) + s\, m^{1/N}(C_1) \geq s\, m^{1/N}(C_1).$$

As before H_s^ε is included in $C_s^{2\varepsilon}$ the set of all 2ε s-intermediate points between C_0 and C_1. As before $\bigcap_{\varepsilon>0} C_s^{2\varepsilon}$ is an non-increasing intersection equal to C_s^0 the set of all the exact s-intermediate points between C_0 and C_1. So

$$m(C_s^0) = \lim_{\varepsilon \to 0} m(C_s^{2\varepsilon})$$

which gives the annonced result. Consequently, on a compact metric measure space where the measure charges all the open balls, the condition $CD(0,N)$ implies $BM(N)$ for all non-empty compact sets.

Remark 2.6. In Polish spaces, Borel measures are regular. It enables us to pass from compact sets to Borel ones. More precisely, if a Polish space satisfies $BM(N,h)$ for all its compact subsets, it also satisfies it for all its Borel subsets. Therefore, if the spaces X_n and X are only Polish (no more compact), the sets $A_{n,i}$ for $i=0,1$ defined as above may be no more compact. However they will be closed, therefore (2.3) will still stay true in this more general context. We can, consequently, drop the assumption of compactness of X_n and X in theorem (2.3) and its corollary (2.4).

Remark 2.7. We can do the same for the Brunn-Minkowski inequality with curvature bounded below by k by using the definition given in [10]. The only additional thing to do is to control the parameter Θ. But, with preceeding notations, we have $|\Theta(C_0,C_1) - \Theta(C_{n,0},C_{n,1})| \leq 2\varepsilon_n$.

Remark 2.8. We can prove also the same theorem for the multiplicative Brunn-Minkowski inequality (1.3).

3. **Discretizations of metric spaces**

Let (M,d,m) be a given Polish measure space. For $h>0$, let $M_h = \{x_i, i \geq 1\}$ be a countable subset of M with $M = \bigcup_{i\geq 1} B_h(x_i)$. Choose $A_i \subset B_h(x_i)$ measurable and mutually disjoint such that $\bigcup_{i \geq 1} A_i = M$ and $x_i \in A_i$. Such a construction always exist (see [2] and the references therein). Consider the measure m_h on M_h given by $m_h(\{x_i\}) = m(A_i)$ for $i \geq 1$. We call (M_h,d,m_h) a h-discretization of (M,d,m).
It is proved in [2] that if $m(M) < \infty$ then

$$(M_h, d, m_h) \xrightarrow{\mathbb{D}} (M, d, m).$$

Theorem 3.1. *Let $h > 0$. If (M, d, m) satisfies $BM(N)$ then (M_h, d, m_h) satisfies $BM(N, 4h)$.*

The proof is based on the two following facts.

Lemma 3.2. (1) *If $H \subset M_h$ then*

$$m(H^h) \geq m_h(H) \tag{3.1}$$

where $H^h = \{x \in M; d(x, H) \leq h\}$.

(2) *If $C \subset M$ measurable then*

$$m_h(C^h) \geq m(C) \tag{3.2}$$

where $C^h = \{x_i \in M_h; d(x_i, C) \leq h\}$.

Proof of lemma 3.2. For the first point, let $H \subset M_h$, we have

$$\begin{aligned} m_h(H) &= \sum_{i; x_i \in H} m(A_i) \\ &= m(\cup_{i; x_i \in H} A_i) \\ &\leq m(H^h) \end{aligned}$$

since the A_i are mutually disjoint and $\cup_{i; x_i \in H} A_i \subset H^h$.
For the second point, let $C \subset M$ measurable, define C^h as above, then

$$\begin{aligned} m_h(C^h) &= \sum_{i; x_i \in C^h} m(A_i) \\ &= m(\cup_{i; x_i \in C^h} A_i) \\ &\geq m(C) \end{aligned}$$

since $\cup_{i; x_i \in C^h} A_i \supset C$. Indeed, if for some j, $C \cap A_j \neq \emptyset$ then there exists $c \in C$ with $d(c, x_j) \leq h$ so $x_j \in C^h$.

□

Proof ot theorem 3.1. Let H_0, H_1 be two compact subsets of M_h and $s \in [0, 1]$. The sets H_0 and H_1 consist of a finite or countable number of points

x_j. Define $C_i \subset M$, for $i = 0, 1$, by $C_i = \{x \in M; \exists x_j \in H_i, d(x_j, x) \leq h\}$. By the first point of the lemma, for $i = 0, 1$,

$$m(C_i) \geq m_h(H_i). \tag{3.3}$$

Let $C_s \subset M$ be the set of all the s-intermediate points between C_0 and C_1 in the entire space M, i.e.

$$C_s = \left\{ x \in M; \exists (c_0, c_1) \in C_0 \times C_1, \begin{array}{ccc} d(x, c_0) & = & s\, d(c_0, c_1) \\ d(x, c_1) & = & (1-s)\, d(c_0, c_1) \end{array} \right\}.$$

$BM(N)$ inequality on M gives us

$$m^{1/N}(C_s) \geq (1-s) m^{1/N}(C_0) + s\, m^{1/N}(C_1). \tag{3.4}$$

As before by triangular inequality, we can see that C_s is included in the set $\tilde{C}_s^{3h}$ of $3h$ s-intermediate points between H_0 and H_1 in the whole space M. Therefore, the set $H_s^{4h} \subset M_h$ of $4h$ s-intermediate points between H_0 and H_1 in the discrete space M_h contains the restriction at M_h of the h dilated of C_s. By the second point of the lemma we have

$$m_h(H_s^{4h}) \geq m(C_s). \tag{3.5}$$

Combining inequalities (3.3), (3.4) and (3.5) ends the proof of the theorem.

□

Remark 3.3. The same proof shows that if (M, d, m) satisfies $BM(N, k)$, then (M_h, d, m_h) satisfies $BM(N, k + 4h)$.

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