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Peter Danchev

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On Some Fully Invariant Subgroups of Summable Groups

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Abstract

We show the inheritance of summable property for certain fully invariant subgroups by the whole group and vice versa. The results are somewhat parallel to these due to Linton (Mich. Math. J., 1975) and Linton-Megibben (Proc. Amer. Math. Soc., 1977). They also generalize recent assertions of ours in (Alg. Colloq., 2009) and (Bull. Allah. Math. Soc., 2008)

1. **Introduction**

Throughout the present paper, all groups into consideration are abelian p-groups, written via the additive record which is the custom when studying such groups, where p is a prime. All other unexplained exclusively characters and definitions are standard and follow essentially the classical reference [4].

A well-known theorem of Nunke [10] (see also [4, v. II, p. 112, Exercise 3]) states that if G is a totally projective group, then so are $p^\alpha G$ and $G/p^\alpha G$ for all ordinals α; and, conversely, if there is an ordinal α such that both $p^\alpha G$ and $G/p^\alpha G$ are totally projective, then G is totally projective. (Note that, in particular, the same claim follows for direct sums of countable groups.) An unpublished observation of Fuchs - E. Walker says that if F is a fully invariant subgroup of the totally projective group G, then both F and G/F are totally projective groups (see cf. [4, v. II, p. 101, Exercise 7] and [6] as well). In [8], Linton and Megibben jointly investigated the more difficult converse question in some special cases for F. Specifically, they proved that if the fully invariant subgroup F of G is of concrete type and if both F and G/F are totally projective, then G itself is totally projective. The general case for F is still unanswered.

Keywords: fully invariant subgroups, λ-large subgroups, summable groups, free valuated vector spaces.
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However, when G is a C_λ-group (that is, $G/p^\alpha G$ is totally projective for each $\alpha < \lambda$) of length λ and F is an unbounded fully invariant subgroup of G, Linton established in ([6, Theorem 5]) that G is totally projective if and only if F is totally projective.

In a subsequent series of articles, we have explored the same kind of claims for an independent class of so-called summable groups (see [9] and [4, v. II, Chapter 84]). In fact, in [1] we show that if G is a group of countable length and if α is an ordinal such that $p^\alpha G$ and $G/p^\alpha G$ are both summable, then G is summable. Further, in [3], we consider the case when length$(G) = \Omega$ by proving that if G is a C_Ω-group of length Ω and if $p^\alpha G$ is summable for some $\alpha < \omega^2$, then G is also summable. After this, we refine in [2] this assertion to an arbitrary countable ordinal α (i.e. $\alpha < \Omega$).

The aim of this manuscript is to remove the condition on G to be a C_Ω-group as well as the countability of length(G) by showing that if $p^\alpha G$ and $G/p^\alpha G$ both are summable for some ordinal number α, then G is summable too; thus generalizing all of the aforementioned attainments. We also concern some partial situations of the general case for the fully invariant subgroup F of G; it is noteworthy that $p^\alpha G$ is a special fully invariant subgroup of G.

2. Main Results

It is well-known that G being summable forces that $p^\alpha G$ is summable for any ordinal α. In this section we shall show that under the condition that $G/p^\alpha G$ is summable the converse statement is true for an arbitrary ordinal α and thus acquire a powerful tool for showing a group is summable.

As already emphasized in the introductory section, we argued in [1] that if length$(G) < \Omega$ and both $p^\alpha G$ with $G/p^\alpha G$ are summable for some $\alpha \leq$ length(G), then G is summable. According to the following example one can expect that the assertion holds valid even for lengths equal to Ω.

Example 2.1. Let $A = G \oplus H$, where G is of countable length whereas H is of length Ω a direct sum of countable groups. We claim that if $p^\alpha A$ and $A/p^\alpha A$ are summable for some $\alpha \leq \Omega$, then A is summable. Indeed, length$(A) = \Omega$ and we see that $\alpha = \Omega$ obviously insures that A is summable. If now $\alpha < \Omega$, we see that $p^\alpha A = p^\alpha G \oplus p^\alpha H$ and $A/p^\alpha A \cong (G/p^\alpha G) \oplus (H/p^\alpha H)$ imply that $p^\alpha G$ and $G/p^\alpha G$ are summable as being direct summands. Thus, by what we have remarked above, G has to be summable, whence so is A (see [9] or [4, v. II, p. 123]) as claimed.

And so, we come to our central result which shows what we happened; it will turn out to be a major step towards proving the desired global theorem.

Theorem 2.2. *Suppose $p^\alpha G$ and $G/p^\alpha G$ are summable groups for some ordinal number α. Then G is summable, provided* $\text{length}(G) \leq \Omega$.

Proof. If $\alpha \geq \Omega$, we are done. For the remaining case $\alpha < \Omega$ we write $G[p] = (p^\alpha G)[p] \oplus V_\alpha$ as valuated vector spaces where V_α is isometric to the socle of some p^α-high subgroup of G. Assume that $\mathcal{H}$ is such a p^α-high subgroup. Since $\mathcal{H} \cong (\mathcal{H} \oplus p^\alpha G)/p^\alpha G$ where the latter group is isotype in $G/p^\alpha G$, a group of countable length, we appeal to [9] or to ([4, v. II, p. 125, Prop. 84.4]) to infer that $\mathcal{H}$ must be summable. Therefore, $\mathcal{H}[p]$ must be free, hence so does V_α. But $(p^\alpha G)[p]$ is free too and henceforth a simple technical argument applies to get that $G[p]$ is free which gives the desired summability of G. □

We now turn to the question: If F is a fully invariant subgroup of G and both F and G/F are summable, does it follow that G is summable as well? (For the corresponding results concerning totally projective groups we refer once again to [6] and [8]). In some instances this is so; for examples:

1) $F = G[p^n]$ for some natural number n.

Then $G/F \cong p^n G$ is summable only when G is summable (see [1] or [3, Lemma 1]).

2) $F = p^\alpha G$ for some ordinal number α.

Then our foregoing theorem allows us to conclude that the above question holds true in that case.

Now, we will make an attempt to examine the general case. Recall that the height $|x|_G$ of a nonzero element x in the reduced group G is defined to be the least ordinal α such that $x \notin p^{\alpha+1}G$. We also set $|0|_G = \infty$. Let $\alpha = (\alpha_i)_{i<\omega}$ be an increasing sequence of ordinals and symbols ∞; that is, for each index i, either α_i is an ordinal or $\alpha_i = \infty$ and $\alpha_i < \alpha_{i+1}$ provided $\alpha_i \neq \infty$. Imitating [8], with each such sequence α we associate the fully invariant subgroup $G(\alpha)$ of the group G defined by

$$G(\alpha) = \{x \in G : |p^i x|_G \geq \alpha_i, \forall i < \omega\}.$$

If G is totally projective, then all of its fully invariant subgroups are of this form (see [4, v. II, Thm. 67.1; p. 112, Exercise 6 and p. 122, Exercise 7]). However, this is an open problem for arbitrary groups. That is why, we specify that all our fully invariant subgroups of G in the sequel are taken of the above form presented.

We have accumulated all the machinery necessary to prove the following.

Theorem 2.3. *Let F denote an unbounded fully invariant subgroup of the C_λ-group G, where G has length $\lambda \leq \Omega$. Then F is a summable group only if G is a summable group.*

Proof. As the text says, we must show that F being summable implies that G is summable. In fact, since $F = G(\alpha)$, where

$$\alpha = (\alpha(0), \alpha(1), \cdots, \alpha(n), \cdots),$$

we set $\beta = \sup\{\alpha(n) : n < \omega\}$. It is not difficult to verify that $p^\omega F = p^\beta G$. Hence $p^\beta G$ is summable. If $\beta < \lambda$, we have that $G/p^\beta G$ is a direct sum of countable groups, whence summable, because β is countable. Hereafter Theorem 2.2 is applicable to get that G is summable. If $\beta = \lambda$, then F is separable summable, thus a direct sum of cyclic groups. Since β is co-final with ω, we obtain that λ is co-final with ω, hence $\lambda < \Omega$ and then G is a direct sum of countable groups (see [9]), whence summable. □

Remark 2.4. The preceding theorem extends ([6, Theorem 5]) to summable groups. Moreover, in that Theorem 5, $\lambda = \mathrm{length}(G)$ should be co-final with ω.

Finally, we can precise the corresponding theorem from [3] (see [2] too).

Theorem 2.5. *If G is a C_μ-group with a λ-large subgroup L such that $\omega \leq \lambda \leq \mu \leq \Omega$, then G is summable precisely when L is summable.*

Proof. "$\Rightarrow$". Foremost, if $\mathrm{length}(G) < \Omega$, we consider two possibilities. Firstly, if $\mathrm{length}(G) = \mu < \Omega$, Hill-Megibben's criterion works to infer that A is a direct sum of countable groups. Hence, Linton's result is applicable to deduce that L is totally projective of countable length, that is a direct sum of countable groups, whence summable. Otherwise, if $\mathrm{length}(G) < \mu$, G is obviously a direct sum of countable groups and again the previous procedure can be applied.

After this, let us assume that $\mathrm{length}(G) = \Omega$. We know by [7] that $p^\omega L = p^\nu G$ for some $\nu \leq \lambda$. Hence $p^\omega L$ is summable. If $\nu < \lambda$, then L

is a Σ-group (see [3]). Therefore, L is summable (see [1] or [3]). In the remaining case when $\nu = \lambda$, we have that $L/p^\omega L = L/p^\lambda A$ is a λ-large subgroup of the totally projective group $A/p^\lambda A$. Thus, in virtue of [7], $L/p^\omega L$ is a direct sum of cyclic groups. As above, L has to be a summable group (see [1] or [3]).

"$\Leftarrow$". As already observed via [7], $p^\omega L = p^\nu G$. Consequently, L being summable secures that so is $p^\nu G$. Since $\nu \leq \lambda \leq \mu$ and $\nu \leq \Omega$, we derive that $G/p^\nu G$ is a direct sum of countable groups, thus summable. Furthermore, we apply Theorem 2.2 to obtain the wanted claim. $\square$

3. **Left-Open Problems**

In conclusion, we pose two questions that we find interesting.

We recollect that a group G is *σ-summable* (see [8]) if $G[p] = \cup_{i<\omega} G_i$, $G_i \subseteq G_{i+1} \leq G[p]$ and, for every $i < \omega$, there exists an ordinal $\alpha_i <$ length(G) so that $G_i \cap p^{\alpha_i} G = 0$. Applying [5], this definition is tantamount to the following one: $G = \cup_{i<\omega} H_i, H_i \subseteq H_{i+1} \leq G$ and, for every $i < \omega$, there exists an ordinal $\beta_i <$ length(G) so that $H_i \cap p^{\beta_i} G = 0$. It is well-known that (see [8]) a σ-summable C_λ-group of length λ is totally projective; note that such a λ is of necessity co-final with ω and thus limit. By analogy, we ask the following.

Problem 3.1. *Does it follow that a σ-summable group G of countable limit length for which $G/p^\alpha G$ is summable for each non-limit $\alpha <$ length(G) is summable? If not, under what additional circumstances this is true?*

Notice that each summable group of countable limit length is σ-summable.

We recall that a group G is *a Σ-group* if some its high subgroup is a direct sum of cyclic groups. In [1] we obtained the criterion that a group G is summable uniquely when G is a Σ-group and $p^\omega G$ is summable. The role of the ordinal ω in this necessary and sufficient condition is crucial. Nevertheless, we ask the following.

Problem 3.2. *Does it follow that a Σ-group G for which $p^\alpha G$ is summable for some $\alpha > \omega$ plus something else will imply that G is summable?*

This is equivalent to find suitable conditions under which a Σ-group G whose power subgroup $p^\gamma G$ is summable will ensure that $p^\delta G$ is summable whenever $\delta < \gamma$.

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Peter Danchev
Dept. of Mathematics and Statistics
Plovdiv University
24 Tzar Assen Str.
4000 Plovdiv, BG
BULGARIA
pvdanchev@yahoo.com