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Emmanuel Germain

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# Non-commutative entropy computations for continuous fields and cross-products

Emmanuel Germain

**Abstract**

We present here two non-commutative situations where dynamical entropy estimates are possible. The first result is concerned with automorphisms of cross-products by an exact group that commute with the group action and generalizes the result known for amenable groups. The second is about continuous fields of C*-algebras and $C(X)$-automorphisms. Each result relies on explicit factorization via matrices.

*Calculs d'entropie non commutative pour des champs continus ou des produits croisés*

**Résumé**

Nous présentons deux situations de géométrie non commutative dans lesquelles des calculs d'entropie dynamique non commutative sont possibles. La première s'interesse aux automorphismes des produits croisés par un groupe exact qui commutent avec l'action du groupe généralisant un résultat connu pour les groupes moyennables. La seconde concerne les champs continues de $C^*$-algèbres et les automorphismes fibrés. Dans chacun des deux cas, il s'agit d'utiliser des factorisations explicites par des algèbres de matrices.

## 1. Introduction

Although exact C*-algebras have a weaker form of factorization through finite dimensional matrix algebras than nuclear C*-algebras, N. Brown [6] was able to show that Voiculescu's initial definition of non-commutative entropy for automorphisms [15] can be extended to this situation and produced the first computations, notably for automorphisms of cross-products of an exact C*-algebra by the group of the integers. Other results, like a striking formula for free product automorphisms of reduced free product

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C*-algebras [5], were obtained proving that it was the correct setting for this kind of entropy.

For group C*-algebra, Ozawa [12] showed about the same time that the reduced C*-algebra of a discrete group is exact if and only if it has an "amenable" action on a compact space. It was long known that this condition was a sufficient one [1]. Indeed since the groupoid associated to the group action on the compact space is amenable, hence its reduced C*-algebra is nuclear, the reduced C*-algebra of the group is a subalgebra of a nuclear one. Therefore Ozawa result is a geometric caracterization of exactness parallel to Kirchberg's caracterization of exact C*-algebras as subalgebras of nuclear C*-algebras.

It was then obvious that in the situation of a reduced cross product of an exact C*-algebra by an exact group a formula for the entropy of an automorphism that commutes with the group action should be available, extending the previous result obtained for amenable groups (cf. [13]) and fullfilling C. Anantharaman's remark in her article [2]. Explicitely, we prove in the first section that if an exact C*-algebra $A$ is endowed with an action of a group $\Gamma$ commuting with an automorphism $\beta$ of $A$, then the entropy of $\beta$ in $A$ is the same as the entropy of its unique extension $\bar{\beta}$ to the cross-product $A \rtimes_r \Gamma$ defined as $\bar{\beta}(a)(g) = \beta(a(g))$ for any element $a$ in the convolution algebra $L^1(\Gamma; A)$.

The crucial ingredient used in the first section to get this result is the existence of explicit matrix factorizations due to the fact that analogues of Folner functions exist for amenable groupoids. Extending the results of [2] section 8 based on a construction of [14], one shows that if $A$ is an exact C*-algebra and $\Gamma$ has an amenable action on a compact space $X$ then $C(X) \otimes A$ is an exact $\Gamma - C(X)$-algebra whose cross-product by $\Gamma$ has factorizations through finite dimensional matrix algebras which can be made out of factorizations of $A$. We show that the ranks of these factorizations are linearly related which yields the entropy comparison we want.

If the algebra $A$ is the algebra of continuous functions on a compact space $E$ with an action of $\Gamma$ such that $B = E/\Gamma$ is again compact then $A \rtimes_r \Gamma$ is fibered over $B$ (it is actually a $C(B)$-algebra) hence the result above can be reinterpreted as computing the entropy of an automorphism of a fibered space whose action factors through the base space $B$ (a "transverse" automorphism as there are transverse differential operators). The second part of this article investigates then the "longitudinal" case, i.e entropy of an automorphism that would act only in the fibers.

It is not clear what is the correct setting for such an appraoch. Of course when the base is discrete, we are dealing with direct sums and it is known that one should take the supremum of the entropy of the automorphisms in each summands (i.e. fibers). But for continuous base space, it must be trickier. First there are two notions of a C*-algebra fibered over a compact space: $C(X)$-algebras and continuous fields, the latter asking for a strictier continuity condition for sections. Then a subtlety arises as it is not true that the whole algebra is exact whenever all fiber algebras are (even for continuous fields see [4]). Therefore we turned our attention to lipschitz continuous fields introduced by Kirchberg and Phillips [10] because, for such continuous fields, explicit matrix factorizations can be realized via the knowledge of factorizations of the fibers. We then found an upper bound for the entropy of an automorphism of such fields that has an extra term which incorporates geometric data (dimension of the base space, lipschitz exponent of the field) and a symbolic dynamics entropy term.

This second part is organized as follows: we define an entropy for linear endomorphisms of the non-commutative polynomials using as a gauge the norm of the non-commutative gradient of a polynomial, we then describe the factorization of lipschitz fields and compute entropy. At last we apply our result to the C*-algebra of the Heisenberg group ( of unipotent upper-triangular $3 \times 3$ matrices with integer coefficients) since it can be seen as a lipschitz continuous fields over the unit circle of the non-commutative tori with exponent $1/2$ as it has been proved by Haagerup and Rordam in [9].

## 2. **Cross-product by exact groups**

An exact discrete group $\Gamma$ is a group such that the reduced cross-product of any exact sequence of $\Gamma$-algebras (i.e. C*-algebras with an action of $\Gamma$ via automorphisms) is again exact. In particular if $E$ is an exact C*-algebra with an action of an exact discrete group $\Gamma$ then the reduced cross-product $E \rtimes_r \Gamma$ is again exact. Indeed let $0 \to I \to A \to B \to 0$ be an exact sequence. Then by exactness of $E$, on gets that $0 \to I \otimes_{min} E \to A \otimes_{min} E \to B \otimes_{min} E \to 0$ is again exact. Endowing $I, A, B$ with a trivial action, it is also a sequence of $\Gamma$-algebras. By definition, its reduced cross product by $\Gamma$ is again exact. Now observing that $(A \otimes_{min} E) \rtimes_r \Gamma$ is $A \otimes_{min} (E \rtimes_r \Gamma)$ for any $A$ ensures that tensoring (for the minimal norm) the original sequence by $E \rtimes_r \Gamma$ leaves it exact.

For discrete groups, exactness need only be checked for the trivial action (see [1]), therefore the reduced C*-algebra $C^*_r(\Gamma)$ is an exact C*-algebra if and only if $\Gamma$ is exact (see [11]). It has recently been proved by Ozawa ([12]) and independantly by Anantharaman ([2]) that it is equivalent to amenability at infinity i.e. the existence of a compact Hausdorff space $X$ with an action of $\Gamma$ such that the action is amenable, a term defined for general groupoids in [1]. Using this amenable action, C. Anantharaman proved that there exists explicit matrix factorizations for the algebra $E \rtimes_r \Gamma$ when $E$ is nuclear (see section 8 of [2]). We extend here this construction to the exact case and use the notations found therein to prove:

**Theorem 2.1.** *Let $E$ be an unital exact $C^*$-algebra with an action $\alpha$ of an exact countable discrete group $\Gamma$. Let $\beta$ be an automorphism of $E$ such that for all $g \in \Gamma$, $\beta$ and $\alpha_g$ commutes, then $\beta$ extends to $\bar{\beta}$ on $E \rtimes_r \Gamma$ and*

$$ht_E(\beta) = ht_{E \rtimes_r \Gamma}(\bar{\beta}).$$

Since $E \subset E \rtimes_r \Gamma$, one already has $ht_E(\beta) \leq ht_{E \rtimes_r \Gamma}(\bar{\beta})$. For the reverse inequality, we will consider an amenable action of $\Gamma$ on a compact Hausdorff set $X$. Now $A = C(X) \otimes E$ is a $\Gamma - C(X)$-algebra meaning that $A$ is a $\Gamma$-algebra (with the diagonal action), a $C(X)$-algebra (actually it is a trivial continuous field) and has the compatibility condition: $g.(fa) = (g.f)(g.a)$ with $g \in \Gamma$, $f \in C(X)$ and $a \in A$. Because $C(X)$ is unital, one has that $E \rtimes_r \Gamma \subset A \rtimes_r \Gamma$.

Consider a faithful representation $\pi_0$ of $A$ in some $B(H)$ such that the action of $\Gamma$ is implemented by a unitary representation noted $u_g$ for $g \in \Gamma$ (take for instance the regular covariant representation). For convenience we will identify $A$ with its image in $B(H)$. Note also that $B(H)$ is endowed with an action of $\Gamma$ (by conjugation with the $u_g$'s) which we will still call $\alpha$ and $A \rtimes_r \Gamma \subset B(H) \rtimes_r \Gamma$. The covariant pair of representations $(\pi, \lambda \otimes 1_H)$ of $B(H) \rtimes_r \Gamma$ in $B(\ell^2(\Gamma) \otimes H)$ is defined as $(\pi(a)\xi)(t) = \alpha_{t^{-1}}(a)\xi(t)$ with $\lambda$ the regular representation of $\Gamma$ in $\ell^2(\Gamma)$.

Following closely the proofs of section 8 of [2], we can reprove the exactness of $E \rtimes_r \Gamma$ as follows:

**Proposition 2.2.** *Let $X$ be a compact space with an amenable action of the discrete group $\Gamma$, and let $A$ be an exact $\Gamma - C(X)$-algebra. Then $A \rtimes_r \Gamma$ is exact.*

Indeed we can use Kirchberg's caracterization of exactness: $A \rtimes_r \Gamma$ is exact if there exists a net of completely positive maps $\Phi_\lambda$ from $A \rtimes_r \Gamma$ to $B(\ell^2(\Gamma) \otimes H)$ that factorize through finte rank matrices and simply converge in norm to the inclusion map. This is exactly what proposition 8.2 of [2] says once we remark that lemma 8.1 can be identically reformulated with $\Phi : A \to B(H)$ a completely positive (or completely bounded) map instead of $\Phi : A \to A$.

**Corollary 2.3.** *If $E$ is an exact $C^*$-algebra with an action of an exact discrete group $\Gamma$ then $E \rtimes_r \Gamma$ is exact.*

Indeed $E \rtimes_r \Gamma$ is a subalgebra of the exact algebra $A \rtimes_r \Gamma$. But this proof allows also to make entropy computations. Recall the definition

**Definition 2.4.** Let $\epsilon > 0$ and $\omega \subset A$ finite. $rcp_A(\pi, \epsilon, \omega)$ is the smallest integer $p$ such that there exists a completely positive contractive $(\epsilon, \omega)$-factorization $A \xrightarrow{\sigma} M_p(\mathbb{C}) \xrightarrow{\tau} B(H)$ of the faithful morphism $A \xrightarrow{\pi} B(H)$, i.e. such that for all $x$ in $\omega$, $||\pi(x) - \tau \circ \sigma(x)|| \leq \epsilon$.

**Lemma 2.5.** *Let $\omega \subset \Gamma$ be a symetric finite set and $O \subset E$ be a finite set of norm 1 elements. Let $\Omega$ be the set $\{au_g, g \in \omega, a \in O\}$ in $E \rtimes_r \Gamma$. Then there exists a finite set $F$ in $\Gamma$ such that*

$$rcp_{E \rtimes_r \Gamma}((\pi, \lambda \otimes 1_H), \epsilon, \Omega) \leq |F| rcp_E(\pi_0, \epsilon/2, \cup_{t \in F} \alpha_{t^{-1}}(O))$$

To prove this we will look carefully at the reformulation of proposition 8.2 in [2].

With the help of a function $f : \Gamma \to C_c(X)$ with finite support $C$ such that $\sup_{x \in X} \sum_{t \in \Gamma} |f(t)(x)|^2 = 1$ and

$$\sup_{x \in X} |\sum_{t \in \Gamma} f(t)(x) \overline{f(s^{-1}t)(s^{-1}x)} - 1| < \epsilon/2$$

for $s \in \omega$ which exists by the amenability of the action on $X$, we define the set $F = \cup_{s \in \omega \cup \{e\}} s^{-1}C = \cup_{s \in \omega \cup \{e\}} sC$.

We then choose a $(\epsilon/2, \cup_{t \in F} \alpha_{t^{-1}}(O))$ factorization $(\sigma, \tau)$ of $E$ through $M_n(\mathbb{C})$ and extends it by Arveson's extension theorem for completely positive maps to a factorization $(\bar{\sigma}, \tau)$ of $A$ through the same $M_n(\mathbb{C})$.

Consider now the two completely positive maps: $\tilde{\sigma}$ from $E \rtimes_r \Gamma$ in $B(\ell^2(F)) \otimes M_n(\mathbb{C})$

$$\tilde{\sigma}(au_g) = (I \otimes \bar{\sigma})\left[(P_F \otimes I_n)(\pi(a)\lambda(g))(P_F \otimes I_n)\right]$$

with $P_F$ the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(F)$ and $\tilde{\tau}$ from the algebra $B(\ell^2(F)) \otimes M_n(\mathbb{C})$ to $B(\ell^2(\Gamma) \otimes H)$

$$\tilde{\tau}(x) = T_f(I \otimes \tau(x))$$

with $T_f$ from $B(\ell^2(\Gamma) \otimes H)$ onto itself as defined is [2].

The composition of the two produces a map $\Psi$ such that $||\Psi(au_g) - \pi(a)\lambda(g)|| < \epsilon$ for all $a \in O$ and $g \in \omega$ because the completely positive map $\Phi = \tau \circ \bar{\sigma}$ has the property that $||\Phi(\alpha_{t^{-1}}(a)) - a|| < \epsilon/2$ for all $a \in O$.

Now the rank of $\tilde{\sigma}$ is $|F|$ multiplied by the rank of $\sigma$ which is what we seek.

**Corollary 2.6.** *Let $\omega \subset \Gamma$ be a finite set and $O \subset E$ be a finite set of norm 1 element. Let $\Omega$ be the set $\{au_g, g \in \omega, a \in O\}$ in $E \rtimes_r \Gamma$. Then there exists a finite set $F$ in $\Gamma$ such that*

$$\begin{aligned}&rcp_{E\rtimes_r\Gamma}((\pi, \lambda \otimes 1_H), \epsilon, \Omega \cup \bar{\beta}(\Omega) \cup \cdots \cup \bar{\beta}^k(\Omega)) \leq \\ &\leq |F| rcp_E(\pi_0, \tfrac{\epsilon}{2}, \cup_{t\in F}\alpha_{t^{-1}}(O) \cup \cdots \cup \beta^k(\cup_{t\in F}\alpha_{t^{-1}}(O))).\end{aligned}$$

Indeed $\beta$ commutes with the action of $\Gamma$, hence $\bar{\beta}(\Omega) = \{au_g,\ g \in \omega, a \in \beta(O)\}$ and $\cup_{t\in F}\alpha_{t^{-1}}(\beta(O)) = \beta(\cup_{t\in F}\alpha_{t^{-1}}(O))$.

Since the entropy $ht_{E\rtimes_r\Gamma}(\bar{\beta})$ is then defined as

$$\sup_{\epsilon>0} \sup_{\Omega\in T} \lim_{n\to\infty} \frac{1}{n} \log(rcp_{E\rtimes_r\Gamma}((\pi, \lambda \otimes 1_H), \epsilon, \Omega \cup \bar{\beta}(\Omega) \cup \cdots \cup \bar{\beta}^n(\Omega))$$

with $T$ the set of all finite subsets of the linear span of elements of the form $au_g$ with $a \in E$, $g \in \Gamma$ by Kolmogorov density property, we have that

$$ht_{E\rtimes_r\Gamma}(\bar{\beta}) \leq ht_E(\beta)$$

keeping in mind that entropy can be computed via the rcp function of any faithful representation.

## 3. Entropy for continuous fields of C*-algebras

A unital continuous field $A$ of C*-algebras over a compact Hausdorff space $X$ is caracterized by two properties. First it is a $C(X)$-algebra, meaning there is a unital morphism of $C(X)$ into the center of $A$. There is thus an action of $C(X)$ on $A$ that we denote as $f.a$ for a function $f$ and an element $a$ of $A$.

Note that the norm in $A$ is given as a supremum. Indeed, for any $x \in X$, let's call $C_x(X)$ the ideal of functions vanishing at $x$. Then $A_x$ is the quotient algebra $A/(C_x(X).A)$ and note $a_x$ the image of $a \in A$ in this quotient. We have the embedding $A \hookrightarrow \Pi_{x\in X} A_x$. (see Blanchard [3])

A $C(X)$-algebra is a continous field if and only if the map $x \mapsto ||a_x||_{A_x}$ is continuous.

We are interested in a $C(X)$-automorphism $\alpha$ of a continuous field $A$, meaning an automorphism such that for any function $f \in C(X)$ and $a \in A$ we have that $f.\alpha(a) = \alpha(f.a)$. Note that $\alpha$ factorizes through all the algebras $A_x$. Let's call $\alpha_x$ the induced automorphism.

For the moment we will study entropy of linear endomorphisms on non-commutative polynomials and propose a definition of symbolic entropy for automorphisms of C*-algebras having a dense finitely generated subalgebra.

### 3.1. Symbolic entropy

Let $\mathbb{C} < X_1 \cdots X_n >$ denotes the set of non-commutative polynomials in $n$ variables.

**Definition 3.1.** If $P \in \mathbb{C} < X_1 \cdots X_n >$ then $\mathcal{J}P \in \mathbb{C} < X_1 \cdots X_n > \otimes \mathbb{C} < X_1 \cdots X_n >$ will denote the non-commutative gradient of $P$ with respect to the variable $X_1, ..., X_n$ and is defined by linearity on generators as follows

$$\mathcal{J}X_i = 1 \otimes 1, \forall i = 1 \cdots n$$

$$\begin{aligned}\mathcal{J}X_{i_1} \cdots X_{i_n} = \; & 1 \otimes X_{i_2} \cdots X_{i_n} + \textstyle\sum_{k=2}^{n-1} X_{i_1} \cdots X_{i_{k-1}} \otimes X_{i_{k+1}} \cdots X_{i_n} \\ & + X_{i_1} \cdots X_{i_{n-1}} \otimes 1\end{aligned}$$

Because the tensors $1\otimes 1$, $1 \otimes X_{j_1} \cdots X_{j_n}$, $X_{i_1} \cdots X_{i_n} \otimes 1$ and $X_{i_1} \cdots X_{i_n} \otimes X_{j_1} \cdots X_{j_n}$ form a basis of $\mathbb{C} < X_1 \cdots X_n > \otimes \mathbb{C} < X_1 \cdots X_n >$, there is an associated $\ell_1$-norm (for which the base elements have norm 1), we will call it $||.||_1$.

The total variation of $P \in \mathbb{C} < X_1 \cdots X_n >$ will then be $||\mathcal{J}P||_1$. Note that on monomials, this gives the total degree of $P$ with respect to $X_1 \cdots X_n$. This name is appropriate because of:

**Proposition 3.2.** *If $\mathcal{A}$ is a complex normed algebra and $\Sigma_i$ for $i = 1, 2$ are two algebra homomorphims from $\mathbb{C} < X_1 \cdots X_n >$ to $\mathcal{A}$ such that $\Sigma_i(X_j)$ is a norm 1 element in $\mathcal{A}$ for $i = 1, 2$ and $j = 1, ..., n$, then for all $P \in \mathbb{C} < X_1 \cdots X_n >$,*

$$||\Sigma_1(P) - \Sigma_2(P)||_{\mathcal{A}} \leq ||\mathcal{J}P||_1 \sup_{i \in \{1,2,..,n\}} ||\Sigma_1(X_i) - \Sigma_2(X_i)||_{\mathcal{A}}$$

The proof is obvious with the remarks that

$$||\Sigma_2(X_{i_1}...X_{i_{k-1}})\Sigma_1(X_{i_k}...X_{i_p}) - \Sigma_2(X_{i_1}...X_{i_k})\Sigma_1(X_{i_{k+1}}...X_{i_p})||_{\mathcal{A}} \leq$$
$$\leq ||\Sigma_1(X_{i_k}) - \Sigma_2(X_{i_k})||_{\mathcal{A}}$$

and $||\Sigma_j(X_{i_1}...X_{i_n})||_{\mathcal{A}} \leq 1$.

Now if $\theta$ is a linear endomorphism of $\mathbb{C} < X_1 \cdots X_n >$, we will define its symbolic entropy as

$$se(\theta) = \sup_{P \in \mathbb{C} < X_1 \cdots X_n >} \limsup_{n \to \infty} \frac{1}{n} \log ||\mathcal{J}(\theta^n P)||_1$$

The above quantity behaves almost as an entropy for we have

**Proposition 3.3.** (1) *$se(\theta^k) = k\, se(\theta), \forall k \geq 0$.*

(2) *If $\theta(P) = QP$ or $PQ$ for some $Q \in \mathbb{C} < X_1 \cdots X_n >$, then $se(\theta) \leq \log ||Q||_1$.*

For 1., one just needs to remark that $se(\theta)$ is the infimum of the constants $\sigma$ such that for all polynomial $P$ there exists a constant $C_P$ such that $||\mathcal{J}(\theta^n(P))||_1 \leq C_P \exp(n\sigma)$. Hence $se(\theta^k) \leq kse(\theta)$ and by considering the maximum of $\{C_P, C_{\theta(P)}..., C_{\theta^{k-1}(P)}\}$ one gets the reverse inequality.

For 2., we of course endow $\mathbb{C} < X_1...X_n >$ with the $\ell_1$-norm for which the monomials have norm 1 which is an algebra norm. Then for the bimodule structure of $\mathbb{C} < X_1...X_n > \otimes \mathbb{C} < X_1...X_n >$ we have that $||Q_1.P.Q_2||_1 \leq ||Q_1||_1||P||_1||Q_2||_1$ with $Q_i \in \mathbb{C} < X_1...X_n >$ and $P \in \mathbb{C} < X_1...X_n > \otimes \mathbb{C} < X_1...X_n >$. Finally note that $\mathcal{J}(QP) = \mathcal{J}(Q).P + Q.\mathcal{J}(P)$ and $\mathcal{J}(Q^n) = \sum_{i=0}^{n-1} Q^i \mathcal{J}(Q).Q^{n-i-1}$. Therefore the result follows from the inequality

$$||\mathcal{J}(Q^n P)||_1 \leq n||Q||_1^{n-1}||\mathcal{J}(Q)||_1||P||_1 + ||Q||_1^n||\mathcal{J}(P)||_1.$$

Note that if $Q$ is a monomial then $se(\theta) = 0$.

Finally we propose this definition for $C^*$-algebra automorphisms:

If $A$ is a unital $C^*$-algebra and $\alpha$ an automorphism, let $F$ be the set of all dense finitely generated subalgebras $\mathcal{A}$ of $A$ such that $\alpha$ induces an automorphism of $\mathcal{A}$.

Now take $G$ as the set of all linear extensions of $\alpha$ i.e. the set of linear endomorphisms $\theta$ of $\mathbb{C} < X_1 \cdots X_n >$ such that there exists an epimorphism $\pi$ from $\mathbb{C} < X_1 \cdots X_n >$ to $\mathcal{A} \in F$ with $\pi(\theta(P)) = \alpha(\pi(P))$ for all polynomials $P$.

**Definition 3.4.** The symbolic entropy of the automorphism $\alpha$ is

$$se(A, \alpha) = \inf_{\theta \in G} se(\theta)$$

The infimum is taken to be $+\infty$ if $F$ is empty.

## 3.2. Exact lipschitz continuous fields over a compact metric space

Suppose $A$ is a unital continuous field over a compact metric space $X$. Let's assume that $A$ is exact (in particular all the $A_x$ are exact since they are quotients). It is then known that $A$ admits a $C(X)$-embedding in some $C(X) \otimes B(\mathcal{H})$. Consider the following definition:

**Definition 3.5.** Suppose $A$ is an exact continuous field on some compact metric space $X$ with metric $d$, we say that $A$ is lipschitz of exponent $L$ if there exists a $C(X)$-linear embedding $\pi$ of $A$ in some $C(X) \otimes B(\mathcal{H})$ such that for all $a \in A$ the map $x \mapsto \pi_x(a_x)$ from $X$ to $B(\mathcal{H})$ is lipschitz with exponent $L$ i.e. for all $a \in A$ there exists a constant $C$ such that

$$||\pi_x(a_x) - \pi_y(a_y)|| \leq C d(x, y)^L.$$

In [4], Blanchard showed that exact continuous fields over a compact space $X$ have $C(X)$-embeddings but in [10] for $X = [0, 1]$ the authors proved the existence of lipschitz embeddings when an intrinsically defined metric function is itself lipschitz (see theorem 2.10 p.83). It is the case for example of the continuous field of the non-commutative tori (reproving a theorem of Haagerup-Rordam, see [9]).

First a minoration.

**Proposition 3.6.** *Suppose $A$ is an $C(X)$-algebra over a compact set $X$, and $\alpha$ is a $C(X)$-automorphism of $A$. Then*

$$ht_A(\alpha) \geq sup_{x\in X'} ht_{A_x}(\alpha_x)$$

*where $X'$ is the set of all such $x \in X$ with $A_x$ commutative.*

Indeed we know topological entropy dominates CNT-entropy [8], therefore $ht_A(\alpha) \geq ht_A^{CNT}(\alpha)$. Since CNT-entropy decreases in quotient, one gets $ht_A(\alpha) \geq sup_{x\in X} ht_{A_x}^{CNT}(\alpha_x)$. Since all entropy definitions coincide in the commutative case, one gets the result.

And now the majoration

**Proposition 3.7.** *Suppose $A$ is an exact lipschitz continuous field of exponent $L$ over a compact metric space $X$ of Hausdorff dimension $N$. Let $\alpha$ be an $C(X)$-automorphism and call $\mathcal{F}(A)$ the set of all finite sets in $A$. Choose also a faithful $C(X)$-homomorphism $\pi$ of $A$ in $C(X) \otimes B(H)$. Then $ht_A(\alpha)$ is bounded by*

$$\sup_{\Omega\in\mathcal{F}(A)} \sup_{\epsilon>0} \lim \sup_{n} \sup_{x\in X} \frac{1}{n} \log(rcp(\pi_x, \epsilon, \Omega_x \cup \cdots \cup \alpha_x^n(\Omega_x))) + \frac{N}{L} se(A, \alpha)$$

*If moreover the automorphism is inner then the term $\frac{N}{L} se(A,\alpha)$can be discarded.*

**Corollary 3.8.** *Under the above hypothesis and if moreover for all $\epsilon > 0$ and $\Omega \in \mathcal{F}(A)$ there exists $y \in X$ such that for all $n$*

$$\sup_{x\in X} \frac{1}{n} \log(rcp(\pi_x, \epsilon, \Omega_x \cup \alpha_x(\Omega_x) \cup \cdots \cup \alpha_x^n(\Omega_x))) \leq$$

$$\leq \frac{1}{n} \log(rcp(\pi_y, \epsilon, \Omega_y \cup \alpha_y(\Omega_y) \cup \cdots \cup \alpha_y^n(\Omega_y)))$$

*then the entropy $ht_A(\alpha)$ is bounded by $\sup_{y\in X} ht_{A_y}(\alpha_y) + \frac{N}{L} se(A, \alpha)$.*

We will see in the last section that it is the case for the continuous field of the non-commutative tori.

*Proof of prop 2.7:*

We assume $A$ is faithfully represented (via a lipschitz ($C(X)$-representation $\pi$) in $C(X) \otimes B(\mathcal{H})$ so that we identify any element of $A$ with a function with value in $B(\mathcal{H})$. Note that $A_x$ embeds then in $B(\mathcal{H})$ (via the representation $\pi_x$) since $a_x$ is the evaluation at $x$ of $a \in A$.

Since $X$ is of Hausdorff dimension $N$, there exists a constant $C_1$ such that when $X$ is covered by balls of radius $\eta$, the smallest number of such balls is bounded by $C_1\eta^{-N}$.

Let $\delta$ be positive and $\mathcal{A}$ be a dense finitely generated algebra in $A$ with an epimorphism $p$ from $\mathbb{C} < X_1...X_q >$ onto $\mathcal{A}$ such that $\alpha$ induces a map $\theta$ of $\mathbb{C} < X_1...X_q >$ for which $se(\theta) \leq se(A,\alpha) + \delta$ and choose $\epsilon > 0$ and a finite set $\Omega$ of norm 1 elements in $\mathcal{A}$. There exists then a constant $C_3$ such that for all integer $k$, $||\mathcal{J}\theta^k(P)||_1 \leq C_3 \exp(k(se(\theta)+\delta))$ for any $P$ in a finite set $\bar{\Omega}$ with $p(\bar{\Omega}) = \Omega$.

By Lipschitz continuity, there exists a constant $C_2$ such that for all $b$ in the generating set $S$ of $\mathcal{A}$, $||b_x - b_y|| \leq C_2 d(x,y)^L$.

Consider $\Omega_n = \Omega \cup \alpha(\Omega) \cup \cdots \cup \alpha^n(\Omega)$, we are going to construct now a factorization for $\Omega_n$ with error bounded by $\epsilon$ of the embedding of $A$ in $C(X) \otimes B(\mathcal{H})$.

Take $\eta = [\frac{\epsilon}{2C_2} \min(1, \frac{1}{C_3} \exp(-n(se(\theta)+\delta)))]^{1/L}$ and cover $X$ with balls of radius $\eta$: $X \subset \cup_{j\in J} B(x_j,\eta)$.

Then there exists $\sigma_j$ from $A_{x_j}$ to $M_{p_j}(\mathbb{C})$ completely contractive and $\tau_j$ from $M_{p_j}(\mathbb{C})$ to $B(\mathcal{H})$ completely contractive such that $\forall a \in \Omega_n$, $||\tau_j \circ \sigma_j(a_{x_j}) - a_{x_j}|| \leq \epsilon/2$ and $p_j = rcp(\pi_{x_j}, \epsilon/2, (\Omega_n)_{x_j})$.

Supppose $(\varphi_j)_{j\in J}$ is a partition of unity associated to the covering and consider $\sigma$ from $A$ to $\oplus_{j\in J} M_{p_j}(\mathbb{C})$ defined as

$$\sigma(a) = \underset{j\in J}{\oplus} \sigma_j(a_{x_j})$$

and $\tau$ from $\oplus_{j\in J} M_{p_j}(\mathbb{C})$ to $C(X) \otimes B(\mathcal{H})$ defined as

$$\tau(\underset{j\in J}{\oplus} z_j) = \sum_{j\in J} \varphi_j \tau_j(z_j)$$

Then for all $a \in \Omega_n$, $||\tau\circ\sigma(a) - a|| \leq \epsilon/2 + \max_{j\in J} \sup_{x\in B(x_j,\eta)} ||a_x - a_{x_j}||$

Now if $a \in \Omega$, $\sup_{x\in B(x_j,\eta)} ||a_x - a_{x_j}|| \leq \epsilon/2$ by definition of $\eta$ and the lipschitz continuity.

Let $P$ in $\mathbb{C} < X_1...X_q >$ be such that $p(P) = a$ and define $\Sigma_1$ and $\Sigma_2$ the homomorphisms of $\mathbb{C} < X_1...X_q >$ in $B(\mathcal{H})$ obtained by composition of $p$ with the evaluation at $x$ or at $x_j$.

Then for all integer $k$,

$$\begin{aligned} ||\alpha_x^k(a_x) - \alpha_{x_j}^k(a_{x_j})|| = \quad & ||\Sigma_1(\theta^k(P)) - \Sigma_2(\theta^k(P))|| \\ & \leq ||\mathcal{J}\theta^k(P)||_1 \sup_{1\leq i\leq q} ||\Sigma_1(X_i) - \Sigma_2(X_i)|| \\ & \leq C_3 \exp(k(se(\theta)+\delta)) \sup_{b\in S} ||b_x - b_{x_j}|| \end{aligned}$$

Thus $||\alpha_x^k(a_x)-\alpha_{x_j}^k(a_{x_j})||\leq \epsilon/2$ for $k\leq n$ and $a\in\Omega$, so $||\tau\circ\sigma(a)-a||\leq\epsilon$ for all $a\in\Omega_n$.

Now the rank of the matrix algebra $\oplus_{j\in J}M_{p_j}(\mathbb{C})$ is bounded by $|J|$ times the $\sup_{j\in J}p_j$, i.e. by

$$C_1(\frac{2C_2C_3}{\epsilon})^{N/L}\exp(n\frac{N}{L}(se(A,\alpha)+\delta))\sup_{x\in X}rcp(\pi_x,\epsilon/2,(\Omega_n)_x)$$

For any faithful representation $C(X)$ in $B(K)$, we have a faithful representation $\pi$ of $C(X)\otimes B(H)$ hence $A$ in $B(K\otimes H)$.

Hence

$$\limsup_{n\to\infty}\frac{1}{n}\log rcp(\pi,\epsilon,\Omega\cup\alpha(\Omega)\cup\cdots\cup\alpha^n(\Omega))\leq\frac{N}{L}[se(A,\alpha)+\delta]+$$
$$+\limsup_{n}\frac{1}{n}\sup_{x\in X}\log(rcp(\pi_x,\epsilon,\Omega_x\cup\alpha_x(\Omega_x)\cup\cdots\cup\alpha_x^n(\Omega_x)))$$

Since $ht_A(\alpha)$ is computed by taking the sup over all finite sets of $\mathcal{A}$ since it is dense, we have that, for all $\delta$ positive, $ht_A(\alpha)-\frac{N}{L}se(A,\alpha)$ is bounded by

$$\delta\frac{N}{L}+\sup_{\Omega\in\mathcal{F}(A)}\sup_{\epsilon>0}\limsup_{n}\frac{1}{n}\sup_{x\in X}\log(rcp(\pi_x,\epsilon,\Omega_x\cup\alpha_x(\Omega_x)\cup\cdots\cup\alpha_x^n(\Omega_x)))$$

which is the result.

In the case of an inner automorphism the proof follows the same lines. Let $u$ be the unitary that implements $\alpha$

After choosing the constant $C_1$ and a finite set $\Omega$ of norm 1 elements in $A$, one consider the constant $C_2$ such that for all $a\in\Omega\cup\{u,u^*\}$, $||a_x-a_y||\leq C_2d(x,y)^L$.

For a given $n$ take $\eta=[\frac{\epsilon}{2(2n+1)C_2}]^{1/L}$ and we cover $X$ by balls of radius $\eta$ and center $x_j$ for $j$ in the finite set $J$.

The definition of $\sigma$ and $\tau$ carries along, we just has to explain the majoration of $||\alpha_x^k(a_x)-\alpha_{x_j}^k(a_{x_j})||$ for $a\in\Omega$ and $k\leq n$:

$$\begin{aligned}||\alpha_x^k(a_x)-\alpha_{x_j}^k(a_{x_j})||\leq\;& ||u_x^ka_x(u_x^*)^k-u_{x_j}^ka_{x_j}(u_{x_j}^*)^k||\\ \leq\;& ||a_x-a_{x_j}||+||u_x^ka_{x_j}(u_x^*)^k-u_{x_j}^ka_{x_j}(u_{x_j}^*)^k||\\ \leq\;& ||a_x-a_{x_j}||+2k||a_{x_j}||.||u_x-u_{x_j}||\\ \leq\;& (2k+1)C_2\eta^L\\ \leq\;& \epsilon/2\end{aligned}$$

Therefore $\sigma$ and $\tau$ gives a $\epsilon$-factorization of the set $\Omega \cup \alpha(\Omega) \cup \cdots \cup \alpha^n(\Omega)$ through matrices of rank $|J| \sup_x rcp(\pi_x, \epsilon/2, \Omega_x \cup \alpha_x(\Omega_x) \cup \cdots \cup \alpha_x^n(\Omega_x)$

And since $\limsup_n \frac{1}{n}\frac{N}{L}\log[\frac{2(2n+1)C_2)}{\epsilon}] = 0$ we get our proposition.

In the case of the continuous field of the non-commutative tori,a more precise computation can be made:

**Theorem 3.9.** *Let $M$ be a matrix in $SL_2(\mathbb{Z})$ with non negative entries and $\alpha_M$ the induced automorphism on the continuous field of the non-commutative tori $A = (A_\theta)_{\theta\in\mathbb{T}}$, then*

$$\sup_{\lambda \in Sp(M)} \log|\lambda| \leq ht_A(\alpha_M) \leq 3. \sup_{\lambda \in Sp(M)} \log|\lambda|.$$

Note that it actually gives a computation for an automorphism of the $C^*$-algebra of the Heisenberg group in $M_3(\mathbb{C})$. Indeed this group is generated by the three matrices $u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now call $U, V, W$ the corresponding three unitaries in the group C*-algebra $A$ (the group is amenable so there is no need to specify a norm). Then since $w$ commutes with $u$ and $v$ and is the commutator of the two, we have that $W$ is in the center of $A$ and $UV = WVU$. So $A$ is a $C(\mathbb{T})$-algebra with $\mathbb{T} = Spec(W)$. But Haagerup and Rordam proved that $A$ is actually a lipschitz continuous field of exponent 1/2.

Now take a matrix $M \in SL_2(\mathbb{N})$, then it induces an automorphism $\alpha_A$ of this field as follows:

$$\alpha_M(U) = U^a V^c, \quad \alpha_M(V) = U^b V^d \quad \alpha_M(W) = W$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

One can check that $\alpha_M(U)\alpha_M(V) = W^{\det M}\alpha_M(V)\alpha_M(U)$ and $\alpha_M^{-1} = \alpha_{M^{-1}}$ so that $\det M = 1$ is the only requirement to get an automorphism of the continuous field as well as of all the $A_\theta$.

Now the lower bound comes from the computation for entropy in $A_0 = C^*(\mathbb{Z}^2)$.

For the upper bound, recall that $A_\theta$ is a cocycle group algebra over the group $\mathbb{Z}^2$. Then we can reinterpret proposition 3.3 of [7]. This proposition actually shows that if one chooses a finite set $\Omega$ of elements of $\mathbb{Z}^2$ (and sees them as monomials in the two generators $U$ and $V$) then $\sup_\theta rcp(\epsilon, \Omega_\theta \cup (\alpha_M)_\theta(\Omega_\theta) \cup \cdots \cup (\alpha_M)^n_\theta(\Omega_\theta))$ is bounded by the amenable $\epsilon$-rank of the set $\Omega \cup \alpha_M(\Omega) \cup \cdots \cup \alpha^n_M(\Omega)$ of elements in $\mathbb{Z}^2$ where $\alpha_M$ is now an automorphism of the group $\mathbb{Z}^2$.

Therefore $ht_A(\alpha_M)$ is bounded by $ha(\alpha_M) + \frac{N}{L} se(A, \alpha_M)$, i.e. by

$$\sup_{\lambda \in Sp(M)} \log|\lambda| + 2se(A, \alpha_M).$$

It remains to compute the symbolic entropy of the automorphism. Consider the dense algebra generated by the six unitaries $U$, $V$, $W$, $U^{-1}$, $V^{-1}$, $W^{-1}$. Since the automorphism leaves $W$ invariant, we only need to concentrate on iterates of polynoùmials in $U$,$V$,$U^{-1}$,$V^{-1}$. Since the image of monomials are monomials and we have an algebra homomorphism, we just have to bound the total degree of iterates of each of the unitaries. Because the coefficients of $M$ are all positive (hence no cancellation need to occur between $U$ and $U^{-1}$ or $V$ and $V^{-1}$ hence no commutativity is required) the degree of the $n$-th iterate is given by the matrix product

$$(1,1,1,1). \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}^n .E$$

where $E$ is a vector of the canonical basis of $\mathbb{N}^4$; $(1,0,0,0)$ representing $U$, $(0,1,0,0)$ representing $V$, and so on. But these quantities are bounded by $C.|\lambda|^n$ where $\lambda$ is the eigenvalue of $M$ of maximal modulus. Hence $se(A, \alpha_M) = \sup_{\lambda \in Sp(M)} \log|\lambda|$.

*Remark 3.10.* In a private communication , N.P. Brown mentionned that a computation of the "dual entropy" (see[7]) of the same automorphism of the field of non-commutative tori, but this time seen as an automorphism of the $C^*$-algebra of the Heisenberg group is possible and that one gets the same upper bound.

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Emmanuel Germain
Institut de Mathematique de Jussieu
Université Paris 7 Denis Diderot et
CNRS
175 rue du Chevaleret
75013 Paris
FRANCE
germain@math.jussieu.fr