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Abstract

The main result of this note is that a finitely generated hyper–(Abelian–by–finite) group $G$ is finite–by-nilpotent if and only if every infinite subset contains two distinct elements $x, y$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$ for some positive integer $n = n(x, y)$ (respectively, $\langle x, x^y \rangle$ is an extension of a group satisfying the minimal condition on normal subgroups by an Engel group).

1. Introduction and results

Let $\mathcal{X}$ be a class of groups. Denote by $(\mathcal{X}, \infty)$ (respectively, $(\mathcal{X}, \infty)^*$) the class of groups $G$ such that for every infinite subset $X$ of $G$, there exist distinct elements $x, y \in X$ such that $\langle x, y \rangle \in \mathcal{X}$ (respectively, $\langle x, x^y \rangle \in \mathcal{X}$). Note that if $\mathcal{X}$ is a subgroup closed class, then $(\mathcal{X}, \infty) \subseteq (\mathcal{X}, \infty)^*$.

In answer to a question of Erdős, B.H. Neumann proved in [16] that a group $G$ is centre–by–finite if and only if $G$ is in the class $(\mathcal{A}, \infty)$, where $\mathcal{A}$ denotes the class of Abelian groups. Lennox and Wiegold showed in [13]...
that a finitely generated soluble group is in the class $(\mathcal{N}, \infty)$ (respectively, $(\mathcal{P}, \infty)$) if and only if it is finite-by-nilpotent (respectively, polycyclic), where $\mathcal{N}$ (respectively, $\mathcal{P}$) denotes the class of nilpotent (respectively, polycyclic) groups. Other results of this type have been obtained, for example in [1]—[3], [4]—[6], [7], [8], [13], [14]—[16], [21], [22] and [23].

We say that a group $G$ has finite depth if the lower central series of $G$ stabilises after a finite number of steps. Thus if $\gamma_n(G)$ denotes the $n^{th}$ term of the lower central series of $G$, then $G$ has finite depth if and only if $\gamma_n(G) = \gamma_{n+1}(G)$ for some positive integer $n$. Denote by $\Omega$ the class of groups which has finite depth. Moreover, if $k$ is a fixed positive integer, let $\Omega_k$ denotes the class of groups $G$ such that $\gamma_k(G) = \gamma_{k+1}(G)$.

Clearly, any group in the class $\mathcal{FN}$ is of finite depth, where $\mathcal{F}$ denotes the class of finite groups. From this and the fact that $\mathcal{FN}$ is a subgroup closed class, we deduce that finite-by-nilpotent groups belong to $(\Omega, \infty)^*$. Here we shall be interested by the converse. In [5], Boukaroura has proved that a finitely generated soluble group in the class $(\Omega, \infty)$ is finite-by-nilpotent. We obtain the same result when $(\Omega, \infty)$ is replaced by $(\Omega, \infty)^*$ and soluble by hyper-(Abelian-by-finite). More precisely we shall prove the following result.

**Theorem 1.1.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group. Then, $G$ is in the class $(\Omega, \infty)^*$ if, and only if, $G$ is finite-by-nilpotent.

Note that Theorem 1.1 improves the result of [12] which asserts that a finitely generated soluble-by-finite group whose subgroups generated by two conjugates are of finite depth, is finite-by-nilpotent.

It is clear that an Abelian group $G$ in the class $(\Omega_1, \infty)^*$ is finite. For if $G$ is infinite, then it contains an infinite subset $X = G \setminus \{1\}$. Therefore there exist two distinct elements $x, y \neq 1$ in $X$ such that $\gamma_1((x, x^y)) = \gamma_2((x, x^y)) = 1$; so $x = 1$, which is a contradiction. From this it follows that a hyper-(Abelian-by-finite) group $G$ in the class $(\Omega_1, \infty)^*$ is hyper-finite as $(\Omega_1, \infty)^*$ is a subgroup and a quotient closed class. But it is not difficult to see that a hyper-(finite) group is locally finite [17, Part 1, page 36]. So $G$ is locally finite. Now if $G$ is infinite, then it contains an infinite Abelian subgroup $A$ [17, Theorem 3.43]. Since $A$ is in the class $(\Omega_1, \infty)^*$, it is finite; a contradiction and $G$, therefore, is finite. As consequence of Theorem 1.1, we shall prove other results on the class $(\Omega_k, \infty)^*$.

**Corollary 1.2.** Let $k$ be a positive integer and let $G$ be a finitely generated hyper-(Abelian-by-finite) group. We have:
A condition on infinite subsets

(i) If $G$ is in the class $(\Omega_k, \infty)^*$, then there exists a positive integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\Omega_2, \infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If $G$ is in the class $(\Omega_3, \infty)^*$, then $G$ is in the class $\mathcal{FN}_3^{(2)}$, where $\mathcal{N}_3^{(2)}$ denotes the class of groups whose 2-generator subgroups are nilpotent of class at most 3.

Let $k$ be a fixed positive integer, denote by $\mathcal{M}$, $\mathcal{E}_k$ and $\mathcal{E}$ respectively the class of groups satisfying the minimal condition on normal subgroups, the class of $k$-Engel groups and the class of Engel groups. Using Theorem 1.1, we will prove the following results concerning the classes $(\mathcal{M}\mathcal{E}, \infty)^*$ and $(\mathcal{M}\mathcal{E}_k, \infty)^*$

**Theorem 1.3.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group. Then, $G$ is in the class $(\mathcal{M}\mathcal{E}, \infty)^*$ if, and only if, $G$ is finite-by-nilpotent.

Note that this theorem improves Theorem 3 of [23] (respectively, Corollary 3 of [5]) where it is proved that a finitely generated soluble group in the class $(\mathcal{C}\mathcal{N}, \infty)^*$ (respectively, $(\mathcal{X}\mathcal{N}, \infty)$) is finite-by-nilpotent, where $\mathcal{C}$ (respectively, $\mathcal{X}$) denotes the class of Chernikov groups (respectively, the class of groups satisfying the minimal condition on subgroups).

**Corollary 1.4.** Let $k$ be a positive integer and let $G$ be a finitely generated hyper-(Abelian-by-finite) group. We have:

(i) If $G$ is in the class $(\mathcal{M}\mathcal{E}_k, \infty)^*$, then there exists a positive integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\mathcal{M}\mathcal{A}, \infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If $G$ is in the class $(\mathcal{M}\mathcal{E}_2, \infty)^*$, then $G$ is in the class $\mathcal{FN}_3^{(2)}$.

Note that these results are not true for arbitrary groups. Indeed, Golod [9] showed that for each integer $d > 1$ and each prime $p$, there are infinite $d$-generator groups all of whose $(d - 1)$-generator subgroups are finite $p$-groups. Clearly, for $d = 3$, we obtain a group $G$ which belongs to the class $(\mathcal{F}, \infty)^*$. Therefore, $G$ belongs to the classes $(\Omega, \infty)^*$, $(\Omega_k, \infty)^*$, $(\mathcal{M}\mathcal{E}, \infty)^*$ and $(\mathcal{M}\mathcal{E}_k, \infty)^*$, but it is not finite-by-nilpotent.

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2. Proofs of Theorem 1.1 and Corollary 1.2

Let $E(\infty)$ the class of groups in which every infinite subset contains two distinct elements $x, y$ such that $[x, n y] = 1$ for a positive integer $n = n(x, y)$. In [15], it is proved that a finitely generated soluble group in the class $E(\infty)$ is finite-by-nilpotent. We will extend this result to finitely generated hyper-(Abelian-by-finite) groups (Proposition 2.5).

Our first lemma is a weaker version of Lemma 11 of [23], but we include a proof to keep our paper reasonably self contained.

**Lemma 2.1.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $(FN, \infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be a finitely generated infinite Abelian-by-finite group in the class $(FN, \infty)$. Hence there is a normal torsion-free Abelian subgroup $A$ of finite index. Let $x$ be a non trivial element in $A$ and let $g$ in $G$. Then the subset $\{x^i g : i \text{ a positive integer}\}$ is infinite, so there are two positive integers $m, n$ such that $\langle x^m g, x^n g \rangle$ is finite-by-nilpotent, hence $\langle x^r, x^n g \rangle$ is finite-by-nilpotent where $r = m - n$. Thus there are two positive integers $c$ and $d$ such that $[x^r, c x^n g]^d = 1$. The element $x$ being in $A$ which is Abelian and normal in $G$, we have $[x^r, c x^n g] = [x^r, c g]^r$; so $[x^r, c g]^{r.d} = 1$. Now $[x, c g]$ belongs to the torsion-free group $A$, so $[x, c g] = 1$. It follows that $x$ is a right Engel element of $G$. Since $G$ is Abelian-by-finite and finitely generated, it satisfies the maximal condition on subgroups; so the set of right Engel elements of $G$ coincides with its hypercentre which is equal to $Z_i (G)$, the $(i + 1)$-th term of the upper central series of $G$, for some integer $i > 0$ [17, Theorem 7.21]. Hence, $A \leq Z_i (G)$; and since $A$ is of finite index in $G$, $G/Z_i (G)$ is finite. Thus, by a result of Baer [10, Theorem 1], $G$ is finite-by-nilpotent. \hfill $\square$

**Lemma 2.2.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $E(\infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be an infinite finitely generated Abelian-by-finite group in the class $E(\infty)$, and let $A$ be an Abelian normal subgroup of finite index in $G$. It is clear that all infinite subsets of $G$ contains two different elements $x, y$ such that $xA = yA$; so $y = xa$ for some $a$ in $A$ and $\langle x, y \rangle = \langle x, a \rangle$. Thus $\langle x, y \rangle$ is a finitely generated metabelian group in the class $E(\infty)$. It follows by the result of Longobardi and Maj [15, Theorem 1], that $\langle x, y \rangle$...
is finite-by-nilpotent. Hence $G$ is in the class $(\mathcal{F}\mathcal{N}, \infty)$. Now, by Lemma 2.1, $G$ is finite-by-nilpotent; as required. □

Lemma 2.3. A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is nilpotent-by-finite.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Since $\mathcal{E}(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that $G$ is not nilpotent-by-finite but every proper homomorphic image of $G$ is in the class $\mathcal{N}\mathcal{F}$. Since $G$ is hyper-(Abelian-by-finite), $G$ contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so we have $G/H$ is in $\mathcal{N}\mathcal{F}$. If $H$ is finite then $G$ is nilpotent-by-finite, a contradiction. Consequently $H$ is Abelian and so $G$ is Abelian-(nilpotent-by-finite) and therefore it is (Abelian-nilpotent)-by-finite. Hence, $G$ is a finite extension of a soluble group; there is therefore a normal soluble subgroup $K$ of $G$ of finite index. Now, $K$ is a finitely generated soluble group in the class $\mathcal{E}(\infty)$; it follows, by the result of Longobardi and Maj [15, Theorem 1], that $K$ is finite-by-nilpotent. By a result of P. Hall [10, Theorem 2], $K$ is nilpotent-by-finite and so $G$ is nilpotent-by-finite, a contradiction. Now, the Lemma is shown. □

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 2.3 has the following consequence:

Corollary 2.4. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Then $G$ satisfies the maximal condition on subgroups.

Proposition 2.5. A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in $\mathcal{E}(\infty)$. According to Corollary 2.4, $G$ satisfies the maximal condition on subgroups. Now, since $\mathcal{E}(\infty)$ is a quotient closed class, we may assume that every proper homomorphic image of $G$ is in $\mathcal{F}\mathcal{N}$, but $G$ itself is not in $\mathcal{F}\mathcal{N}$. Our group $G$ being hyper-(Abelian-by-finite), contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so by hypothesis $G/H$ is in the class $\mathcal{F}\mathcal{N}$. If $H$ is finite, then $G$ is finite-by-nilpotent, a contradiction. Consequently $H$ is Abelian and so $G$ is in the class $\mathcal{A}(\mathcal{F}\mathcal{N})$, hence $G$ is in $(\mathcal{A}\mathcal{F})\mathcal{N}$. Now, since $G$ satisfies the maximal condition on
subgroups, it follows from Lemma 2.2, that $G$ is in $(\mathcal{FN})\mathcal{N}$, so it is in $\mathcal{F}(\mathcal{NN})$. Consequently, there is a finite normal subgroup $K$ of $G$ such that $G/K$ is soluble. The group $G/K$, being a finitely generated soluble group in the class $\mathcal{E}(\infty)$, is in $\mathcal{FN}$, by the result of Longobardi and Maj [15, Theorem 1]. So $G$ is in the class $\mathcal{F}\mathcal{N}$, which is a contradiction and the Proposition is shown. □

The remainder of the proof of Theorem 1.1 is adapted from that of Lennox’s Theorem [11, Theorem 3]

**Lemma 2.6.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. If $G$ is residually nilpotent, then $G$ is in the class $\mathcal{FN}$.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$ and assume that $G$ is residually nilpotent. Let $X$ be an infinite subset of $G$, there are two distinct elements $x$ and $y$ of $X$ such that $\langle x, x^y \rangle \in \Omega$. It follows that there exists a positive integer $k$ such that $\gamma_k(\langle x, x^y \rangle) = \gamma_{k+1}(\langle x, x^y \rangle)$. The group $\langle x, x^y \rangle$, being a subgroup of $G$, is residually nilpotent, so $\bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1$. Hence $\gamma_k(\langle x, x^y \rangle) = \bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1$. Since $\langle x, x^y \rangle = \langle [y, x], x \rangle$; $\gamma_k([y, x], x) = 1$, thus $[y, k, x] = 1$. We deduce that $G$ is a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. It follows, by Proposition 2.5, that $G$ is in the class $\mathcal{FN}$, as required. □

**Lemma 2.7.** If $G$ is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$, then it is nilpotent-by-finite.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in $(\Omega, \infty)^*$. Since finitely generated nilpotent-by-finite groups are finitely presented and $(\Omega, \infty)^*$ is a quotient closed class of groups, by [17, Lemma 6.17], we may assume that every proper quotient of $G$ is nilpotent-by-finite, but $G$ itself is not nilpotent-by-finite. Since $G$ is hyper-(Abelian-by-finite), it contains a non-trivial normal subgroup $K$ such that $K$ is finite or Abelian; so $G/K$ is in $\mathcal{NF}$. In this case, $K$ is Abelian and so $G$ is in the class $\mathcal{A}(\mathcal{NF})$ and therefore it is in the class $\mathcal{AN}\mathcal{F}$. Consequently, $G$ has a normal subgroup $N$ of finite index such that $N$ is Abelian-by-nilpotent. Moreover, $N$ being a subgroup of finite index in a finitely generated group, is itself finitely generated, and so $N$ is a finitely generated Abelian-by-nilpotent group. It follows, by a result of Segal [19,
Corollary 1], that $N$ has a residually nilpotent normal subgroup of finite index. Thus, $G$ has a residually nilpotent normal subgroup $H$, of finite index. Therefore, $H$ is residually nilpotent and it is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. So, by Lemma 2.6, $H$ is in the class $\mathcal{FN}$, hence $H$ is in the class $\mathcal{NF}$. Thus $G$ is in the class $\mathcal{NF}$, a contradiction which completes the proof.

**Lemma 2.8.** Let $G$ be a finitely generated group in the class $(\Omega, \infty)^*$ which has a normal nilpotent subgroup $N$ such that $G/N$ is a finite cyclic group. Then $G$ is in the class $\mathcal{FN}$.

**Proof.** We prove by induction on the order of $G/N$ that $G$ is in the class $\mathcal{FN}$. Let $n = |G/N|$; if $n = 1$, then $G = N$ and $G$ is nilpotent. Now suppose that $n > 1$ and let $q$ be a prime dividing $n$. Since $G/N$ is cyclic, it has a normal subgroup of index $q$. Thus $G$ has a normal subgroup $H$ of index $q$ containing $N$. Since $|H/N| < |G/N|$, then by the inductive hypothesis, $H$ is in the class $\mathcal{FN}$. Let $T$ be the torsion subgroup of $H$. Since $H$ is finitely generated, $T$ is finite. So $H/T$ is a finitely generated torsion-free nilpotent group. Therefore, by Gruenberg [18, 5.2.21], $H/T$ is residually a finite $p$-group for all primes $p$ and hence, in particular, $H/T$ is residually a finite $q$-group. But $H$ has index $q$ in $G$ from which we get that $G/T$ is residually a finite $q$-group [20, Exercise 10, page 17]. This means that $G/T$ is residually nilpotent. It follows, by Lemma 2.6, that $G/T$ is in the class $\mathcal{FN}$. So $G$ itself is in $\mathcal{FN}$. □

**Proof of Theorem 1.1.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. Hence, by Lemma 2.7, $G$ is in the class $\mathcal{NF}$. Let $K$ be a normal nilpotent subgroup of $G$ such that $G/K$ is finite. Since $K$ is a finitely generated nilpotent group, it has a normal torsion-free subgroup of finite index [18, 5.4.15 (i)]. Thus, $G$ has a normal torsion-free nilpotent subgroup $N$ of finite index. Let $x$ be a non-trivial element of $G$. Since $N$ is finitely generated, $\langle N, x \rangle$ is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. Furthermore, $\langle N, x \rangle/N$ is cyclic. Therefore, by Lemma 2.8, $\langle N, x \rangle$ is in the class $\mathcal{FN}$. Consequently, there is a finite normal subgroup $H$ of $\langle N, x \rangle$ such that $\langle N, x \rangle/H$ is nilpotent. Therefore $\gamma_{k+1}(\langle N, x \rangle) \leq H$ for some positive integer $k$; so $\gamma_{k+1}(\langle N, x \rangle)$ is finite. Hence, there is a positive integer $m$ such that $[g, x]^m = 1$, for all $g \in N$. Since $[g, x]$ is an element of the torsion-free group $N$, we get that $[g, x] = 1$. Thus, $g$ is a right Engel element of $G$; so $N \subseteq R(G)$,
where \( R(G) \) denotes the set of right Engel elements of \( G \). Moreover, since \( G \) is a finitely generated nilpotent-by-finite group, it satisfies the maximal condition on subgroups. Therefore, from Baer [17, Theorem 7.21], \( R(G) \) coincides with the hypercentre of \( G \) which equal to \( Z_n(G) \) for some positive integer \( n \). Thus \( N \leq Z_n(G), \) so \( Z_n(G) \) is of finite index in \( G \). It follows, by a result of Baer [10, Theorem 1], that \( G \) is in the class \( \mathcal{F}N^\ast \).

Proof of Corollary 1.2. (i) Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega_k, \infty)^*\); from Theorem 1.1, \( G \) is in the class \( \mathcal{F}N \). Let \( H \) be a normal finite subgroup of \( G \) such that \( G/H \) is nilpotent. It is clear that \( G/H \) is in the class \((\Omega_k, \infty)^*\). Let \( X \) be an infinite subset of \( G/H \); there are therefore two distinct elements \( \bar{x} = xH, \bar{y} = yH \) \((x, y \in G)\) of \( X \) such that \( \langle \bar{x}, \bar{x}^y \rangle \subseteq \Omega_k \), so \( \gamma_k(\langle \bar{x}, \bar{x}^y \rangle) = \gamma_k(\langle \bar{x}, \bar{x}^y \rangle) \). Now, since \( \langle \bar{x}, \bar{x}^y \rangle \) is nilpotent, there is an integer \( i \) such that \( \gamma_i(\langle \bar{x}, \bar{x}^y \rangle) = 1 \); so \( \gamma_k(\langle \bar{x}, \bar{x}^y \rangle) = 1 \). Since \( \langle \bar{x}, \bar{x}^y \rangle = \langle [\bar{y}, \bar{x}], \bar{x} \rangle \), we have \( \gamma_k(\langle [\bar{y}, \bar{x}], \bar{x} \rangle) = 1 \) and thus \( [\bar{y}, \bar{x}] = 1 \). Consequently, \( G/H \) is in the class \( \mathcal{E}_k(\infty) \) of groups in which every infinite subset contains two distinct elements \( g, h \) such that \( [g, k, h] = 1 \). The group \( G/H \), being a finitely generated soluble group in the class \( \mathcal{E}_k(\infty) \); it follows by a result of Abdollahi [2, Theorem 3], that there is an integer \( c = c(k) \), depending only on \( k \), such that \( (G/H)/Z_c(G/H) \) is finite. By a result of Baer [10, Theorem 1], \( \gamma_{c+1}(G/H) = \gamma_{c+1}(G)H/H \) is finite; and since \( H \) is finite, \( \gamma_{c+1}(G) \) is finite. According to a result of P. Hall [10, 1.5], \( G/Z_c(G) \) is finite.

(ii) If \( G \) is in the class \((\Omega_2, \infty)^*\), then by Theorem 1.1 \( G \) is finite-by-nilpotent. Therefore, \( G \) has a finite normal subgroup \( H \) such that \( G/H \) is nilpotent. Since \( G/H \) is in the class \((\Omega_2, \infty)^*\), it is in the class \( \mathcal{E}_2(\infty) \). Hence, by Abdollahi [1, Theorem], \( (G/H)/Z_2(G/H) \) is finite, so \( \gamma_3(G/H) \) is finite. Since \( H \) is finite, \( \gamma_3(G) \) is finite. It follows, by P. Hall [10, 1.5], that \( G/Z_2(G) \) is finite.

(iii) Now if \( G \) is in the class \((\Omega_3, \infty)^*\), then by Theorem 1.1 \( G \) has a finite normal subgroup \( H \) such that \( G/H \) is nilpotent. Since \( G/H \) is in the class \((\Omega_3, \infty)^*\), it is in the class \( \mathcal{E}_3(\infty) \). Hence, by Abdollahi [2, Theorem 1] \( G/H \) is in the class \( \mathcal{F}N_3^\ast \); consequently \( G \) is in the class \( \mathcal{F}N_3^\ast \). \( \square \)

3. Proofs of Theorem 1.3 and Corollary 1.4

We start by showing a weaker version of Theorem 1.3:
Lemma 3.1. A finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{MN}, \infty)^*\) is finite-by-nilpotent.

Proof. Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{MN}, \infty)^*\), and let \(X\) be an infinite subset of \(G\). There are therefore two distinct elements \(x, y\) of \(X\) such that \(\langle x, x^y \rangle\) is in the class \(\mathcal{M}\), so there exists a normal subgroup \(N\) of \(\langle x, x^y \rangle\) such that \(N\) is in \(\mathcal{M}\) and \(\langle x, x^y \rangle/N\) is nilpotent. Now, \(\gamma_{i+1}(\langle x, x^y \rangle) \leq N\) for some positive integer \(i\), therefore \(\gamma_{i+1}(\langle x, x^y \rangle) \geq \gamma_{i+2}(\langle x, x^y \rangle) \geq ...\) is an infinite descending sequence of normal subgroups of \(N\); however \(N\) is in \(\mathcal{M}\), therefore there exists a positive integer \(n \geq i + 1\) such that \(\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)\). Hence, \(G\) is in the class \((\Omega, \infty)^*\); it follows, by Theorem 1.1, that \(G\) is finite-by-nilpotent. \(\Box\)

Lemma 3.2. A finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{ME}, \infty)^*\) is nilpotent-by-finite.

Proof. Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{ME}, \infty)^*\). Since \((\mathcal{ME}, \infty)^*\) is a closed quotient class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that \(G\) is not nilpotent-by-finite, but every proper homomorphic image of \(G\) is nilpotent-by-finite. Since \(G\) is hyper-(Abelian-by-finite), there exists a non-trivial normal subgroup \(H\) of \(G\) such that \(H\) is finite or Abelian; so we have \(G/H\) is nilpotent-by-finite. If \(H\) is finite then \(G\) is nilpotent-by-finite, a contradiction. Consequently \(H\) is Abelian and so \(G\) is Abelian-by-(nilpotent-by-finite) and therefore it is \((\text{Abelian-by-nilpotent})\)-by-finite. Hence, \(G\) is a finite extension of a soluble group. Let \(K\) be a normal soluble subgroup of \(G\) of finite index. Clearly, \(K\) is in \((\mathcal{ME}, \infty)^*\), and since all soluble Engel group coincides with its Hirsch-Plotkin radical which is locally nilpotent [17, Theorem 7.34], we deduce that \(K\) is in the class \((\mathcal{MN}, \infty)^*\); it follows by Lemma 3.1 that \(K\) is finite-by-nilpotent. According to a result of P. Hall [10, Theorem 2], \(K\) is nilpotent-by-finite. Thus, \(G\) is nilpotent-by-finite, a contradiction. The proof is now complete. \(\Box\)

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 3.2 has the following consequence:

Corollary 3.3. Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{ME}, \infty)^*\). Then \(G\) satisfies the maximal condition on subgroups.
Proof of Theorem 1.3. It is clear that all finite-by-nilpotent groups are in the class \((\mathcal{ME}, \infty)^*\). Conversely, let \(G\) be a finitely generated hyper-
(Abelian-by-finite) group in \((\mathcal{ME}, \infty)^*\). According to Corollary 3.3, \(G\) satisfies the maximal condition on subgroups. Since Engel groups satisfying the maximal condition on subgroups are nilpotent \([18, 12.3.7]\), we deduce that \(G\) is in the class \((\mathcal{MN}, \infty)^*\). It follows, by Lemma 3.1, that \(G\) is in the class \(\mathcal{FN}\); as required. \(\square\)

Proof of Corollary 1.4. (i) Let \(G\) be a finitely generated hyper-
(Abelian-by-finite) group in the class \(\mathcal{F\!N} \ldots\); from Theorem 1.3, \(G\) is in the class \(\mathcal{FN}\). Let \(N\) be a normal finite subgroup of \(G\) such that \(G/N\) is nilpotent. Since \(G/N\) is nilpotent and finitely generated, its torsion subgroup \(T/N\) is finite, so \(T\) is finite and \(G/T\) is a torsion-free nilpotent group. Clearly, the property \((\mathcal{ME}, \infty)^*\) is inherited by \(G/T\), and since \(G/T\) is torsion-free and soluble, it belongs to \((\mathcal{E}, \infty)^*\) \([17, \text{Theorem } 5.25]\). Let \(\bar{X}\) be an infinite subset of \(G/T\); there are therefore two distinct elements \(\bar{x} = xT, \bar{y} = yT \ (x, y \in G)\) of \(\bar{X}\) such that \(\langle \bar{x}, \bar{x}^{-1}\bar{y}\rangle\) is a \(k\)-Engel group. Since \(\langle \bar{x}, \bar{x}^{-1}\bar{y}\rangle = \langle [\bar{y}, \bar{x}], \bar{x}\rangle\), we have \([\bar{y}, \bar{x}], k \bar{x}] = 1\). Hence, \(G/T\) is in the class \(\mathcal{E}_{k+1}(\infty)\). The group \(G/T\), being a finitely generated soluble group in the class \(\mathcal{E}_{k+1}(\infty)\); it follows by a result of Abdollahi \([2, \text{Theorem } 3]\), that there is an integer \(c = c(k)\), depending only on \(k\), such that \((G/T)/Z_c(G/T)\) is finite. By a result of Baer \([10, \text{Theorem } 1]\), \(\gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T\) is finite; and since \(T\) is finite, \(\gamma_{c+1}(G)\) is finite. According to a result of P. Hall \([10, 1.5]\), \(G/Z_c(G)\) is finite.

(ii) If \(G\) is in the class \((\mathcal{MA}, \infty)^* = (\mathcal{ME}_1, \infty)^*\), then by Theorem 1.3, \(G\) is finite-by-nilpotent. We proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_2(\infty)\). Hence, by Abdollahi \([1, \text{Theorem } 1]\), \((G/T)/Z_2(G/T)\) is finite, so \(\gamma_3(G/T)\) is finite. Since \(T\) is finite, \(\gamma_3(G)\) is finite. It follows, by P. Hall \([10, 1.5]\), that \(G/Z_2(G)\) is finite.

(iii) Now if \(G\) is in the class \((\mathcal{ME}_2, \infty)^*\), we proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_3(\infty)\). Hence, by Abdollahi \([2, \text{Theorem } 1]\) \(G/T\) is in the class \(\mathcal{F\!N}^{(2)}_3\); consequently \(G\) is in the class \(\mathcal{F\!N}^{(2)}_3\). \(\square\)

References

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