Annales mathématiques Blaise Pascal

Camillo Melzi

Large time estimates for non-symmetric heat kernel on the affine group

Annales mathématiques Blaise Pascal, tome 9, nº 1 (2002), p. 63-78

<http://www.numdam.org/item?id=AMBP_2002__9_1_63_0>

ANNALES MATHEMATIQUES BLAISE PASCAL 9, 63-78 (2002)

Large time estimates for non-symmetric heat kernel on the affine group

Camillo Melzi

Abstract

We consider the heat kernel ϕ_t associated to the left invariant Laplacian with a drift term, on the affine group of the line. We obtain a large time upper estimate for ϕ_t.

1 Introduction

Let G be the two dimensional Lie group of affine transformations on $\mathbb{R}$, $\sigma : \xi \mapsto y\xi + x$ ($\xi \in \mathbb{R}$) with $0 < y = e^t \in \mathbb{R}_+^*$ and $t, x \in \mathbb{R}$. This group is the only non abelian two dimensional Lie group, and it can be seen as the semidirect product $\mathbb{R} \rtimes \mathbb{R}_+^*$ since

$$\sigma_1\sigma_2 : \xi \mapsto y_1y_2\xi + x_2y_1 + x_1,$$

where the action of $\mathbb{R}_+^*$ on $\mathbb{R}$ is $x \mapsto xy$.

The Lie algebra $\mathfrak{g}$ of G is spanned by the left-invariant vector fields

$$X = y\frac{\partial}{\partial x} \qquad \text{and} \qquad Y = y\frac{\partial}{\partial y}.$$

A left invariant distance d on G, called the control distance, is associated to these vector fields (cf. [9]). We denote by $|g| = d(e, g)$, where $e = (0, 1)$ is the identity element of G.

The group G is not unimodular, indeed $d^r g = y^{-1}dxdy$ is the right Haar measure whereas $d^l g = y^{-2}dxdy$ is the left Haar measure. The modular function $m(g) = d^r g/d^l g$ is thus $m(g) = y,\ g = (x, y)$.
It is a solvable Lie group, and the lack of unimodularity implies that G is of exponential growth.

One can consider on G the Laplacian with drift term

$$\begin{aligned} L &= -(X^2+Y^2)+\mu X+\nu Y \\ &= -(y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+y\frac{\partial}{\partial y})+\mu y\frac{\partial}{\partial x}+\nu y\frac{\partial}{\partial y}, \end{aligned} \tag{1.1}$$

where μ and ν are real numbers.

The operator L generates a diffusion semigroup e^{-tL} (cf. [2], [7]). We denote by ϕ_t the kernel of e^{-tL} with respect to the left Haar measure on G, i.e.

$$T_t f(g) = \int_G \phi_t(\xi^{-1}g)f(\xi)d^l\xi, \qquad t>0,\ g\in G,\ f\in C_0^\infty(G).$$

Many authors have studied the heat kernel associated to a driftless Laplacian on various Lie groups and Riemannian manifolds (cf. for instance [6], and references there, for an interesting survey). In our setting, the affine group, an explicit formula of the (driftless) kernel is known (see e.g. [4] and (2.4) below).

Note that $G \simeq \mathbb{H}^2$ (the hyperbolic space) as a manifold. Further the control distance coincides with the Riemannian distance and the left invariant measure is the Riemannian measure. Since the action of G on itself is isometric the vectors fields X, Y form on the tangent space a basis at every point of G. Finally $|\nabla f|^2 = (Xf)^2+(Yf)^2$ coincides with the square of the norm of the Riemannian gradient.

The first order terms of (1.1) are not invariant under the isometry group of $\mathbb{H}^2$ but they are invariant under the action of G. This makes it natural to study L as an invariant operator on G.

The small time behavior of the kernel associated to Laplacians with drift has been studied by Varopoulos in [11] and [12]. In our setting we have the upper estimate (cf. [15], [14], [11] and [12])

$$\phi_t(g) \le Ct^{-1}y^{-1/2}\exp(-\frac{|g|^2}{Ct}), \qquad 0<t<1,\ g=(x,y)\in G. \tag{1.2}$$

In the setting of Lie groups of polynomial growth, a large time upper estimate for the kernel has been obtained by Alexopoulos (cf. [1]).

In this paper we study the large time behavior of ϕ_t on the affine group. Note that the non unimodularity of the affine group and the consequent exponential volume growth lead to additional difficulties. The techniques used in this paper are therefore completely different from those used in [1]. We obtain that if $\nu = 0$ then the kernel ϕ_t satisfies the estimate:

Theorem 1.1: *There exists $C, c > 0$ such that*

$$\phi_t(g) \leq Ct^{-3/2} y^{-1/2} \exp(-\frac{|g|^2}{ct}), \qquad t > 1,\ g = (x, y) \in G. \tag{1.3}$$

If we conjugate the Laplacian (1.1) with the multiplicative character on G

$$\chi(g) = y^{\nu/2}, \qquad g = (x, y) \in G,$$

we obtain the operator

$$\chi^{-1} L(\chi \cdot) = -(X^2 + Y^2) + \mu X + \frac{\nu^2}{4}.$$

Therefore (1.3) implies the following upper estimate for the kernel of the Laplacian (1.1):

Theorem 1.2: *There exists $C, c > 0$ such that*

$$\phi_t(g) \leq Ct^{-3/2} e^{-\nu^2 t/4} y^{\nu/2 - 1/2} \exp(-\frac{|g|^2}{ct}), \qquad t > 1,\ g = (x, y) \in G.$$

From now on we consider the Laplacian (1.1) with $\nu = 0$, that we denote by L_0.

Throughout this paper positive constants are denoted either by the letters c or C. These may differ from one line to another.

Acknowledgement. *I would like to thank S. Meda, who suggested me the problem, and A. Setti for several inspiring discussions during the preparation of this paper.*

2 Preliminaries

The left invariance of the distance d forces us to work with the left Haar measure. The driftless Laplacian

$$\begin{aligned}\Delta &= -(X^2+Y^2)\\ &= -(y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+y\frac{\partial}{\partial y})\end{aligned}$$

is formally self-adjoint with respect to $d^r g$. It follows that the conjugated Δ with the modular function m gives rise to an operator

$$\begin{aligned}\tilde{\Delta} &= m^{1/2}\Delta(m^{-1/2}\cdot)\\ &= -y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-\frac{1}{4}\end{aligned}$$

which is formally self-adjoint with respect to $d^l g$.

Let us remark that if we conjugate the vector field X with m we have

$$m^{1/2}X(m^{-1/2}\cdot)=X,$$

and it follows that the left-invariant field X is anti-symmetric with respect to $d^l g$ too.

Let

$$\begin{aligned}\tilde{L_0} &= m^{1/2}L_0(m^{-1/2}\cdot) \qquad (2.4)\\ &= -(y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+\frac{1}{4})+\mu y\frac{\partial}{\partial x}\end{aligned}$$

the conjugated Laplacian with the modular function m.

It is easy to see that also $\tilde{L_0}$ is a left-invariant operator on G.

Let $\tilde{T_t}=e^{-t\tilde{L_0}}$ be the diffusion semigroup generated by $\tilde{L_0}$ and let $\tilde{\phi_t}$ be its kernel with respect to $d^l g$. By (2.4) we have

$$\tilde{T_t}=m^{1/2}T_t(m^{-1/2}\cdot)$$

and therefore

$$\tilde{\phi_t}(g)=m^{1/2}(g)\phi_t(g). \qquad (2.5)$$

Because of the anti-symmetry of the field X, the adjoint operator $\tilde{L_0}^*$ of (2.4) on $L^2(G, d^l g)$ is given by

$$\begin{aligned}\tilde{L_0}^* &= \tilde{\Delta} - \mu X \\ &= -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - \mu y \frac{\partial}{\partial x} - \frac{1}{4}.\end{aligned} \tag{2.6}$$

We denote by $\tilde{T}_t^* = e^{-t\tilde{L_0}^*}$ the semigroup generated by $\tilde{L_0}^*$ and by $\tilde{\phi}_t^*$ its kernel with respect to $d^l g$. We have that

$$\tilde{\phi}_t(g) = \tilde{\phi}_t^*(g^{-1}).$$

In the following of this paper we shall prove the upper Gaussian estimate for the kernel $\tilde{\phi}_t$

$$\tilde{\phi}_t(g) \leq Ct^{-3/2} \exp(-\frac{|g|^2}{Ct}), \qquad t > 1,\ g \in G.$$

Then (1.3) follows from (2.5).

We begin by estimating $\left\|\tilde{\phi}_t\right\|_\infty = \sup_{g\in G} \tilde{\phi}_t(g)$. We shall now define by h_t (resp. $\tilde{h}_t$) the convolution kernel of $e^{-t\Delta}$ (resp. $e^{-t\tilde{\Delta}}$). We have

$$\tilde{h}_t(g) = m^{1/2}(g) h_t(g). \tag{2.7}$$

It is known that the kernel h_t satisfies the on-diagonal upper estimates (cf. [3], cf. also [13], [14])

$$h_t(e) \leq Ct^{-3/2}, \qquad t > 1,$$

and (cf. [15])

$$h_t(e) \leq Ct^{-1}, \qquad 0 < t < 1. \tag{2.8}$$

Because of the symmetry of the kernel $\tilde{h}_t$ and of (2.7) and (2.8)

$$\begin{aligned}\sup_{g\in G} \tilde{h}_t(g) &= \tilde{h}_t(e) \\ &= h_t(e) \leq Ct^{-3/2}, \qquad t > 0.\end{aligned} \tag{2.9}$$

We prove the following lemma, whose proof is inspired by the well-known Nash-Varopoulos method.

Lemma 2.1: *We have that*

$$\left\|\tilde{\phi}_t\right\|_\infty \leq Ct^{-3/2}, \qquad t > 0.$$

PROOF: Let $f \in C_0^\infty(G)$, $\|f\|_1 = 1$, where by $\|\cdot\|_p$ we denote the norm in $L^p(G, d^l g)$.
Because of the anti-symmetry of the field X with respect to $d^l g$ we have that

$$\begin{aligned} \frac{d}{dt}\left\|\tilde{T}_t f\right\|_2^2 &= -2(L_0 \tilde{T}_t f, \tilde{T}_t f) \\ &= -2(\tilde{\Delta}\tilde{T}_t f, \tilde{T}_t f). \end{aligned} \tag{2.10}$$

It follows from (2.9) that

$$\left\|e^{-t\tilde{\Delta}}\right\|_{1\to\infty} \leq Ct^{-3/2}, \qquad t > 0,$$

where by $\|R\|_{p\to q}$ we denote the norm of the operator $R : L^p(G, d^l g) \to L^q(G, d^l g)$. Therefore (cf. [15, theorem II.5.2]) we have that

$$\|f\|_6^2 \leq C\left\|\tilde{\Delta}^{\frac{1}{2}} f\right\|_2^2 = C(\tilde{\Delta} f, f). \tag{2.11}$$

Then, by (2.10) and (2.11) (with $\tilde{T}_t f$ instead of f)

$$\frac{d}{dt}\left\|\tilde{T}_t f\right\|_2^2 \leq -C\left\|\tilde{T}_t f\right\|_6^2.$$

Making use of the fact that $\left\|\tilde{T}_t\right\|_{1\to 1} \leq 1$, by Hölder inequality, we obtain

$$\left\|\tilde{T}_t f\right\|_2^{2+\frac{4}{3}} \leq \left\|\tilde{T}_t f\right\|_6^2 \left\|\tilde{T}_t f\right\|_1^{\frac{4}{3}} \leq \left\|\tilde{T}_t f\right\|_6^2,$$

and therefore

$$\frac{d}{dt}\left\|\tilde{T}_t f\right\|_2^2 \leq -C\left\|\tilde{T}_t f\right\|_2^{2+\frac{4}{3}}.$$

This proves that $\left\|\tilde{T}_t f\right\|_2 \leq Ct^{-\frac{3}{4}}$ so that, recalling that $\|f\|_1 = 1$,

$$\left\|\tilde{T}_t\right\|_{1\to 2} \leq Ct^{-\frac{3}{4}}, \qquad t > 0.$$

Applying the same argument to the adjoint operator $\tilde{T}_t^*$ we obtain the estimate

$$\left\|\tilde{T}_t^*\right\|_{1\to 2} \le Ct^{-\frac{3}{4}}, \qquad t > 0.$$

Then

$$\begin{aligned}\left\|\tilde{T}_{2t}\right\|_{1\to\infty} &\le \left\|\tilde{T}_t\right\|_{1\to 2}\left\|\tilde{T}_t\right\|_{2\to\infty}\\ &\le \left\|\tilde{T}_t\right\|_{1\to 2}\left\|\tilde{T}_t^*\right\|_{1\to 2} \le Ct^{-\frac{3}{2}}, \qquad t > 0.\end{aligned}$$

□

3 Proof of the theorem

In this section we shall prove the upper Gaussian estimate for the kernel $\tilde{\phi}_t$

$$\tilde{\phi}_t(g) \le Ct^{-3/2}\exp(-\frac{|g|^2}{Ct}), \qquad t > 1,\ g \in G.$$

The method we shall use to deduce the above Gaussian estimate directly from lemma 2.1 appeared first in the work of Ushakov [10].
We proceed as in [5] (cf. also [8]) and write

$$\tilde{\phi}_{2t}(g) = \int_G \tilde{\phi}_t(\eta^{-1}g)\tilde{\phi}_t(\eta)d^l\eta.$$

By the change of variables $\eta^{-1}g = \xi^{-1}$

$$\begin{aligned}\tilde{\phi}_{2t}(g) &= \int_G \tilde{\phi}_t(\xi^{-1})\tilde{\phi}_t(g\xi)d^l\xi\\ &= \exp(-\frac{|g|^2}{2Ct})\int_G \exp(\frac{|g|^2}{2Ct})\tilde{\phi}_t(\xi^{-1})\tilde{\phi}_t(g\xi)d^l\xi.\end{aligned}$$

The left invariance of the control distance implies that $|g|^2 \le 2(|\xi^{-1}|^2 + |g\xi|^2)$. Substituing and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\tilde{\phi}_{2t}(g) &\le \exp(-\frac{|g|^2}{2Ct})\int_G \tilde{\phi}_t(\xi^{-1})\exp(\frac{|\xi^{-1}|^2}{Ct})\tilde{\phi}_t(g\xi)\exp(\frac{|g\xi|^2}{Ct})d^l\xi\\ &\le \exp(-\frac{|g|^2}{2Ct})(\int_G \tilde{\phi}_t^2(\xi^{-1})\exp(\frac{2\,|\xi^{-1}|^2}{Ct})d^l\xi)^{1/2}\\ &\times(\int_G \tilde{\phi}_t^2(g\xi)\exp(\frac{2\,|g\xi|^2}{Ct})d^l\xi)^{1/2}.\end{aligned}$$

By the left invariance of $d^l\xi$

$$\int_G \tilde{\phi}_t^2(g\xi)\exp(\frac{2\,|g\xi|^2}{Ct})d^l\xi = \int_G \tilde{\phi}_t^2(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi \tag{3.12}$$

and, since $|\xi^{-1}| = |\xi|$

$$\int_G \tilde{\phi}_t^2(\xi^{-1})\exp(\frac{2\,|\xi^{-1}|^2}{Ct})d^l\xi = \int_G \tilde{\phi}_t^{*2}(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi, \tag{3.13}$$

where $\tilde{\phi}_t^*$ is the kernel of the semigroup $e^{-t\tilde{L_0}^*}$, generated by the adjoint operator (2.6).

The idea of the proof (cf. [5]) is to show that if the constant C in the integrals (3.12) and (3.13) is large enough we have

$$\int_G \tilde{\phi}_t^2(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi \le Ct^{-3/2}, \qquad t > 1 \tag{3.14}$$

and

$$\int_G \tilde{\phi}_t^{*2}(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi \le Ct^{-3/2}, \qquad t > 1.$$

Note that the operator $\tilde{L_0}$ and its adjoint $\tilde{L_0}^*$ are operators of the same kind, so it suffices to prove (3.14).

We begin by showing that (3.14) is a consequence of the following estimate: there exists $C_0 > 0$ such that for every $R > 0$ and $t > 1$

$$\int_{|\xi|\ge R} \tilde{\phi}_t^2(\xi)d^l\xi \le C_0 t^{-3/2}\exp(-\frac{R^2}{C_0 t}). \tag{3.15}$$

Indeed

$$\int_G \tilde{\phi}_t^2(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi = \int_{|\xi|<R} \tilde{\phi}_t^2(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi$$
$$+\sum_{k\ge 0}\int_{2^kR\le|\xi|<2^{k+1}R} \tilde{\phi}_t^2(\xi)\exp(\frac{2\,|\xi|^2}{Ct})d^l\xi.$$

On one hand, we have that

$$\begin{aligned}\int_{|\xi|<R} \tilde{\phi}_t^2(\xi) \exp(\frac{2|\xi|^2}{Ct}) d^l\xi &\leq \exp(\frac{2R^2}{Ct}) \int_G \tilde{\phi}_t^2(\xi) d^l\xi \\ &\leq \exp(\frac{2R^2}{Ct}) \left\|\tilde{\phi}_t\right\|_\infty \int_G \tilde{\phi}_t(\xi) d^l\xi \\ &\leq \exp(\frac{2R^2}{Ct}) t^{-3/2},\end{aligned}$$

and on the other hand, applying (3.15), we obtain

$$\begin{aligned}\int_{2^kR\leq|\xi|<2^{k+1}R} \tilde{\phi}_t^2(\xi) \exp(\frac{2|\xi|^2}{Ct}) d^l\xi &\leq \exp(\frac{2^{2k+3}R^2}{Ct}) \int_{|\xi|\geq 2^kR} \tilde{\phi}_t^2(\xi) d^l\xi \\ &\leq Ct^{-3/2} \exp(\frac{2^{2k+3}R^2}{Ct} - \frac{2^{2k}R^2}{C_0t}) \\ &\leq Ct^{-3/2} \exp(-\frac{2^{2k+3}R^2}{Ct})\end{aligned}$$

provided $C > 16C_0$. Thus

$$\int_G \tilde{\phi}_t^2(\xi) \exp(\frac{2|\xi|^2}{Ct}) d^l\xi \leq C \exp(\frac{2R^2}{Ct}) t^{-3/2} + Ct^{-3/2} \sum_{k\geq 0} \exp(-\frac{2^{2k+3}R^2}{Ct}).$$

Choosing $R^2 = Ct$ we deduce (3.14).

Proof of the estimate (3.15). Let $R > 0,\ s > 0$, and $\eta(t,g),\ 0 < t < s,\ g \in G$ the function defined by

$$\eta(t,g) = \begin{cases} \frac{(|g|-R)^2}{4(1+\mu^2)(t-s)} & |g| < R \\ 0 & |g| \geq R. \end{cases}$$

We have

$$\begin{aligned}\frac{\partial\eta}{\partial t} + (1+\mu^2)|\nabla\eta|^2 &= -\frac{(|g|-R)^2}{4(1+\mu^2)(t-s)^2} + \frac{(|g|-R)^2}{4(1+\mu^2)(t-s)^2} |\nabla d(e,\cdot)|^2 \\ &= -\frac{(|g|-R)^2}{4(1+\mu^2)(t-s)^2}(1 - |\nabla d(e,\cdot)|^2) \leq 0,\end{aligned}$$

where $|\nabla f|^2 = (Xf)^2 + (Yf)^2$. Since $|\nabla d|^2 \leq 1$ we obtain

$$\frac{\partial \eta}{\partial t} \leq -(1+\mu^2)\,|\nabla \eta|^2 \tag{3.16}$$

Let $F(t)$ be the function defined by

$$F(t) = \int_G \tilde{\phi}_t^2(g) e^{\eta(t,g)} d^l g = \int_G \phi_t^2(g) e^{\eta(t,g)} d^r g.$$

We claim that $F(t)$ is a decreasing function. Indeed

$$\begin{aligned} F'(t) &= 2\int_G \phi_t \frac{\partial \phi_t}{\partial t} e^\eta d^r g + \int_G \phi_t^2 e^\eta \frac{\partial \eta}{\partial t} d^r g \\ &= -2\int_G \phi_t e^\eta L\phi_t d^r g + \int_G \phi_t^2 e^\eta \frac{\partial \eta}{\partial t} d^r g \\ &= -2\int_G \phi_t e^\eta \Delta\phi_t d^r g - 2\mu \int_G \phi_t e^\eta X\phi_t d^r g + \int_G \phi_t^2 e^\eta \frac{\partial \eta}{\partial t} d^r g. \end{aligned}$$

Since $X = [Y, X]$ we have that

$$F'(t) = -2\int_G \phi_t e^\eta \Delta\phi_t d^r g - 2\mu \int_G \phi_t e^\eta (YX\phi_t - XY\phi_t) d^r g + \int_G \phi_t^2 e^\eta \frac{\partial \eta}{\partial t} d^r g.$$

Using the anti-symmetry of the fields X, Y with respect to $d^r g$ and (3.16) we estimate

$$\begin{aligned} F'(t) &= -2\int_G (X(\phi_t e^\eta)X\phi_t + Y(\phi_t e^\eta)Y\phi_t) d^r g \\ &\quad - 2\mu \int_G (Y(\phi_t e^\eta)X\phi_t - X(\phi_t e^\eta)Y\phi_t) d^r g + \int_G \phi_t^2 e^\eta \frac{\partial \eta}{\partial t} d^r g \\ &\leq -2\int_G (e^\eta (X\phi_t)^2 + \phi_t e^\eta X\eta X\phi_t + e^\eta (Y\phi_t)^2 + \phi_t e^\eta Y\eta Y\phi_t) d^r g \\ &\quad - 2\mu \int_G (\phi_t e^\eta Y\eta X\phi_t - \phi_t e^\eta X\eta Y\phi_t) d^r g - (1+\mu^2) \int_G \phi_t^2 e^\eta |\nabla \eta|^2 d^r g \\ &\leq - \int_G e^\eta ((X\phi_t + \phi_t X\eta)^2 + (Y\phi_t + \phi_t Y\eta)^2 \\ &\quad + (X\phi_t + \mu\phi_t Y\eta)^2 + (Y\phi_t - \mu\phi_t X\eta)^2) d^r g \leq 0. \end{aligned}$$

Let

$$H(R,t) = \int_{|g|\geq R} \tilde{\phi}_t^2(g) d^l g.$$

Then

$$H(R,t) = \int_{|g|\geq R} \tilde{\phi}_t^2(g) e^{\eta(t,g)} d^l g \leq \int_G \tilde{\phi}_t^2(g) e^{\eta(t,g)} d^l g = F(t),$$

and since F is decreasing for every $0 < \tau < t$, and $0 < R' < R$, we have

$$\begin{aligned} H(R,t) &\leq \int_G \tilde{\phi}_t^2(g) e^{\eta(t,g)} d^l g \\ &\leq \int_G \tilde{\phi}_\tau^2(g) e^{\eta(\tau,g)} d^l g \\ &\leq \int_{|g|\geq R'} \tilde{\phi}_\tau^2(g) e^{\eta(\tau,g)} d^l g + \int_{|g|<R'} \tilde{\phi}_\tau^2(g) e^{\eta(\tau,g)} d^l g \\ &\leq \int_{|g|\geq R'} \tilde{\phi}_\tau^2(g) d^l g + \int_{|g|<R'} \tilde{\phi}_\tau^2(g) \exp(\frac{(R'-R)^2}{4(1+\mu^2)(\tau - s)}) d^l g \\ &\leq H(R',\tau) + \exp(\frac{(R'-R)^2}{4(1+\mu^2)(\tau-s)}) \int_G \tilde{\phi}_\tau^2(g) d^l g \end{aligned}$$

and therefore

$$H(R,t) \leq H(R',\tau) + \exp(\frac{(R'-R)^2}{4(1+\mu^2)(\tau-s)}) \left\| \tilde{\phi}_\tau \right\|_\infty, \qquad 0<\tau<t<s,\ R'<R.$$

Passing to the limit as $s \to t^+$ we obtain

$$H(R,t) \leq H(R',\tau) + \exp(\frac{(R'-R)^2}{4(1+\mu^2)(\tau-t)}) \left\| \tilde{\phi}_\tau \right\|_\infty, \qquad 0<\tau<t,\ R'<R. \tag{3.17}$$

Let

$$t_k = \frac{t}{2^k}, \qquad R_k = \frac{R}{2} + \frac{R}{k+2}, \qquad k = 0,1,2,\cdots$$

then applying (3.17) to the successive couples (R_k, t_k), (R_{k+1}, t_{k+1}) we have

$$H(R_k,t_k) \leq H(R_{k+1},t_{k+1}) + \exp(\frac{(R_{k+1}-R_k)^2}{4(1+\mu^2)(t_{k+1}-t_k)}) \left\| \tilde{\phi}_{t_{k+1}} \right\|_\infty \tag{3.18}$$

and, applying (3.18) for $k = 0, 1, \cdots, N-1$, that

$$\begin{aligned} H(R,t) &\leq H(R_1,t_1) + \exp(\frac{(R-R_1)^2}{4(1+\mu^2)(t_1-t)}) \left\|\tilde{\phi}_{t_1}\right\|_\infty \\ &\leq H(R_2,t_2) + \exp(\frac{(R_2-R_1)^2}{4(1+\mu^2)(t_2-t_1)}) \left\|\tilde{\phi}_{t_2}\right\|_\infty \\ &\quad + \exp(\frac{(R-R_1)^2}{4(1+\mu^2)(t_1-t)}) \left\|\tilde{\phi}_{t_1}\right\|_\infty \\ &\leq \cdots \\ &\leq H(R_N,t_N) + \sum_{k=0}^{N-1} \exp(\frac{(R_{k+1}-R_k)^2}{4(1+\mu^2)(t_{k+1}-t_k)}) \left\|\tilde{\phi}_{t_{k+1}}\right\|_\infty . \end{aligned} \tag{3.19}$$

Moreover

$$H(R_N,t_N) = \int_{|g|\geq R_N} \tilde{\phi}^2_{t_N}(g) d^l g \leq \int_{|g|\geq R/2} \tilde{\phi}^2_{t_N}(g) d^l g,$$

and by (1.2) and (2.5) it follows that if N is large enough that $t_N < 1$

$$\begin{aligned} H(R_N,t_N) &\leq C t_N^{-2} \int_{|g|\geq R/2} \exp(-\frac{|g|^2}{ct_N}) d^l g \\ &\leq C t_N^{-2} \int_{|g|\geq R/2} \exp(-\frac{|g|^2}{2ct_N}) \exp(-\frac{|g|^2}{2ct_N}) d^l g \\ &\leq C t_N^{-2} \exp(-\frac{R^2}{8ct_N}) \int_{|g|\geq R/2} \exp(-\frac{|g|^2}{2ct_N}) d^l g \\ &\leq C t_N^{-2} \exp(-\frac{R^2}{8ct_N}) \int_{|g|\geq R/2} \exp(-\frac{|g|^2}{2c}) d^l g \\ &\leq C t_N^{-2} \exp(-\frac{R^2}{8ct_N}) \int_G \exp(-\frac{|g|^2}{2c}) d^l g \\ &\leq C t_N^{-2} \exp(-\frac{R^2}{8ct_N}). \end{aligned}$$

Since

$$\lim_{N\to\infty} C t_N^{-2} \exp(-\frac{R^2}{8ct_N}) = 0,$$

passing to the limit as $N \to \infty$ in (3.19) we obtain

$$H(R,t) \leq \sum_{k=0}^{\infty} \exp(\frac{(R_{k+1}-R_k)^2}{4(1+\mu^2)(t_{k+1}-t_k)}) \left\| \tilde{\phi}_{t_{k+1}} \right\|_{\infty}. \tag{3.20}$$

By lemma 2 we have that

$$\left\| \tilde{\phi}_{t_{k+1}} \right\|_{\infty} \leq C \frac{(2^{3/2})^k}{t^{3/2}} \leq C \frac{A^k}{t^{3/2}}, \qquad A > 1$$

and by definition

$$R_k - R_{k+1} = (\frac{1}{k+2} - \frac{1}{k+3})R \geq \frac{R}{(k+3)^2}$$

and that

$$t_k - t_{k+1} = \frac{t}{2^k} - \frac{t}{2^{k+1}} = \frac{t}{2^{k+1}}.$$

Thus (3.20) yields

$$\begin{aligned} H(R,t) &\leq \frac{C}{t^{3/2}} \sum_{k=0}^{\infty} A^k \exp(-\frac{R^2}{4(1+\mu^2)(k+3)^4} \frac{2^{k+1}}{t}) \\ &\leq \frac{C}{t^{3/2}} \sum_{k=0}^{\infty} \exp(-(k+3)(-\log A + \alpha \frac{R^2}{t})) \\ &\leq \frac{C}{t^{3/2}} \exp(-3(-\log A + \alpha \frac{R^2}{t})) \sum_{k=0}^{\infty} \exp(-k(-\log A + \alpha \frac{R^2}{t})) \\ &\leq \frac{C}{t^{3/2}} \exp(-3\alpha \frac{R^2}{t}) \sum_{k=0}^{\infty} \exp(-k(-\log A + \alpha \frac{R^2}{t})), \end{aligned}$$

where $\alpha = \min \left\{ \frac{2^{k+1}}{4(1+\mu^2)(k+3)^5}, \ k = 0, 1, \cdots \right\}$.

If we suppose that R and t are such that

$$-\log A + \alpha \frac{R^2}{t} \geq \log A$$

then

$$\begin{aligned} H(R,t) &\leq \frac{C}{t^{3/2}} \exp(-3\alpha \frac{R^2}{t}) \sum_{k=0}^{\infty} \exp(-k \log A) \\ &\leq \frac{C}{t^{3/2}} \exp(-3\alpha \frac{R^2}{t}). \end{aligned}$$

Otherwise we have

$$-\log A + \alpha\frac{R^2}{t} < \log A,$$

i.e.

$$\alpha\frac{R^2}{t} < 2\log A$$

and in this case it is enough to observe that

$$\begin{aligned} H(R,t) &\leq \int_G \tilde{\phi}_t^2(g)d^l g \leq \left\|\tilde{\phi}_t\right\|_\infty \\ &\leq \frac{C}{t^{3/2}} \leq \frac{C}{t^{3/2}}\exp(-\frac{\alpha R^2}{t}), \end{aligned}$$

and the proof of (3.15) is complete.

4 Final remark

In section 3 we adapted to a non-symmetric case the method decribed in [5]. We managed to overcome the difficulties imposed by the presence of the drift term X, thanks to its special nature.

First, the anti-symmetry of X with respect to the left Haar measure too allowed us to evaluate the two integrals (3.14) in the same way.
Second, since $X = [X, Y]$ we are able to prove that $F(t)$ is a deceasing function.

Assuming that an upper estimate of the form $\left\|\tilde{\phi}_t\right\|_\infty \leq Ct^{-\delta}$ holds for some $\delta > 0$ and every $t > 1$, this method could be generalized in an natural way to the semi-direct product $G = N \rtimes \mathbb{R}_+^*$, where N is a stratified group and $\mathbb{R}_+^*$ acts on N by dilations. Let us remark that the proof of Lemma 2.1 works provided the local dimension of the group is less or equal to 3.

As far as we can see this method does not work in more general contexts.

References

[1] G. Alexopoulos. Sublaplacians on groups of polynomial growth. *Mem. Amer. Math. Soc.*, to appear.

[2] R. Azencott. Géodésiques et diffusions en temps petit. *Astérisque*, 84-85:17–31, 1981.

[3] Ph. Bougerol. Comportement asymptotique des puissances de convolution d'une probabilité sur un espace symétrique. *Astérisque*, 74:29–45, 1980.

[4] E.B. Davies and N. Mandouvalos. Heat bounds on hyperbolic space and Kleinian groups. *Proc. London Math. Soc. (3)*, 52:182–208, 1988.

[5] A. Grigor'yan. Gaussian upper bounds for the heat kernel on arbitrary manifolds. *J. Differential Geometry*, 45:33–52, 1997.

[6] A. Grigor'yan. Estimates of heat kernels on Riemannian manifolds. In B. Davies and Yu. Safarov, editors, *Spectral Theory and Gemetry. Edinburgh 1998*, pages 140–225. London Math. Soc. Lectures Nte Series 273, Cambridge Univ. Press, 1998.

[7] G. A. Hunt. Semi-groups of measures on Lie groups. *A.M.S.*, 81:264–293, 1956.

[8] S. Mustapha. Gaussian estimates for heat kernels on Lie groups. *Math. Proc. Camb. Phil. Soc.*, 128:45–64, 2000.

[9] A. Nagel, E. Stein, and M. Wainger. Balls and metrics defined by vector fields. *Acta Math.*, 155:103–147, 1985.

[10] V.I. Ushakov. Stabilization of solutions of the third mixed problem for a second order parabolic equation in a non-cyclic domain (Engl. trans.). *Math. USSR Sb.*, 39:87–105, 1981.

[11] N. Varopoulos. Small time gaussian estimates of heat diffusion kernels. Part I: The semi-group technique. *Bull. Sc. Math.*, 113:253–277, 1989.

[12] N. Varopoulos. Small time gaussian estimates of heat diffusion kernels. Part II: The theory of large deviations. *J. Funct. Anal.*, 93:1–33, 1990.

[13] N. Varopoulos. Analysis on Lie groups. *Rev. Mat. Iberoamericana*, 12:791–917, 1996.

[14] N. Varopoulos and S. Mustapha. Forthcoming book. Cambridge University Press.

[15] N. Varopoulos, L. Saloff-Coste, and Th. Coulhon. *Analysis and geometry on groups.* Cambridge Univ. Press, Cambridge, 1992.

Camillo Melzi
Universita dell'Insubria
Dipartimento di Scienze
Chimiche Fisiche e Matematiche
via Valleggio 11
22100, Como
Italy
melzi@uninsubria.it