ANNALES MATHÉMATIQUES BLAISE PASCAL

MOHAMED AKKOUCHI

A note on d'Alembert's functional equation

Annales mathématiques Blaise Pascal, tome 8, nº 1 (2001), p. 1-6

<http://www.numdam.org/item?id=AMBP_2001__8_1_1_0>

Ann. Math. Blaise Pascal, Vol. 8, N° 1, 2001, pp 1-6

A NOTE ON

D'ALEMBERT'S FUNCTIONAL EQUATION

Mohamed AKKOUCHI

Département de Mathématiques.
Faculté des Sciences-Semlalia.
Université Cadi Ayyad.
Avenue du Prince My. Abdellah. B.P. 2390.
40.000, Marrakech. MAROC (MOROCCO).

RÉSUMÉ : Dans cette note, nous montrons que toute fonction mesurable réelle bornée vérifiant l'équation fonctionnelle de D'Alembert sur $\mathbb{R}^n$ est nécessairement continue et périodique. Nous montrons aussi que toute fonction presque-périodique à valeurs complexes satisfaisant l'équation fonctionnelle de D'Alembert sur $\mathbb{R}^n$ doit être réelle et égale à la partie réelle d'un caractère du groupe commutatif $\mathbb{R}^n$. Ainsi, nous retrouvons un résultat bien connu concernant l'équation fonctionnelle de D'Alembert sur $\mathbb{R}^n$ (voir par exemple [2], [4] et [14]).

ABSTRACT : In this note, we show that every measurable bounded real valued function satisfying D'Alembert's functional equation on $\mathbb{R}^n$ must be periodic. We show that every complex valued solution of D'Alembert's functional equation on $\mathbb{R}^n$ which is almost-periodic must be real valued and given by the real part of a character of the locally compact abelian group $\mathbb{R}^n$. By this way, we recapture a well known result concerning D'Alembert's functional equation (see for example [2], [4] and [14]).

Mathematics Subject Classification (1991). 39B05 and 43A90.
Keywords. D'Alembert's functional equation. Almost-periodic functions.

1. INTRODUCTION

1.1 D'Alembert's functional equation was first introduced in [4]. It is given by

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad x, y \in \mathbb{R}^n. \tag{1}$$

This equation has been intensively studied and generalized. The classical case $n = 1$ and some of its applications is discussed in detail in Aczèl [1], section 2.4 and in Aczèl and Dhombres [2] Ch. 8. Many papers were concerned by generalizing (1) to general locally compact abelian groups (see Ljubenova [13], O'Connor [14]) or to non abelian groups (see Corovei [7], Gajda [8], Kannapan [11], Penney and Rukhin [16], Székelyhidi [17]) or to other contexts like Gelfand pairs (see Stetkaer [18] and [19], Akkouchi et al [3]), where relationships with Spherical functions and Representation theory are investigated.

1.2 In this note, we work in $\mathbb{R}^n$. We use the notion of almost-periodicity to determine the measurable real bounded solutions of (1). Our method is based on two fundamental observations. The first one is that, if f is any complex valued function which is almost-periodic and satisfying (1), then f must be real and given by $f(x) = \cos < x, \xi >$, for all $x \in \mathbb{R}^n$, where $<,>$ is the usual inner product of $\mathbb{R}^n$. This is proved in the next section. The second observation is that, if f is any real measurable and essentially bounded function verifying (1), then f must be periodic, continuous and its modulus is bounded by one. This is proved in the third section. Combining the two results, we determine the real bounded solutions of (1) and recapture, by this way, a classical result concerning equation (1) (see for example [2], [4] and [14]).

2.

2.1 Throughout this paper, $\mathbb{R}^n$ is the usual locally compact Abelian group. its Haar measure will be denoted by dx or $dx_1...dx_n$. We set $< x, y >:= x_1y_1 + ... + x_ny_n$, for all $x, y \in \mathbb{R}^n$. The Banach space of (all complex) continuous and bounded fuctions on $\mathbb{R}^n$ will be denoted by $C_b(\mathbb{R}^n)$. We denote $AP(\mathbb{R}^n)$ the Banach subspace of $C_b(\mathbb{R}^n)$ formed by the almost-periodic functions on $\mathbb{R}^n$ (see for example [5], [6], [9] and [12]). We know that $AP(\mathbb{R}^n)$ is a closed subalgbra of $L^\infty(\mathbb{R}^n)$ and that every $f \in AP(\mathbb{R}^n)$ is uniformly continuous on $\mathbb{R}^n$.

2.2 For every $f \in AP(\mathbb{R}^n)$, we shall denote the mean of f by $M(f)$. We recall that $M(f)$ is the complex number given by

$$M(f) := \lim_{T\to\infty} \frac{1}{2^nT^n} \int_{-T}^{T} ... \int_{-T}^{T} f(x_1, x_2, ..., x_n)\, dx_1 dx_2 ... dx_n. \tag{2}$$

For each element $\xi \in \mathbb{R}^n$, we set $\hat{f}(\xi) := M(f(.)e^{-i<.,\xi>})$. The norm spectrum $\sigma(f)$ is the set of all elemets $\xi \in \mathbb{R}^n$, verifying $\hat{f}(\xi) \neq 0$. It is well known that $\sigma(f)$ is countable and that $\sigma(f) = \emptyset$, if and only if f is identically zero on $\mathbb{R}^n$.

2.3 A trigonometric polynomial is any function P defined on $\mathbb{R}^n$ by

$$P(x) = \sum_{1}^{m} c_j e^{i<x,\xi_j>}, \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where $c_j \in \mathbb{C}$, $\xi_j \in \mathbb{R}^n$, for $j = 1, ..., m$ and $m \in \mathbb{N} \setminus \{0\}$.

Using Lemma 5.18 in [12], p. 165, one can deduce the following important result.

2.4 Lemma : *Given a finite number of points $\xi_1, ..., \xi_N \in \mathbb{R}^n$ and a number $\epsilon \in]0,1[$, there exists a trigonometric polynomial P having the following properties :*

(i) $P(x) \geq 0$, *for all* $x \in \mathbb{R}^n$,

(ii) $M(P) = 1$, *and*

(i) $\hat{P}(\xi_j) > 1 - \epsilon$, *for* $j = 1, ..., N$.

With all these considerations, we are ready to state our first main result.

2.5 Theorem : *Let $f \in AP(\mathbb{R}^n) \setminus \{0\}$ be a solution of (1). Then there exists a unique $\xi \in \mathbb{R}^n$ such that $f(x) = \cos < x, \xi >$ for all $x \in \mathbb{R}^n$. In particular f is real valued.*

Proof : Suppose that $f \in AP(\mathbb{R}^n) \setminus \{0\}$ is satisfying (1), and take an element $\xi \in \sigma(f)$. By Lemma 2.4, there exists a trigonometric polynomial F (depending on f and ξ) such that $F(x) \geq 0$ for all $x \in \mathbb{R}^n$, $M(F) = 1$, and $\hat{F}(\xi) > \frac{1}{2}$. We introduce the function G defined for all $x \in \mathbb{R}^n$, by

$$G(x) := M(f(x - .)F(.)) = \lim_{T \to \infty} \frac{1}{2^n T^n} \int_{-T}^{T} \cdots \int_{-T}^{T} f(x-y)F(y)\, dy. \tag{4}$$

Then, a short computation will show that G is also a trigonometric polynomial and that is given in the form

$$G(x) = \sum_{\eta \in \Xi} \hat{f}(\eta)F(\eta)e^{i<x,\eta>}, \quad \forall x \in \mathbb{R}^n, \tag{5}$$

where Ξ is a finite subset of $\mathbb{R}^n$ containing ξ. An easy computation will show that f and G are verifying the following functional equation :

$$G(x+y) + G(x-y) = 2G(x)f(y), \quad \forall x, y \in \mathbb{R}^n. \tag{6}$$

This is on one hand, on the other hand, it is easy to obtain by a computation that

$$G(x+y) + G(x-y) = 2\sum_{\eta \in \Xi} \hat{f}(\eta)F(\eta)e^{i<x,\eta>} \cos < y, \eta >, \quad \forall x \in \mathbb{R}^n. \tag{7}$$

From (6) and (7), we get in particular that

$$\hat{f}(\xi)F(\xi)[f(y) - \cos < y, \xi >] = 0, \quad \forall y \in \mathbb{R}^n. \tag{8}$$

Since $\hat{f}(\xi)F(\xi) \neq 0$, then $f(y) = \cos < y, \xi >$ for all $y \in \mathbb{R}^n$. Thus, our theorem is proved. $\square$

3.

3.1 In this section, we investigate qualitative properties of the measurable bounded solutions of (1). More precisely, we shall prove the following theorem :

3.2 Theorem : *Let $f \in L^\infty(\mathbb{R}^n) \setminus \{0\}$ be a be a real valued function satisfying (1). Then (i) $f \in C_b(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, moreover (ii) f is periodic, and (iii)* $\sup_{x\in\mathbb{R}^n} |f(x)| \leq 1$.

Proof : (i) One may find a $C^\infty-$function h with compact support verifying $< f, h >:= \int_{\mathbb{R}^n} f(x)h(x)\,dx \neq 0$. Multiplying both members of (1) bt h and integrating, we obtain

$$f = \frac{1}{2 < f, h >} f * [h + \check{h}], \tag{9}$$

where, $*$ designates the convolution and $\check{h}(x) := h(-x)$. The equality (9) proves that $f \in C_b(\mathbb{R}^n)$ and that f is a $C^\infty-$function.

(iii) We suppose that $|f(x)| \leq C$ for all x in $\mathbb{R}^n$ for some positive constant $C > 0$. By using (1), we have

$$|f(x)|^2 \leq \frac{1}{2}|f(2x)| + \frac{1}{2}, \quad \forall x \in \mathbb{R}^n,$$

which implies that

$$\begin{aligned} |f(x)| &\leq \limsup_{k\to\infty} \left[|f(2^k x)|\right]^{2^{-k}} \\ &\leq \limsup_{k\to\infty} C^{2^{-k}} = 1, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

(ii) It remains to show that f is periodic. We begin by pointing out (since f is not identicaly zero) that $f(0) = 1$ and that f is even.

a) We may suppose that f is not identically equal to one on $\mathbb{R}^n$. In this case, f has at least a zero. Indeed, (following an idea of O'Connor in [14], see also [2]) we consider an element y_0 such that $f(y_0) < 1$, and we choose a natural number $m \in \mathbb{N}$ such that $f(y_0) < \cos(\frac{\pi}{2^m}) < 1 = f(0)$. Since f is continuous on the connected metric space $\mathbb{R}^n$, then there exists at least one element z_0 in $\mathbb{R}^n$ such that $f(z_0) = \cos(\frac{\pi}{2^m})$. From (1), we deduce that

$$f(2z_0) = 2f(z_0)^2 - 1 = \cos(\frac{\pi}{2^{m-1}}). \tag{10}$$

By the same manner, we have

$$f(2^2 z_0) = \cos(\frac{\pi}{2^{m-2}}), \tag{11}$$

and so on. At the end, we obtain

$$f(2^m z_0) = \cos(\pi) = -1. \tag{12}$$

Since $f0) = 1$, and since f is continuous on the connected metric space $\mathbb{R}^n$, then there exists at least one $t_0 \in \mathbb{R}^n$ such that $f(t_0) = 0$.

b) We set $g(x) := f(t_0 - x)$ for all x in $\mathbb{R}^n$. By using equation (1), we get

$$2g(x)g(y) = f(2t_0 - x - y) + f(y - x), \quad \forall x, y \in \mathbb{R}^n. \tag{13}$$

Replacing x by $t_0 - x$ and y by t_0 in the equation (1), we get $f(2t_0 - x) = -f(x)$ for every $x \in \mathbb{R}^n$. Therefore, we have $f(x - 2t_0) = -f(x)$ for all element x in $\mathbb{R}^n$. In particular, we deduce that $|f|$ is periodic and that $2t_0$ is a period for it, and that $f([2k + 1]t_0) = 0$, for every integer k. Now, equation (13) becomes

$$2g(x)g(y) = f(y - x) - f(x + y), \quad \forall x, y \in \mathbb{R}^n, \tag{14}$$

from which we deduce that

$$f(t + s) - f(t) = 2f(t_0 + \frac{s}{2})f(t_0 - t - \frac{s}{2}), \quad \forall t, s \in \mathbb{R}^n. \tag{15}$$

We take $s = 4t_0$ in (15), then we get

$$f(t + 4t_0) = f(t), \quad \forall t \in \mathbb{R}^n. \tag{16}$$

The last equality shows that f is periodic. $\square$

We conclude from Theorems 2.5 and 3.2 that we have given a new proof to the following classical result (see [2], [4], [14], ...) by a method consisting of utilizing the concept of almost-periodicity due to H. Bohr (see [5] and [6]). More precisely, we have

3.3 Theorem : *Let $f \in L^\infty(\mathbb{R}^n) \setminus \{0\}$ be a real valued function satisfying D'Alembert's functional equation. Then there exists a unique $\xi \in \mathbb{R}^n$ such that $f(x) = \cos < x, \xi >$ for all $x \in \mathbb{R}^n$.*

REFERENCES

[1] J. ACZÈL, *Vorlesungen über Funktionalgleichungen une ihre Anwendungen*, Birkhäuser, basel, 1961.

[2] J. ACZÈL AND J. DHOMBRES, *Functional Equations in Severables Variables*, Encyclopedia of Mathematics and its applications. Cambridge University Press, 1989.

[3] M. AKKOUCHI, A. BAKALI AND H. EL QORACHI, D'Alembert's functional equation on compact Gel'fand pairs, *Maghreb Math. Rev.* (To appear).

[4] J. D'ALEMBERT, Mémoire sur les principes de la mécanique, *Hist. Acad. Sci.*, Paris, 1769, pp. 278 - 286.

[5] H. BOHR, *Almost Periodic Functions,* Chelsea, New York, 1951.

[6] C. CORDUNEANU, *Almost Periodic Functions,* Interscience Publishers, New York, London, Toronto, 1968.

[7] I. COROVEI, The cosine functional equation for nilpotent groups, *Aequationes Mathematicae,* 15, (1977) 99 - 106.

[8] Z. GAJDA, On functional equations associated with characters of unitary reprsentations of groups, *Aequationes Mathematicae,* 44, (1992) 109 - 121.

[9] E. HEWITT AND K.A. ROSS, *Abstract harmonic analysis, VOL. 1 (VOL. 2)* Springer-Verlag, Berlin, 1979. (1970).

[10] PL. KANNAPPAN, On the functional equation $f(x+y)+f(x-y)=2f(x)f(y)$, *Amer. Math. Monthly,* 72 (1965) 374 - 377.

[11] PL. KANNAPPAN, On the functional equation $f(xy)+f(xy^{-1})=2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc,* 19 (1968) 69 - 74.

[12] Y. KATZNELSON, *An Introduction to Harmonic Analysis,* Dover Publications, Inc. New York, 1976 (Second edition).

[13] E. T. LJUBENOVA, On D'Alembert's functional equation on an Abelian group, *Aequationes Mathematicae,* 22, (1981) 54 - 55.

[14] THOMAS A. O'CONNOR, A solution of D'Alembert's functional equation on a locally compact group, *Aequationes Mathematicae,* 15, (1977) 235 - 238.

[15] F. J. PAPP, The D'Alembert functional equation on groups, *Amer. Math. Monthly,* 92, (1985) 273 - 275.

[16] R. C. PENNY AND A. L. RUKHIN, D'Alembert's functional equation on groups, *Proc. Amer. Math. Soc.* , 77, (1979) 73 - 80.

[17] L. SZÉKELYHIDI, Almost periodicity and functional equations, *Aequationes Mathematicae,* 26, (1983) 163 - 175.

[18] H. STETKAER, D'Alembert's equation and spherical functions, *Aequationes Mathematicae,* 48, (1994) 220 - 227.

[19] H. STETKAER, Functional equations on abelian groups with involution, *Aequationes Mathematicae,* 54, (1997) 144 - 172.