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# A Method for Constructing Orthonormal Bases for Non-Archimedean Banach Spaces of Continuous Functions

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## Abstract

Let  $K$  be a local field, let  $M$  be a regular compact in  $K$  with diameter 1 and let  $C(M \rightarrow K)$  be the Banach space of continuous functions from  $M$  to  $K$ . Our aim is to find orthonormal bases for  $C(M \rightarrow K)$  by taking powers, products and infinite linear combinations of functions in  $C(M \rightarrow K)$ . To construct the orthonormal bases, we use very well distributed sequences in  $M$ .

## 1. Introduction

In this paper we construct orthonormal bases for non-Archimedean Banach spaces of continuous functions. The method used here generalises the results from reference [9]. For the convenience of the reader, we start by recalling some definitions and some previous results. All these results can be found in [1], chapter 1, sections 1 and 2, and chapter 2, sections 5 and 6. For additional information we refer the reader to [1]. We remark that the notations used in this section are sometimes different from the notations used in [1]. The notations used in this section can also be found in [3]. Throughout this paper,  $\mathbb{N}$  denotes the set of natural numbers, and  $\mathbb{N}_0$  is the set of natural numbers without zero.

**Definition 1.1** A countable projective system of finite sets  $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$  consists of

- 1) a sequence  $(M_n)$  of finite sets
- 2) mappings  $\varphi_{k,n}$  of  $M_n$  in  $M_k$  defined for  $k \leq n$  and such that for  $k \leq n \leq m$  we have

$$\varphi_{k,m} = \varphi_{k,n} \circ \varphi_{n,m}$$

and  $\varphi_{n,n}$  is the identity mapping on  $M_n$ .

We denote by  $M = \varprojlim M_k$  the projective limit of this system.  $M$  is compact and ultrametric, since it is the projective limit of the sequence of sets  $M_n$  equipped with the

discrete topology.  $M = \{(x_i) \in \prod_{i \in \mathbb{N}} M_i \mid \forall i \leq j \in \mathbb{N} : x_i = \varphi_{i,j}(x_j)\}$ . On  $M \times M$  we define the function  $v(x, y)$  as follows :  $v(x, y) = \sup\{i \in \mathbb{N} \mid x_k = y_k \ \forall k \leq i\}$  if  $x \neq y$ ;  $v(x, x) = +\infty$ .  $v$  satisfies the ultrametric inequality  $v(x, z) \geq \inf\{v(x, y), v(y, z)\}$  and this for all  $x, y, z \in M$ . The function  $d(x, y) = \alpha^{v(x,y)}$  ( $\alpha \in (0, 1)$ ,  $\alpha$  fixed) is an ultrametric on  $M$ , which induces the projective limit topology.

### Definition 1.2

A countable projective system of finite sets  $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$  is called a *regular projective system* if it satisfies

- 1)  $\#M_0 = 1$ ,
- 2) for all  $i \leq j \in \mathbb{N}$  :  $\varphi_{i,j}$  is surjective,
- 3) there exist  $q_1, q_2, \dots \in \mathbb{N} \setminus \{0, 1\}$  such that for all  $n \in \mathbb{N}_0$ , for all  $\omega \in M_{n-1}$  :  $\#\{\varphi_{n-1,n}^{-1}(\omega)\} = q_n$ .

In this case we put  $N_0 = \#M_0 = 1$ ,  $N_n = \#M_n = q_1 q_2 \dots q_n$ .

From 3), definition 1.2, it immediately follows that, for  $M = \varprojlim M_k$ ,

*Each closed ball in  $M$  with radius  $\alpha^n$  is the finite disjoint union of  $q_{n+1}$  closed balls with radius  $\alpha^{n+1}$ . (\*)*

*Let us now look at something more general.*

*Let  $M$  be a set with an integer valued function  $v$  defined on  $M \times M$  which satisfies for all  $x, y, z \in M$*

$$v(x, y) = v(y, x)$$

$$v(x, y) = +\infty \Leftrightarrow x = y$$

$$v(x, z) \geq \inf\{v(x, y), v(y, z)\}.$$

*We say that  $M$  is valued by  $v$ . The function  $d(x, y) = \alpha^{v(x,y)}$  ( $\alpha \in (0, 1)$ ,  $\alpha$  fixed) is an ultrametric on  $M$ . If  $M$  is compact, then each closed ball with radius  $\alpha^n$  is a finite disjoint union of balls with radius  $\alpha^{n+1}$ .  $M$  is then called a valued compact.*

*Let  $M$  be a valued compact such that the closed balls satisfy (\*). Let  $B_b(r)$  denote the 'closed' ball with center  $b$  and radius  $r$ . We introduce*

- the equivalence relation  $\pi_k$  on  $M$ , defined for  $k \geq 0$  by

$$x \pi_k y \iff B_x(\alpha^k) = B_y(\alpha^k)$$

- the quotient  $M_k$  of  $M$  by  $\pi_k$ , and the canonical projection  $pr_k$  of  $M$  on  $M_k$ ,
- the mapping  $\varphi_{k,n}$  of  $M_n$  on  $M_k$  defined for  $k < n$  by

$$\varphi_{k,n}(\omega) = pr_k(pr_n^{-1}(\omega)) \text{ for } \omega \in M_n$$

and  $\varphi_{n,n}$  is the identity mapping on  $M_n$ .

The system  $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$  is then a regular projective system, and its projective limit is isomorphic to  $M$ . If the closed balls of a valued compact  $M$  satisfy condition (\*), then we call  $M$  a *regular valued compact*. Let us consider the following examples :

**Example 1.1** Let  $A_1, \dots, A_n, \dots$  be non-empty finite sets and let  $q_n \geq 2$  be the the cardinality of  $A_n$ . The products  $M_n = A_1 \times \dots \times A_n$ , ( $M_0$  consisting of one element), equipped with the canonical projections from  $M_n$  on  $M_k$  defined by  $(x_1, \dots, x_n) \rightarrow (x_1, \dots, x_k)$ ,  $k \leq n$ ,  $x_i \in A_i$ , form a regular projective system. Conversely, every regular projective system is isomorphic to such a system.

**Example 1.2** Let  $p$  be a prime number.  $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$  is a regular valued compact, with  $q_n = p$ ,  $N_n = p^n$ .

**Example 1.3** Let  $p$  be an odd prime number. The unit circle  $\{x \in \mathbb{Z}_p \mid |x| = 1\}$ , where  $|\cdot|$  denotes the p-adic valuation, is also a regular valued compact, with  $q_1 = p - 1$ ,  $q_n = p$  if  $n \geq 2$ . To see this put in example 1.1  $A_1 = \{1, 2, \dots, p - 1\}$ ,  $A_n = \{0, 1, \dots, p - 1\}$  if  $n \geq 2$ .

Now we give the definition of very well distributed sequences. Therefore, let  $A$  be a finite set,  $\#A = N$ , and  $M = \varprojlim M_k$  a regular valued compact.

**Definition 1.3**

A sequence  $u : \mathbb{N} \rightarrow A$  is *well distributed* if for all  $n \in \mathbb{N}_0$ , for all  $a \in A : \#\{i < nN \mid u_i = a\} = n$ .

A sequence  $u : \mathbb{N} \rightarrow M$  is called *very well distributed* if for all  $n \in \mathbb{N} : \varphi_n \circ u : \mathbb{N} \rightarrow M_n$  is well distributed, where  $\varphi_n : M \rightarrow M_n$  is the canonical projection.

A very well distributed sequence  $u$  in  $M$  is always injective, and lays dense in  $M$ .

**Example 1.4** If  $p$  is a prime number, then  $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$  is a regular valued compact (example 1.2). The sequence  $(u_i)$ ,  $u_i = i$  for all  $i$ , is a very well distributed sequence in  $\mathbb{Z}_p$ .

For more details concerning regular valued compacts and very well distributed sequences, we refer the reader to [1].

Let  $K$  be a local field (i.e. a locally compact, non-trivially, non-Archimedean valued field)

with valuation  $|\cdot|$  and logarithmic valuation  $v$  and let  $k$  be the finite residue class field of  $K$ . We have  $|x| = \alpha^{v(x)}$ , where  $\alpha = |\pi|$ ,  $|\pi|$  the generator of the value group of  $K$ .  $K$  is an ultrametric space by putting  $v(x, y) = v(x - y)$ ,  $d(x, y) = |x - y|$ . The closed unit ball of  $K$  is a regular valued compact.

**Definition 1.4**

A compact part  $M$  of the closed unit ball satisfying (\*) is called a *regular compact* of  $K$ .

$C(M \rightarrow K)$  denotes the non-Archimedean Banach space of continuous functions from  $M$  to  $K$ , equipped with the supremum norm  $\|\cdot\|_\infty : \|f\|_\infty = \sup\{|f(x)| \mid x \in M\}$ . An orthonormal basis for  $C(M \rightarrow K)$  is defined as follows :

**Definition 1.5**

A sequence  $e_0, e_1, e_2, \dots$  of elements of  $C(M \rightarrow K)$  is called an *orthonormal basis* for  $C(M \rightarrow K)$  if every element  $f$  of  $C(M \rightarrow K)$  has a unique representation  $f = \sum_{i=0}^{+\infty} x_i e_i$  where  $x_i \in K$  and  $|x_i| \rightarrow 0$  if  $i \rightarrow \infty$ , and if  $\|f\|_\infty = \max_{0 \leq i} \{|x_i|\}$ .

The aim of this paper is to construct orthonormal bases for  $C(M \rightarrow K)$ , by taking powers, products and infinite linear combinations of functions in  $C(M \rightarrow K)$ . To construct the orthonormal bases, we use very well distributed sequences in  $M$ . In section 2 we prove some preliminary lemmas, and in section 3 we prove the main theorem of this paper. We also give some examples and in particular we obtain some well-known orthonormal bases.

**2. Preliminary Lemmas**

Let  $K$  be a local field with valuation  $|\cdot|$  and finite residue class field  $k$  and let  $M$  be a regular compact in  $K$ . We will assume that the diameter of  $M$  equals 1. We put  $\alpha = |\pi|$ ,  $\pi$  the generator of the value group of  $K$ . Throughout sections 2 and 3,  $(u_n)$  denotes an arbitrary but fixed very well distributed sequence in  $M$ . First we introduce some functions on  $M$  which are going to play an important role in this paper.

We define sequences  $(q_n)$  and  $(\phi_n)$  on  $M$  as follows :

$$q_0(x) = \phi_0(x) = 1,$$

$$q_n(x) = \frac{(x - u_0) \dots (x - u_{n-1})}{(u_n - u_0) \dots (u_n - u_{n-1})} \quad n \geq 1,$$

$$\phi_n(x) = 1 \text{ if } x \in B_{u_n}(r_n), \text{ where } r_n = \alpha^{i+1} \text{ if } N_i \leq n < N_{i+1},$$

$$\text{otherwise } \phi_n(x) = 0 \quad n \geq 1.$$

The functions  $q_n$  and  $\phi_n$  are clearly continuous. The functions  $q_n$  were introduced by Amice (see [1], sections 2.4 and 6). The sequence  $(q_n)$  forms an orthonormal basis for  $C(M \rightarrow K)$  ([1], section 6.2). It is clear that  $\|\phi_n\|_\infty = 1$  for all  $n$ . Since  $(q_n)$  forms an orthonormal basis we have that  $\|q_n\|_\infty = 1$  for all  $n$ . It is easy to see that for all  $n$

$$q_n(u_n) = \phi_n(u_n) = 1,$$

$$q_n(u_j) = \phi_n(u_j) = 0 \text{ if } j < n.$$

We only have to prove  $\phi_n(u_j) = 0$  if  $j < n$ . To see this, suppose that  $N_i \leq n < N_{i+1}$ . Then  $\phi_n$  is the characteristic function of the ball  $B_{u_n}(r_n)$  where  $r_n = \alpha^{i+1}$ . There are  $N_{i+1}$  disjoint balls with radius  $\alpha^{i+1}$ , namely the balls with centers  $u_0, u_1, \dots, u_j, \dots, u_n, \dots, u_{N_{i+1}-1}$ . So  $|u_j - u_n| > \alpha^{i+1}$  and we conclude that  $\phi_n(u_j) = 0$  if  $j < n$ .

For the functions  $q_n$  and  $\phi_n$  we can prove the following (see [9], lemmas 1 and 2)

**Lemma 2.1**

- 1) If  $x, y \in M$ ,  $|x - y| \leq \alpha^t$  then  $|q_n(x) - q_n(y)| \leq \alpha$  if  $0 \leq n < N_t$ .
- 2) If  $x, y \in M$ ,  $|x - y| \leq \alpha^t$  then  $\phi_n(x) = \phi_n(y)$  if  $0 \leq n < N_t$ .

**Proof**

- 1) This can be found in [1] (p. 135, lemma 4).
- 2) Let  $x, y \in M$  such that  $|x - y| \leq \alpha^t$  and let  $0 \leq n < N_t$ . Then  $|x - y| \leq \alpha^t \leq r_n$ . So the elements  $x$  and  $y$  either belong both to  $B_{u_n}(r_n)$  and then  $\phi_n(x) = \phi_n(y) = 1$ , or none of them is in  $B_{u_n}(r_n)$  and then  $\phi_n(x) = \phi_n(y) = 0$ .  $\square$

Let  $(\psi_i)$  be a sequence of functions where  $\psi_i$  can be equal to the function  $q_i$  or to the function  $\phi_i$  (this can be different for every index  $i$ ). For instance,  $(q_0, \phi_1, q_2, \phi_3, \dots, q_{2i}, \phi_{2i+1}, \dots)$  and  $(q_0, q_1, \phi_2, \dots, q_{3i}, q_{3i+1}, \phi_{3i+2}, \dots)$  are sequences of this type. We then have, for all  $n$ ,

$$\|\psi_n\|_\infty = 1, \quad \psi_n(u_n) = 1, \quad \psi_n(u_k) = 0 \text{ if } k < n.$$

Let us consider functions of the following form :

$$\sum_{i=0}^{\infty} a_i \psi_i, \quad a_i \in K \text{ for all } i, \quad |a_i| \rightarrow 0 \text{ if } i \rightarrow \infty.$$

It is clear that these functions are continuous since  $a_i \psi_i$  tends to zero uniformly.

For functions of this type we can prove the following lemmas :

**Lemma 2.2**

Let  $n_0$  be a natural number and let  $f$  be the function

$$f = \sum_{i=0}^{\infty} a_i \psi_i, \quad a_i \in K \text{ for all } i, \quad |a_i| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

where the coefficients  $a_i$  satisfy

$$|a_{n_0}| = 1 \text{ and } |a_k| < 1 \text{ for } k \neq n_0.$$

Then we have that  $|f(u_{n_0})| = 1$  and  $|f(u_k)| < 1$  if  $0 \leq k < n_0$ . Furthermore,  $\|f\|_{\infty} = 1$ .

**Proof**

We clearly have  $|a_i \psi_i(u_{n_0})| < 1$  if  $i \neq n_0$ ,  $|a_{n_0} \psi_{n_0}(u_{n_0})| = 1$  and so we have  $|f(u_{n_0})| = \max_{i \geq 0} \{|a_i \psi_i(u_{n_0})|\} = 1$ . In an analogous way we have, for  $k < n_0$ ,  $|a_i \psi_i(u_k)| < 1$  if  $i \neq n_0$ ,  $|a_{n_0} \psi_{n_0}(u_k)| = 0$  and so  $|f(u_k)| \leq \max_{i \geq 0} \{|a_i \psi_i(u_k)|\} < 1$ . Since  $\|\psi_i\|_{\infty} = 1$  for all  $i$  it now immediately follows that  $\|f\|_{\infty} = 1$ .  $\square$

**Lemma 2.3**

Let the function  $f$  be as defined in lemma 2.2.

Let  $x, y \in M$  such that  $|x - y| \leq \alpha^t$ . Assume  $0 \leq n_0 < N_t$ . Then for every  $j \in \mathbb{N}$ , we have

$$|f(x)^j - f(y)^j| \leq \alpha.$$

**Proof**

For  $j=1$  we have  $|f(x) - f(y)| \leq \max_{i \geq 0} \{|a_i| |\psi_i(x) - \psi_i(y)|\} \leq \alpha$  by lemma 2.1 and the fact that  $|a_i| \leq \alpha$  for  $i \neq n_0$ . The case  $j = 0$  is trivial and for  $j > 1$  we find

$$|f(x)^j - f(y)^j| = |f(x) - f(y)| \cdot \left| \sum_{s=0}^{j-1} f(x)^s f(y)^{j-1-s} \right| \leq \alpha.$$

$\square$

In lemmas 2.4 and 2.5, we introduce a function  $g$  which is an infinite linear combination of powers of functions of the previous type.

**Lemma 2.4**

Let  $(f_n)$  be a sequence of continuous functions, where for every  $n$ ,  $f_n$  is of the form

$$f_n = \sum_{i=0}^{\infty} a_{n,i} \psi_i, \quad a_{n,i} \in K, \quad |a_{n,i}| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with  $|a_{n,n}| = 1$  and  $|a_{n,i}| < 1$  if  $i \neq n$ .

Let  $n_0$  be a natural number and let  $g$  be a continuous function of the form

$$g = \sum_{i=0}^{\infty} c_i (f_i)^{m_i}, \quad c_i \in K, \quad |c_i| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with  $|c_{n_0}| = 1$ ,  $|c_i| < 1$  if  $i \neq n_0$  and  $m_i \in \mathbb{N}_0$  for all  $i$ .

Then  $|g(u_{n_0})| = 1$ ,  $|g(u_j)| < 1$  if  $0 \leq j < n_0$ . Furthermore,  $\|g\|_{\infty} = 1$ .

**Proof**

We remark that  $g$  is continuous. From lemma 2.2 it follows that, for all  $n$ ,  $|f_n(u_n)| = 1$ ,  $\|f_n\|_{\infty} = 1$  and since  $|c_i| \leq \alpha$  for  $i \neq n_0$  we have, for  $i \neq n_0$ ,  $|c_i f_i(u_{n_0})^{m_i}| < |c_{n_0} f_{n_0}(u_{n_0})^{m_{n_0}}| = 1$  so  $|g(u_{n_0})| = \max_{i \geq 0} \{|c_i f_i(u_{n_0})^{m_i}|\} = 1$ .

Further, if  $j < n_0$ ,  $|g(u_j)| \leq \max_{i \geq 0} \{|c_i f_i(u_j)^{m_i}|\} < 1$  since  $|c_i| \leq \alpha$  for  $i \neq n_0$  and  $|f_{n_0}(u_j)| < 1$  (lemma 2.2). It now follows immediately that  $\|g\|_{\infty} = 1$ , since  $\|f_j\|_{\infty} = 1$  for all  $j$ .  $\square$

**Lemma 2.5**

Let the function  $g$  be as defined in lemma 2.4.

Let  $x, y \in M$  such that  $|x - y| \leq \alpha^t$ . Assume  $0 \leq n_0 < N_t$ . Then for every  $j \in \mathbb{N}$ , we have

$$|g(x)^j - g(y)^j| \leq \alpha.$$

**Proof**

Since  $|c_i| \leq \alpha$  for  $i \neq n_0$ , we have  $|g(x) - g(y)| \leq \max_{i \geq 0} \{|c_i| |f_i(x)^{m_i} - f_i(y)^{m_i}|\} \leq \alpha$  by lemma 2.3. If  $j > 1$  then

$$|g(x)^j - g(y)^j| = |g(x) - g(y)| \left| \sum_{s=0}^{j-1} g(x)^s g(y)^{j-1-s} \right| \leq \alpha$$

since  $\|g\|_{\infty} = 1$  (lemma 2.4). So  $|g(x)^j - g(y)^j| \leq \alpha$  for all  $j \in \mathbb{N}$  (the case  $j = 0$  is trivial).  $\square$

### 3. Orthonormal bases for $C(M \rightarrow K)$

We are going to use the lemmas in section 2 to construct orthonormal bases for  $C(M \rightarrow K)$  with the aid of the following result. Let  $k$  be the finite residue class field of  $K$ .

If  $f$  is an element of  $C(M \rightarrow K)$  with  $\|f\|_\infty \leq 1$ , let  $\bar{f}$  denote the canonical projection of  $f$  on  $C(M \rightarrow k)$ . Then we have the following ([6], lemme 1)

A sequence  $(e_i)$  of elements of  $C(M \rightarrow K)$  forms an orthonormal basis for  $C(M \rightarrow K)$  if and only if

- 1)  $\|e_i\|_\infty \leq 1$  for all  $i$
- 2)  $(\bar{e}_i)$  forms an algebraic basis for the  $k$ -vectorspace  $C(M \rightarrow k)$ .

This allows us to prove the main result of this paper. Let  $(\psi_{1,i})$  and  $(\psi_{2,i})$  be sequences of functions where  $\psi_{l,i}$  ( $l = 1$  or  $2$ ) is equal to the function  $q_i$  or to the function  $\phi_i$ .

Remark that this can be different for every index  $i$ , just as in section 2. As an example of such a sequence we have  $(\phi_0, \phi_1, q_2, \phi_3, \phi_4, q_5, \dots, \phi_{3i}, \phi_{3i+1}, q_{3i+2}, \dots)$

#### Theorem

Let  $(f_{1,n})$ ,  $(f_{2,n})$ ,  $(g_{1,n})$  and  $(g_{2,n})$  be sequences of continuous functions of the following form :

for  $l = 1$  or  $l = 2$ , for every  $n \geq 0$ ,  $f_{l,n}$  is of the form

$$f_{l,n} = \sum_{i=0}^{\infty} a_{l,n,i} \psi_{l,i}, \quad a_{l,n,i} \in K, \quad |a_{l,n,i}| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with  $|a_{l,n,n}| = 1$  and  $|a_{l,n,i}| < 1$  if  $i \neq n$ ,

and for  $l = 1$  or  $l = 2$ , for every  $n \geq 0$ ,  $g_{l,n}$  is of the form

$$g_{l,n} = \sum_{i=0}^{\infty} c_{l,n,i} (f_{l,i})^{m_{l,n,i}}, \quad c_{l,n,i} \in K, \quad |c_{l,n,i}| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with

$$|c_{l,n,n}| = 1, \quad |c_{l,n,i}| < 1 \text{ if } i \neq n,$$

and

$$m_{l,n,i} \in \mathbb{N}_0 \text{ for all } i.$$

If  $(k_n)$  is a sequence in  $\mathbb{N}$  and if  $(j_n)$  is a sequence in  $\mathbb{N}_0$ , then the sequence  $((g_{1,n})^{k_n} (g_{2,n})^{j_n})$  forms an orthonormal basis for  $C(M \rightarrow K)$ .

**Proof**

The functions  $f_{1,n}, f_{2,n}, g_{1,n}$  and  $g_{2,n}$  are clearly continuous. We remark that, by lemma 2.4,  $|g_{1,n}(u_n)| = |g_{2,n}(u_n)| = 1, |g_{1,n}(u_i)| < 1$  and  $|g_{2,n}(u_i)| < 1$  for all  $n$  in  $\mathbb{N}$  and for all  $i, i < n$ , and  $\|((g_{1,n})^{k_n}(g_{2,n})^{j_n})\|_\infty = 1$ . By the remark above, it suffices to prove that  $\overline{((g_{1,n})^{k_n}(g_{2,n})^{j_n})}$  forms an algebraic basis for  $C(M \rightarrow k)$ . Let  $C_t$  be the subspace of  $C(M \rightarrow k)$  of the functions constant on closed balls with radius  $\alpha^t$ . Since  $C(M \rightarrow k) = \bigcup_{t \geq 0} C_t$  it suffices to prove that  $\overline{((g_{1,n})^{k_n}(g_{2,n})^{j_n}) | n < N_t}$  forms a basis for  $C_t$ .  $M$  is the union of  $N_t$  disjoint balls with centers  $u_n, 0 \leq n < N_t$ , radius  $\alpha^t$ . Let  $\chi_i$  denote the characteristic function of the ball with center  $u_i$ .

For  $x, y \in M$  satisfying  $|x - y| \leq \alpha^t$ , for  $0 \leq n < N_t$  we have

$$|g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}| \leq \alpha.$$

This can be seen as follows :

$$\begin{aligned} & |g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}| \\ & \leq \max\{|g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(x)^{k_n} g_{2,n}(y)^{j_n}|, |g_{1,n}(x)^{k_n} g_{2,n}(y)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}|\} \\ & \leq \max\{|g_{1,n}(x)^{k_n}| |g_{2,n}(x)^{j_n} - g_{2,n}(y)^{j_n}|, |g_{2,n}(y)^{j_n}| |g_{1,n}(x)^{k_n} - g_{1,n}(y)^{k_n}|\} \leq \alpha \text{ by lemma 2.5.} \end{aligned}$$

It now follows that

$$\begin{aligned} \overline{(g_{1,n}(x))^{k_n}(g_{2,n}(x))^{j_n}} &= \sum_{i=0}^{N_t-1} \chi_i(x) \overline{(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}} \\ &= \sum_{i=n}^{N_t-1} \chi_i(x) \overline{(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}} \end{aligned}$$

since  $|(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}| < 1$  if  $i < n$  (lemma 2.4) and hence the transition matrix from  $(\chi_n | n < N_t)$  to  $\overline{((g_{1,n})^{k_n}(g_{2,n})^{j_n}) | n < N_t}$  is triangular. Since  $|(g_{1,n}(u_n))^{k_n}(g_{2,n}(u_n))^{j_n}| = 1$  (lemma 2.4),  $\overline{((g_{1,n})^{k_n}(g_{2,n})^{j_n}) | n < N_t}$  forms a basis for  $C_t$ .  $\square$

In the next corollary  $(\psi_i)$  is, as before, a sequence of functions where  $\psi_i$  is equal to the function  $q_i$  or to the function  $\phi_i$  (this can be different for every index  $i$ ).

**Corollary**

Let  $(f_n)$  and  $(g_n)$  be sequences of functions defined as follows : for all  $n$ ,

$$f_n = \sum_{i=0}^{\infty} a_{n,i} \psi_i, \quad a_{n,i} \in K, \quad |a_{n,i}| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with  $|a_{n,n}| = 1$  and  $|a_{n,i}| < 1$  if  $i \neq n$ .

$$g_n = \sum_{i=0}^{\infty} c_{n,i} (f_i)^{m_{n,i}}, \quad c_{n,i} \in K, \quad |c_{n,i}| \rightarrow 0 \text{ if } i \rightarrow \infty,$$

with  $|c_{n,n}| = 1$ ,  $|c_{n,i}| < 1$  if  $i \neq n$  and  $m_{n,i} \in \mathbb{N}_0$  for all  $i$ .

If  $(j_n)$  is a sequence in  $\mathbb{N}_0$ , then

1) the sequence  $((g_n)^{j_n})$  forms an orthonormal basis for  $C(M \rightarrow K)$ ,

2) the sequence  $((f_n)^{j_n})$  forms an orthonormal basis for  $C(M \rightarrow K)$ .

In particular, the sequences  $(g_n)$  and  $(f_n)$  form orthonormal bases for  $C(M \rightarrow K)$ .

### Proof

1) Apply the theorem above with  $k_n = 0$  for all  $n$ .

2) Put in 1) for all  $n$   $m_{n,n} = c_{n,n} = 1$  and  $c_{n,i} = 0$  if  $i \neq n$ .

If we put  $j_n = 1$  for all  $n$ , it follows that  $(g_n)$  and  $(f_n)$  form orthonormal bases for  $C(M \rightarrow K)$ .  $\square$

**Remark 3.1** If we look at the proof of the theorem and at the proofs of the lemmas in section 2, it is not difficult to extend the theorem and to find more orthonormal bases for  $C(M \rightarrow K)$ . However, the notations get more and more complicated.

**Remark 3.2** If we put in the theorem  $a_{2,0,0} = 1$ ,  $a_{2,0,i} = 0$  for  $i \neq 0$ , then  $f_{2,0} = 1$  so for  $m_{2,n,0} = 0$ ,  $n = 0, 1, \dots$  the theorem also holds. If we then put  $c_{2,0,0} = 1$ ,  $c_{2,0,i} = 0$  for  $i \neq 0$ , then  $g_{2,0} = 1$  and so the theorem holds for  $j_0 = 0$ . In analogous way we see that the corollary is also valid for  $m_{n,0} = 0$ ,  $n = 0, 1, \dots$  and  $j_0 = 0$ .

Let us now look at some examples. Examples 3.1, 3.2 and 3.3 can also be found with the results of [9].

**Example 3.1** The sequence  $((p_n)^{j_n})$  ( $j_n \in \mathbb{N}_0$ ) forms an orthonormal basis for  $C(M \rightarrow K)$ ,

where for all  $n$ ,  $p_n$  is a function defined as follows :  $p_n = \sum_{i=0}^n a_{n,i} \phi_i$ , with  $|a_{n,n}| = 1$  and

$|a_{n,i}| < 1$  if  $0 \leq i < n$ ,  $a_{n,i} \in K$ . To see this, apply the corollary above for  $f_n = \sum_{i=0}^n a_{n,i} \psi_i$

with  $\psi_i = \phi_i$ . In particular, the sequence  $(\phi_n)$  forms an orthonormal basis.

**Example 3.2** Let for all  $n$ ,  $p_n$  be a polynomial of degree  $n$  defined as follows :  $p_n = \sum_{i=0}^n a_{n,i} q_i$

with  $|a_{n,n}| = 1$  and with  $|a_{n,i}| < 1$  if  $0 \leq i < n$  ( $a_{n,i} \in K$ ). If we apply the corollary above for  $f_n = \sum_{i=0}^n a_{n,i} \psi_i$  with  $\psi_i = q_i$  for all  $i$ , then the sequence  $((p_n)^{j_n})$  ( $j_n \in \mathbb{N}_0$ ) forms an orthonormal basis for  $C(M \rightarrow K)$ . In particular, the sequence  $((q_n)^{j_n})$  forms an orthonormal basis. If we put  $j_n = 1$  for all  $n$  then the sequence  $(q_n)$  also forms an orthonormal basis.  $(q_n)$  is known as Amice's basis (see also [1], p. 143, theorem 1).

**Example 3.3** Put  $K = \mathbb{Q}_p$ , the field of the  $p$ -adic numbers and  $M = \mathbb{Z}_p$ , the ring of the  $p$ -adic integers, and let  $|\cdot|$  be the  $p$ -adic valuation on  $\mathbb{Q}_p$ . From examples 1.2 and 1.4 we know that  $\mathbb{Z}_p$  is a regular compact and that  $(u_n), u_n = n$  for all  $n$ , is a very well distributed sequence in  $\mathbb{Z}_p$ .

From example 3.1 it follows that the sequence  $(\phi_n)$  defined on  $M$  by

$$\phi_0(x) = 1 \quad \text{for all } x \text{ in } M$$

$$\phi_n(x) = 1 \text{ if and only if } x \in B_{u_n}(r_n) \text{ (} n \geq 1 \text{), where } r_n = \alpha^{i+1} \text{ if } p^i \leq n < p^{i+1}$$

forms an orthonormal basis for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$ . Then,  $(\phi_n)$  is known as van der Put's basis ([4], example 7.2).

Define the sequence of polynomials  $\left(\binom{x}{k}\right)$  by  $\binom{x}{0} = 1, \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$

if  $k \geq 1$ . From example 3.2 it follows that the sequences  $\left(\binom{x}{k}\right)$  (Mahler's basis, [5]),

and  $\left(\left(\binom{x}{k}\right)^s\right)$  (Caenepeel, [2]) form orthonormal bases for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$  ( $s \in \mathbb{N}_0$ ).

**Example 3.4** Put  $K = \mathbb{Q}_p, M = \mathbb{Z}_p$  and let  $|\cdot|$  be the  $p$ -adic valuation on  $\mathbb{Q}_p$ .  $V_q$  is the closure of the set  $\{aq^n | n = 0, 1, 2, \dots\}$  where  $a$  and  $q$  are two units of  $\mathbb{Z}_p, q$  not a root of unity.

Let  $m$  be the smallest integer such that  $q^m \equiv 1 \pmod{p}$  ( $1 \leq m \leq p-1$ ). There exists a  $k_0$  such that  $q^m \equiv 1 \pmod{p^{k_0}}, q^m \not\equiv 1 \pmod{p^{k_0+1}}$ . If  $(p, k_0) = (2, 1)$ , i.e.  $q \equiv 3 \pmod{4}$ , then there exists a natural number  $N$  such that  $q = 1 + 2 + 2^2\varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots, \varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{N-1} = 1, \varepsilon_N = 0$ . Then we have ([7], lemmas 4 and 5)

1) Let  $q^m \equiv 1 \pmod{p^{k_0}}, q^m \not\equiv 1 \pmod{p^{k_0+1}}$  with  $(p, k_0) \neq (2, 1)$ .

Then  $V_q = \bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p | |x - aq^r| \leq p^{-k_0}\}$ .

2) Let  $q \equiv 3 \pmod{4}, q = 1 + 2 + 2^2\varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots, \varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{N-1} = 1, \varepsilon_N = 0$ .

Then  $V_q = \{x \in \mathbb{Z}_2 \mid |x - a| \leq 2^{-(N+3)}\} \cup \{x \in \mathbb{Z}_2 \mid |x - aq| \leq 2^{-(N+3)}\}$ .

The method used in this paper can be applied to construct orthonormal bases for the Banach space  $C(V_q \rightarrow \mathbb{Q}_p)$  of continuous functions from  $V_q$  to  $\mathbb{Q}_p$ . A theorem concerning this item can be found in [8].

**Example 3.5** Put  $K = \mathbb{Q}_p$ ,  $M = \mathbb{Z}_p$  and let  $|\cdot|$  be the p-adic valuation on  $\mathbb{Q}_p$ . With the theorem and the corollary above it is not so difficult to find orthonormal bases for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$ . Let  $\binom{x}{n}$  denote Mahler's basis (example 3.3). Put  $f_n = \sum_{i=0}^{\infty} p^i \binom{x}{i+n} = \sum_{k=n}^{\infty} p^{k-n} \binom{x}{k}$ . Then the sequences  $(f_n)$ ,  $((f_n)^n)$  and  $(f_0, f_1, (f_2)^2, \dots, (f_n)^n, \dots)$  form orthonormal bases for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$ . Put  $g_n = \sum_{i=0}^{\infty} p^i \binom{x}{i+n}^n = \sum_{k=n}^{\infty} p^{k-n} \binom{x}{k}^n$ . Then  $(g_n)$ ,  $((g_n)^n)$  and  $(g_0, g_1, (g_2)^2, \dots, (g_n)^n, \dots)$  are orthonormal bases for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$ . Let  $(\phi_n)$  be van der Put's basis for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$  (example 3.3). Put  $f_n = \phi_n + \sum_{i \geq 0, i \neq n} p^{2^i} \phi_i$ , then  $(f_n)$  and  $((f_n)^{2^n})$  are orthonormal bases for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$ .

Another orthonormal basis for  $C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)$  forms the sequence  $(f_n)$ , where  $f_n = \psi_n + \sum_{\substack{i \geq 0 \\ i \neq n}} (p)^{p^i} \psi_i$ , with  $\psi_{2i}(x) = \phi_{2i}(x)$ ,  $\psi_{2i+1}(x) = \binom{x}{2i+1}$  for all  $n, i \in \mathbb{N}$ .

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