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3-filiform Lie algebras of dimension 8[1]

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Abstract

We give, up to isomorphism and in dimension 8, all the 3-filiform Lie algebras (whose Goze's invariant is $(n-3,1,1,1)$).

1 Introduction

The classification of finite dimensional complex Lie algebras is an open problem. Only the seven (or less) dimensional nilpotent Lie algebras are classified. A general classification seems very difficult. In fact, a recent result of Goze [11] shows that the general classification of $2p$ or $2p+1$-dimensional Lie algebras is equivalent to the linear classification of $(2,1)$-tensors in $\mathbf{C}^p$. This implies that the Jacobi conditions do not reduce the difficult problem of classification of the $(2,1)$-tensors.

Except the 7-dimensional case, we know also the classification of filiform algebras up to dimension 11 ([2], [8], [10]) or the general classification of 2-abelian filiform Lie algebras [9]. The results of Khakimdjanov [12] are very important for these classifications.

Cabezas, Gómez and Jiménez-Merchán [4], [6] and [5] generalize the notion of filiform algebra to *p-filiform* algebra, which correspond to nilpotent algebras of Goze's invariant $(n-p,1,\dots,1)$ where $n=dim(\mathfrak{g})$); hence, the filiform algebras are the 1-filiform algebras and the quasi-filiform algebras are the 2-filiform algebras. The authors above mentioned also give the classification for high values of p (that is, close to the dimension of the algebra), more exactly for the integer values of p between $n-4$ and $n-2$.

In [3], the $(n-5)$-filiform Lie algebras with maximal derived subalgebra are classified. In this way our first goal in this paper is to give an explicit description of the 3-filiform Lie algebra in 8-dimension.

Goze's invariant or *characteristic sequence* of the nilpotent Lie algebra $\mathfrak{g}$, denoted by $c(\mathfrak{g})$, is defined to be $sup\{c(X): X\in\mathfrak{g}-[\mathfrak{g},\mathfrak{g}]\}$, where $c(X)$ is the sequence, in decreasing order, of dimensions of characteristic subspaces of the nilpotent operator $ad(X)$. Thus, the filiform, quasifiliform and abelian Lie algebras of dimension n have as their Goze's invariant $(n-1,1)$, $(n-2,1,1)$ and $(1,1,\dots,1)$, respectively.

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2 Notation and Terminology

The notions and notations used in this paper are defined in [11].

Let $\mathfrak{g}$ be a 3-filiform Lie algebra of dimension 8. We consider a characteristic vector $X_0 \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$. An adapted basis [11] is given by $\{X_0, X_1, X_2, X_3, X_4, X_5, Y_1, Y_2\}$ where $\{X_1, X_2, X_3, X_4, X_5, Y_1, Y_2\}$ is a Jordan basis of $ad(X_0)$. It satisfies $[X_0, X_i] = X_{i+1}$, $i = 1, \ldots, 4$ and $[X_0, Y_j] = 0$, $j = 1, 2$. We denote by AL3F the set of complex 3-filiform Lie algebras of dimension 8, AL3F(k) the subset of AL3F constituted of Lie algebras whose derived subalgebra is of dimension k, AL3F(k,l) the subset of AL3F(k) for which elements satisfy $dim(\mathcal{Z}(\mathfrak{g})) = l$, AL3F(k,-,m) the subset of AL3F(K) with $dim\mathcal{D}^2(\mathfrak{g}) = m$.

If $\mathfrak{g} \in$ AL3F will be given by the following brackets

$$\begin{array}{ll} [X_0, X_i] & 1 \le i \le 4 \\ [X_i, X_{i+1}] & 1 \le i \le 3 \\ [X_1, Y_j] & 1 \le j \le 2 \\ [Y_1, Y_2] & \end{array}$$

An easy computation (using Jacobi's identity) shows that the remaining brackets can be found from the above mentioned.

In what follows, when we use subindexs i and j, then respective ranges of variation will be $1 \le i \le 4$ and $1 \le j \le 2$, though we do not indicate it. The laws will be denoted by $\mu^s_{(5,1,1,1)}$ or by $\mu^{21,\lambda}_{(5,1,1,1)}$, $\lambda \in \mathbf{C}_2$, where $\mathbf{C}_2 = \mathbf{C}/R$, being R the equivalence relation defined by $uRv \iff u = \pm v$.

3 3-filiform complex Lie algebras of dimension 8

1. Decomposable case

Proposition 3.1. *Let $\mathfrak{g}$ be a 6-dimension filiform Lie algebra. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}^2$ is a 8-dimension 3-filiform Lie algebra.*

Let $\mathfrak{g}$ be a 7-dimension 2-filiform Lie algebra. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}$ is a 8-dimension 3-filiform Lie algebra.

The proof is obvious.

As we know the 6-dimensional Lie algebras and the 7-dimensional 2-filiform Lie algebras, we can deduce the complete classification of decomposable 8-dimension 3-filiform Lie algebras.

2. Non decomposable case.

Now, we consider only 3-filiform non-decomposable Lie algebras.

Lemma 3.2. *[3] If $\mathfrak{g} \in$ AL3F there is an adapted basis satisfying*

$$(1) \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + dX_5 + \alpha_1 Y_1 + \alpha_2 Y_2 \\ [X_2, X_3] = -eX_5 - \beta_1 Y_1 - \beta_2 Y_2 \\ [X_1, Y_j] = a_{3j} X_3 + a_{2j} X_4 + a_{1j} X_5 \\ [Y_1, Y_2] = bX_5 \end{cases}$$

with the restrictions following $\alpha_k a_{3j} = 0$; $2a_{32}e - \alpha_1 b = 0$; $2a_{31}e + \alpha_2 b = 0$; $\beta_k a_{3j} = 0$; $\beta_k a_{2j} c = 0$; $\beta_k a_{1j} = 0$; $\beta_k b = 0$; $\beta_1 a_{21} + \beta_2 a_{22} = 0$, $1 \le k \le 2$

We can deduce that there exists three families of AL3F, pairwise non-isomorphic, whose laws can be expressed, in a suitable adapted basis by

AL3F(6) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + Y_1 \\ [X_2, X_3] = -Y_2 \\ [X_1, Y_1] = a_{21} X_4 \\ ca_{21} = 0 \end{cases}$$

AL3F(5) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + Y_1 \\ [X_2, X_3] = -eX_5 - \beta Y_1 \\ [X_1, Y_j] = a_{2j} X_4 + a_{1j} X_5 \\ \beta a_{1j} = 0;\ \beta a_{21} = 0; \\ \beta a_{22} c = 0 \end{cases}$$

AL3F(4) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + dX_5 \\ [X_2, X_3] = -eX_5 \\ [X_1, Y_j] = a_{3j} X_3 + a_{2j} X_4 + a_{1j} X_5 \\ [Y_1, Y_2] = bX_5 \\ a_{3j} e = 0 \end{cases}$$

In fact, let $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$. We have

$$dim(\mathcal{D}^1(\mathfrak{g})) = 4 + rank(A)$$

a) $rank(A) = 2$. We consider the change of basis

$$\begin{cases} Y_1' = dX_5 + \alpha_1 Y_1 + \alpha_2 Y_2 \\ Y_2' = eX_5 + \beta_1 Y_1 + \beta_2 Y_2 \end{cases}$$

The relation (1) can be reduced to $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = d = e = 0$. This gives the first family AL3F(6).

KINANI

b) $rank(A) = 1$. Always we can supposed $\alpha_1 \neq 0$. The change of basis

$$\begin{cases} Y_1' = dX_5 + \alpha_1 Y_1 + \alpha_2 Y_2 \\ Y_2' = Y_2 \end{cases}$$

permits to consider $(\alpha_1, \alpha_2) = (1, 0)$ and $\beta_2 = 0$. We obtain AL3F(5).

c) $rank(A) = 0$. We find AL3F(4).

Theorem 3.3. *[3] If $\mathfrak{g} \in$ AL3F(6), then it is isomorphic to one of the algebras, pairwise non-isomorphic, that will be denoted by μ^s, with $1 \le s \le 3$.*

Lemma 3.4. *There exist three subfamilies of AL3F(5), pairwise non isomorphic, whose laws can be expressed, in a suitable adapted basis by*

AL3F(5, 3) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + Y_1 \\ [X_2, X_3] = -eX_5 - \beta Y_1 \end{cases}$$

AL3F(5, 2) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + Y_1 \\ [X_2, X_3] = -eX_5 - \beta Y_1 \\ [X_1, Y_j] = a_{2j} X_4 + a_{1j} X_5 \\ a_{21} a_{12} - a_{22} a_{11} = 0, \\ \exists i, j :\ a_{ij} \neq 0 \\ \beta a_{1j} = 0,\ \beta a_{21} = 0, \\ \beta a_{22} c = 0 \end{cases}$$

AL3F(5, 1) :

$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + Y_1 \\ [X_2, X_3] = -eX_5 \\ [X_1, Y_j] = a_{2j} X_4 + a_{1j} X_5 \\ a_{12} a_{21} - a_{22} a_{11} \neq 0 \end{cases}$$

Proof: It is easy to check that the dimension of the center of any Lie algebra $\mathfrak{g}$ with $\mathfrak{g} \in AL3F(5)$ depends of the rank of the matrix $B = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$. Thus, we will consider three cases:

(1) If $rank(B) = 0$ we lead to AL3F(5,3).

(2) If $rank(B) = 1$ this implicates that $a_{21}a_{12} - a_{22}a_{11} = 0$ with any $a_{ij} \neq 0$ obtaining of this form the family AL3F(5,2).

(3) If $rank(B) = 2$ then $a_{21}a_{12} - a_{22}a_{11} \neq 0$ from we can assert that $(a_{21}, a_{11}, a_{12}) \neq (0,0,0)$ what together to the above restrictions lead to $\beta = 0$, and thus we obtain the family AL3F(5,1). □

We can note that if $\mathfrak{g} \in$ AL3F(5,3) then $\mathfrak{g}$ is decomposable of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbf{C}$ with $\mathfrak{g}_1$ of dimension 7. These algebras are will known (see § 3.1). We find the algebras $\mu^s_{(5,1,1)} \oplus \mathbf{C}$ for $s = 1$ to 8.

Consider the case AL3F(5,2). We note that if $a_{22} = a_{12} = 0$ then $a_{11} \neq 0$ or $a_{21} \neq 0$. The corresponding algebras are decomposable.

Theorem 3.5. *If $\mathfrak{g} \in$ AL3F(5,2), then it is isomorphic to one of the algebras, pairwise non-isomorphic, of laws $\mu^s_{(5,1,1,1)}$ with $12 \leq s \leq 15$ and the decomposable Lie algebras $\mu^s_{(5,1,1)} \oplus \mathbf{C}$, $9 \leq s \leq 18$, $s \neq 14$ and $\mu^{14,\lambda}_{(5,1,1)} \oplus \mathbf{C}$, $\lambda \in \mathbf{C}_2$.*

Proof: The nullity of β is an invariant. In fact, $dim(\mathcal{C}^3(\mathfrak{g}))$ is 2 if $\beta = 0$ and 3 if $\beta \neq 0$.

Case 1: ($\beta = 0$) Making generic changes of basis,

$$X'_0 = P_0X_0 + P_1X_1 + P_2X_2 + P_3X_3 + P_4X_4 + P_5X_5 + P_6Y_1 + P_7Y_2$$

$$X'_1 = Q_0X_0 + Q_1X_1 + Q_2X_2 + Q_3X_3 + Q_4X_4 + Q_5X_5 + Q_6Y_1 + Q_7Y_2$$

$$Y'_2 = S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3 + S_4X_4 + S_5X_5 + S_6Y_1 + S_7Y_2$$

the condition of change of basis for to remain ($X'_5 \neq 0$) is

$$(P_0 + P_1e)(P_0^2 + P_1^2a_{21})(P_0Q_1 - P_1Q_0) \neq 0$$

and for to remain into the family, ($[X'_1, X'_3] = c'X'_5$), we have to impose that: $P_0Q_0 + P_1Q_1a_{21} = 0$, and the changes will be completed particularizing for some concrete values of any parameter and the respective restrictions.

The new parameters are:

$$c' = \frac{Q_1c}{P_0^2 + P_1^2a_{21}} + \frac{P_1(Q_0c + Q_1a_{11})}{(P_0 + P_1e)(P_0^2 + P_1^2a_{21})};$$

$$e' = \frac{(P_0e - P_1a_{21})(P_0Q_1 - P_1Q_0)}{(P_0 + P_1e)(P_0^2 + P_1^2a_{21})}; \qquad a'_{21} = \frac{(P_0Q_1 - P_1Q_0)^2}{(P_0^2 + P_1^2a_{21})^2}a_{21}.$$

We observe that the nullities of a_{21} and $e^2 + a_{21}$ are invariants of the algebra, indeed

$$e'^2 + a'_{21} = \frac{(P_0Q_1 - P_1Q_0)^2}{(P_0 + P_1e)^2(P_0^2 + P_1^2a_{21})}(e^2 + a_{21})$$

• If $a_{21} = 0$, to remain into the family: $P_1S_7a_{22} + P_0S_3 = 0$ and $P_0S_4 + P_1S_3c + P_1S_4e + P_1S_6a_{11} + P_1S_7a_{12} - P_2S_3e + P_2S_7a_{22} = 0$, and the condition of change of basis is $P_0Q_1(P_0 + P_1e)S_7 \neq 0$.

Thus, the parameters remain:

$$c' = \frac{Q_1c}{P_0^2} + \frac{P_1Q_1a_{11}}{P_0^2(P_0 + P_1e)}; \qquad e' = \frac{Q_1e}{P_0 + P_1e}$$

$$a'_{11} = \frac{Q_1^2a_{11}}{P_0(P_0 + P_1e)^2}; \qquad a'_{22} = \frac{S_7a_{22}}{P_0^3};$$

$$a'_{12} = \frac{S_6a_{11} + S_7a_{12}}{P_0^2(P_0 + P_1e)^2} - \frac{P_1S_7a_{22}(3P_0c + P_1a_{11})}{P_0^5(P_0 + P_1e)} - \frac{P_1^3S_7ce^2a_{22}}{P_0^5(P_0 + P_1e)^2}$$

Furthermore, the nullity of $ce + a_{11}$ is an invariant. In fact,

$$c'e' + a'_{11} = \frac{Q_1^2}{P_0^2(P_0 + P_1e)}(ce + a_{11})$$

By the above changes of basis we observe that the nullities of e, a_{11} and a_{22} are invariants. Taking into account that the dimension of the center is 1 together to fact that $a_{21} = 0$, we arrive at $a_{22}a_{11} = 0$ with any $a_{ij} \neq 0$.

Now, we show in a table the configuration of the parameters

$$\left\{\begin{array}{l}
e = 0 \left\{\begin{array}{l}
a_{11} = 0 \left\{\begin{array}{l}
c = 0 \left\{\begin{array}{l} a_{22} = 0 \longrightarrow a_{12} \neq 0 \\ a_{22} \neq 0 \left\{\begin{array}{l} a_{12} = 0 \\ a_{12} \neq 0 \end{array}\right. \end{array}\right. \\
c \neq 0 \left\{\begin{array}{l} a_{22} = 0 \longrightarrow a_{12} \neq 0 \\ a_{22} \neq 0 \ (1) \longrightarrow a_{12} = 0 \end{array}\right.
\end{array}\right. \\
a_{11} \neq 0 \ (2) \longrightarrow c = 0; \quad a_{22} = 0; \quad a_{12} = 0
\end{array}\right. \\
e \neq 0 \left\{\begin{array}{l}
a_{11} = 0 \left\{\begin{array}{l}
c = 0 \left\{\begin{array}{l} a_{22} = 0 \longrightarrow a_{12} \neq 0 \\ a_{22} \neq 0 \left\{\begin{array}{l} a_{12} = 0 \\ a_{12} \neq 0 \end{array}\right. \end{array}\right. \\
c \neq 0 \left\{\begin{array}{l} a_{22} = 0 \longrightarrow a_{12} \neq 0 \\ a_{22} \neq 0 \ (3) \left\{\begin{array}{l} ea_{12} + ca_{22} \neq 0 \ (\longrightarrow a_{12} = 0) \\ ea_{12} + ca_{22} = 0 \ (\longrightarrow a_{12} = -\frac{ca_{22}}{e}) \end{array}\right. \end{array}\right.
\end{array}\right. \\
a_{11} \neq 0 \ (4) \left\{\begin{array}{l} ce + a_{11} \neq 0 \longrightarrow c = 0; \quad a_{22} = 0; \quad a_{12} = 0 \\ ce + a_{11} = 0 \longrightarrow c = -\frac{a_{11}}{e}; \ a_{22} = 0; \ a_{12} = 0 \end{array}\right.
\end{array}\right.
\end{array}\right.$$

where the cases (1), (2), (3) and (4) are described below

(1) $e = a_{11} = 0, \quad ca_{22} \neq 0$

We choose $P_1 = \frac{P_0 a_{12}}{3ca_{22}}$ in order to obtain $a_{12} = 0$.

(2) $e = 0, \quad a_{11} \neq 0$

By choosing $P_1 = -\frac{P_0 c}{a_{11}}$ always can be supposed $c = 0$ and $a_{22}a_{11} = 0 \longrightarrow a_{22} = 0$. Thus, taking $S_6 = -\frac{S_7 a_{12}}{a_{11}}$ we lead to $a'_{12} = 0$.

(3) $a_{11} = 0, \quad eca_{22} \neq 0$

It is easily seen that $e'a'_{12} + c'a'_{22} = \dfrac{Q_1 S_7}{P_0^2(P_0 + P_1 e)^3}(ea_{12} + ca_{22})$ and so, its nullity is an invariant. Thus, if $ea_{12} + ca_{22} \neq 0$, by a suitable choosing of $P_1 \neq -\frac{P_0}{e}$ can be obtained $a_{12} = 0$. Otherwise, if $ea_{12} + ca_{22} = 0$ then $a_{12} = -\dfrac{ca_{22}}{e}$.

(4) $ea_{11} \neq 0, \qquad c' = \dfrac{Q_1(P_0 c + P_1(ce + a_{11}))}{P_0^2(P_0 + P_1 e)}$ and, so, the nullity of c depends on the nullity of $ce + a_{11}$. If $ce + a_{11} = 0$, the nullity of c is an invariant and if $ce + a_{11} \neq 0$, by substituting $P_1 = -\frac{P_0 c}{ce + a_{11}}$ we obtain $c = 0$ and in the same way $S_6 = -\frac{S_7 a_{12}}{a_{11}}$ lead to $a_{12} = 0$.

At this point, is a laborious but simple process to verify that the only algebras or families of algebras with $a_{21} = 0$, pairwise non-isomorphic, are the correspond to the enunciate, previous changes of scale when they are needed.

• If $a_{21} \neq 0$, the change of basis given by $(X'_0 = X_0,\ X'_1 = X_1,\ Y'_1 = Y_1,\ Y'_2 = a_{21}Y_2 - a_{22}Y_1)$ together to the restriction $a_{21}a_{12} - a_{22}a_{11} = 0$, let us suppose $a_{22} = a_{12} = 0$. In this case, we can note that any algebra is decomposable of the form $\mu^s_{(5,1,1)} \oplus \mathbf{C}$ for $s = 12$ to $s = 18$, $s \neq 14$, and $\mu^{14,\lambda}_{(5,1,1)} \oplus \mathbf{C}$ with $\lambda \in \mathbf{C}_2$.

Case 2: $(\beta \neq 0)$ Taking into account the restrictions, we obtain $a_{11} = a_{12} = a_{21} = 0$, $a_{22} \neq 0 \Longrightarrow c = 0$ and doing an suitable change of basis $(X'_0 = X_0;\ X'_1 = X_1 - \dfrac{e}{\beta a_{22}}Y_2)$ we can supposed $e = 0$. In this way, we arrive at $\mu^{15}_{(5,1,1,1)}$, previous change of scale. □

Theorem 3.6. *If* $\mathfrak{g} \in AL3F(5,1)$*, then it is isomorphic to one of the algebras, pairwise non-isomorphic, of laws* $\mu^s_{(5,1,1,1)}$ *with* $16 \leq s \leq 25$ *and* $s \neq 21$ *or to one of the type* $\mu^{21,\lambda}_{(5,1,1,1)}$ *with* $\lambda \in \mathbf{C}_2$.

Proof: By the generic changes of basis used in Theorem 3.5 can be proved that the nullity of a_{21} is an invariant.

• If $a_{21} = 0 \longrightarrow Q_0 = 0$, and from the condition of change of basis we can arrive at $a_{11}a_{22} \neq 0$ and thanks to the change of basis $(Y'_1 = Y_1,\ Y'_2 = a_{11}Y_2 - a_{12}Y_1)$ can be supposed $a_{12} = 0$. The others parameters remain:

$$c' = \frac{(P_0 + P_1 e)Q_1 c + P_1 Q_1 a_{11}}{P_0^2(P_0 + P_1 e)}; \qquad e' = \frac{Q_1 e}{P_0 + P_1 e}$$

under the restrictions:

$$P_0 Q_1 (P_0 + P_1 e) \neq 0;$$
$$P_0^3 S_6 a_{11} - 3P_0 P_1 S_7 (P_0 + P_1 e) ca_{22} - P_1^2 S_7 (P_0 + P_1 e) a_{11} a_{22} - P_1^3 S_7 ce^2 a_{22} = 0$$

We observe that the nullities of e and $ce + a_{11}$ are invariants.

· if $e = 0$, choosing $P_1 = -\frac{P_0 c}{a_{11}}$ $(a_{11} \neq 0)$ we obtain $c = 0$ and

· if $e \neq 0$,

$*$ $ce + a_{11} \neq 0$, choosing $P_1 = -\frac{P_0 c}{ce+a_{11}}$ we obtain $c = 0$,

$*$ $ce + a_{11} = 0$, then $c = -\frac{a_{11}}{e}$.

After all the above considerations, we can consider the following configuration for the parameters:

$$\left(\begin{array}{c} a_{21} = 0 \Longrightarrow a_{11}a_{22} \neq 0; \\ \Longrightarrow a_{12} = 0 \end{array} \right) \left\{ \begin{array}{l} e = 0 \longrightarrow c = 0 \\ \\ e \neq 0 \left\{ \begin{array}{l} c = 0 \\ \\ c = -\dfrac{a_{11}}{e} \end{array} \right. \end{array} \right.$$

obtaining the algebras $\mu^s_{(5,1,1,1)}$, with $16 \leq s \leq 18$, previous changes of scale.

• If $a_{21} \neq 0$, using a similar reasoning of Theorem 3.5 the nullity of $e^2 + a_{21}$ is invariant and the change $(Y_1' = Y_1,\ Y_2' = a_{21}Y_2 - a_{22}Y_1)$ let us suppose $a_{22} = 0$ and so $a_{12} \neq 0$.

· If $e^2 + a_{21} \neq 0$, choosing $P_1 = \frac{P_0 e}{a_{21}}$, we obtain $e = 0$.

By doing again the changes of basis together to the imposition of remain into the family, that is $e = 0$, we have that $Q_0 = P_1 = 0$, leading to

$$\mathrm{c}' = \frac{\mathrm{Q}_1 \mathrm{c}}{\mathrm{P}_0^2}; \qquad \mathrm{a}_{11}' = \frac{\mathrm{Q}_1^2 \mathrm{a}_{11}}{\mathrm{P}_0^3}$$

obtaining $\mu^s_{(5,1,1,1)}$, with $19 \leq s \leq 20$ and $\mu^{21,\lambda}_{(5,1,1,1)}$, with $\lambda \in \mathrm{C}_2$ previous changes of scale when they are needed.

· If $e^2 + a_{21} = 0 \longrightarrow a_{21} = -e^2$; $e \neq 0$. It is easy to see that the changes of basis used in the proof of the Theorem 3.5 can be adapted because of there is not necessary to consider the vector Y_2 thanks to the fact that $Y_2 \notin \mathcal{C}^1(\mathfrak{g})$ and $a_{12} \neq 0$. The nullities of $a_{11} - ec$ and $a_{11} + 3ec$ hold, and now repeating the cases considered in Theorem 3.5 we obtain the algebras $\mu^s_{(5,1,1,1)}$, with $22 \leq s \leq 25$. □

Lemma 3.7. *There exist two subfamilies of AL3F(4), pairwise non-isomorphic, whose laws can be expressed, in a suitable adapted basis by*

$AL3F(4,-,0)$:

$$\left\{ \begin{array}{l} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + dX_5 \\ [X_1, X_3] = cX_5 \\ [X_1, Y_1] = a_{31}X_3 + a_{21}X_4 + a_{11}X_5 \\ [X_1, Y_2] = a_{22}X_4 + a_{12}X_5 \\ [X_2, Y_1] = a_{31}X_4 + a_{21}X_5 \\ [X_2, Y_2] = a_{22}X_5 \\ [X_3, Y_1] = a_{31}X_5 \\ [Y_1, Y_2] = bX_5 \end{array} \right.$$

$AL3F(4,-,1)$:

$$\left\{ \begin{array}{l} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = cX_4 + dX_5 \\ [X_1, X_3] = cX_5 \\ [X_1, X_4] = eX_5 \\ [X_2, X_3] = -eX_5 \\ [X_1, Y_1] = a_{21}X_4 + a_{11}X_5 \\ [X_1, Y_2] = a_{22}X_4 + a_{12}X_5 \\ [X_2, Y_1] = a_{21}X_5 \\ [X_2, Y_2] = a_{22}X_5 \\ [Y_1, Y_2] = bX_5 \qquad e \neq 0 \end{array} \right.$$

Proof: By tacking into account the restrictions of the family of laws AL3F(4), can be supposed $a_{32} = 0$ (doing changes of basis when they are needed). The nullity of e is an invariant, in fact $dim(\mathcal{D}^2(\mathfrak{g}))$ is 0 if $e = 0$ and 1 if $e \neq 0$. □

Theorem 3.8. *If $\mathfrak{g} \in$ AL3F(4,-,0), then it is isomorphic to one of the algebras, pairwise non-isomorphic, of laws $\mu^s_{(5,1,1,1)}$ with $26 \leq s \leq 41$, and the decomposable Lie algebras $\mu^s_{(5,1)} \oplus \mathbf{C}^2$, $1 \leq s \leq 3$, $\mu^s_{(5,1,1)} \oplus \mathbf{C}$, $19 \leq s \leq 27$.*

Proof: Similar to precedents. □

Theorem 3.9. *If $\mathfrak{g} \in$ AL3F(4,-,1), then it is isomorphic to one of the algebras, pairwise non-isomorphic, of laws $\mu^s_{(5,1,1,1)}$ with $42 \leq s \leq 47$, and the decomposable Lie algebras $\mu^s_{(5,1)} \oplus \mathbf{C}^2$, $4 \leq s \leq 5$, $\mu^s_{(5,1,1)} \oplus \mathbf{C}$, $28 \leq s \leq 33$.*

Proof: Similar to precedents. □

4 List of laws

Continuously we will explicit the laws of each one of the algebras with Goze's invariant $(5,1,1,1)$ in order to simplify their placing. We remind that it is only necessary to know the following brackets:

$$\begin{array}{ll} [X_0, X_i] & 1 \leq i \leq 4 \\ [X_i, X_{i+1}] & 1 \leq i \leq 3 \\ [X_1, Y_j] & 1 \leq j \leq 2 \\ [Y_1, Y_2] & \end{array}$$

the remaining brackets can be found using Jacobi's identity.

The used notations in the list of algebras can be see in [7]. By commodity, we include the list of laws of any $(n-5)$-filiform Lie algebra of dimension 6 and 7 (whose Goze's invariant are $(5,1)$ ([10], [13]) and $(5,1,1)$ ([1]), respectively).

AL3F(6):

$\mu^1_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -Y_2 \end{cases}$$

$\mu^2_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -Y_2 \\ [X_1, Y_1] = X_4 \end{cases}$$

$\mu^3_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -Y_2 \end{cases}$$

AL3F(5):

AL3F(5,3):

$$\mu^s_{(5,1,1)} \oplus \mathbf{C}, \qquad 1 \leq s \leq 8$$

AL3F(5,2):

$\mu^4_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^5_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^6_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_2] = X_4 + X_5 \end{cases}$$

$\mu^7_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^8_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^9_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{10}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^{11}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_4 + X_5 \end{cases}$$

$\mu^{12}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{13}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^{14}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_2] = X_4 - X_5 \end{cases}$$

$\mu^{15}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -Y_1 \\ [X_1, Y_2] = X_4 \end{cases}$$

$$\mu^s_{(5,1,1)} \oplus \mathbf{C}, \qquad 9 \leq s \leq 18,\ i \neq 14$$

$$\mu^{14,\lambda}_{(5,1,1)} \oplus \mathbf{C}, \qquad \lambda \in \mathbf{C}_2$$

AL3F(5,1):

$\mu^{16}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_1] = X_5 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^{17}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_1] = X_5 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^{18}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = -X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_1] = X_5 \\ [X_1, Y_2] = X_4 \end{cases}$$

$\mu^{19}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_1] = X_4 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{20}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_1, Y_1] = X_4 + X_5 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{21,\lambda}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_1, Y_1] = X_4 + \lambda X_5 \\ [X_1, Y_2] = X_5 \\ \lambda \in \mathbf{C}_2 \end{cases}$$

$\mu^{22}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_1] = -X_4 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{23}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_1] = -X_4 + X_5 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{24}_{(5,1,1,1)}$:
$$\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_1, X_2] = X_4 + Y_1 \\ [X_2, X_3] = -X_5 \\ [X_1, Y_1] = -X_4 - 3X_5 \\ [X_1, Y_2] = X_5 \end{cases}$$

$\mu^{25}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=Y_1 \\ [X_2,X_3]=-X_5 \\ [X_1,Y_1]=-X_4+X_5 \\ [X_1,Y_2]=X_5 \end{cases}$$

AL3F(4):

AL3F(4,-,0):

$\mu^{26}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{27}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{28}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{29}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_4 \\ [X_1,Y_2]=X_5 \end{cases}$$

$\mu^{30}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_4 \\ [X_1,Y_2]=X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{31}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{32}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_1,Y_1]=X_4 \\ [X_1,Y_2]=X_5 \end{cases}$$

$\mu^{33}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_1,Y_1]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{34}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_3 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{35}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_5 \end{cases}$$

$\mu^{36}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{37}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_5 \\ [X_1,Y_1]=X_3 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{38}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_5 \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_5 \end{cases}$$

$\mu^{39}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_5 \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$\mu^{40}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_4 \end{cases}$$

$\mu^{41}_{(5,1,1,1)}$:
$$\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,Y_1]=X_3 \\ [X_1,Y_2]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$$

$$\mu^{s}_{(5,1)} \oplus \mathbf{C}^2, \qquad 1 \le s \le 3$$
$$\mu^{s}_{(5,1,1)} \oplus \mathbf{C}, \qquad 19 \le s \le 27$$

AL3F(4,-,1):

$\mu^{42}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_2,X_3]=-X_5 \\ [X_1,Y_1]=X_4 \\ [X_1,Y_2]=X_5 \end{cases}$
$\mu^{43}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_2,X_3]=-X_5 \\ [X_1,Y_1]=X_4 \\ [X_1,Y_2]=X_5 \end{cases}$
$\mu^{44}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_2,X_3]=-X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$

$\mu^{45}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_2,X_3]=-X_5 \\ [X_1,Y_1]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$
$\mu^{46}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_2,X_3]=-X_5 \\ [Y_1,Y_2]=X_5 \end{cases}$
$\mu^{47}_{(5,1,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_2,X_3]=-X_5 \\ [X_1,Y_1]=X_4 \\ [Y_1,Y_2]=X_5 \end{cases}$

$$\mu^{s}_{(5,1)} \oplus \mathbf{C}^2, \qquad 4 \le s \le 5$$
$$\mu^{s}_{(5,1,1)} \oplus \mathbf{C}, \qquad 28 \le s \le 33$$

5 Appendix

We will explicit the laws of each one of the algebras with Goze's invariant $(5,1)$ and $(5,1,1)$ in order to simplify their placing. We write only essential brackets (see § 4).

5.1 Lie algebras of Goze's invariant $(5,1)$

(see, [10], [13])

$\mu^{1}_{(5,1)}$: $\{ [X_0,X_i]=X_{i+1}$
$\mu^{2}_{(5,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_5 \end{cases}$
$\mu^{3}_{(5,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \end{cases}$

$\mu^{4}_{(5,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_2,X_3]=-X_5 \end{cases}$
$\mu^{5}_{(5,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4 \\ [X_2,X_3]=-X_5 \end{cases}$

5.2 Lie algebras of Goze's invariant $(5,1,1)$

(see, [1])

$\mu^{1}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=Y \end{cases}$
$\mu^{2}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=Y \\ [X_2,X_3]=-X_5 \end{cases}$
$\mu^{3}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4+Y \end{cases}$

$\mu^{4}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=X_4+Y \\ [X_2,X_3]=-X_5 \end{cases}$
$\mu^{5}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=Y \\ [X_2,X_3]=-Y \end{cases}$
$\mu^{6}_{(5,1,1)}$: $\begin{cases} [X_0,X_i]=X_{i+1} \\ [X_1,X_2]=Y \\ [X_2,X_3]=-X_5-Y \end{cases}$

$\mu^{7}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4+Y\\ [X_2,X_3]=-Y\end{cases}$$

$\mu^{8}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4+Y\\ [X_2,X_3]=-\sqrt{2}X_5-Y\end{cases}$$

$\mu^{9}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{10}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{11}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=-X_4+Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{12}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_1,Y]=X_4\\ [X_2,Y]=X_5\end{cases}$$

$\mu^{13}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_1,Y]=X_4+X_5\end{cases}$$

$\mu^{14,\lambda}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4+Y\\ [X_1,Y]=X_4+\lambda X_5\end{cases}$$

$\mu^{15}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=-X_4\end{cases}$$

$\mu^{16}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4+Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=-X_4+X_5\end{cases}$$

$\mu^{17}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4+Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=-X_4-3X_5\end{cases}$$

$\mu^{18}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=Y\\ [X_2,X_3]=-X_5\\ [X_1,Y]=-X_4+X_5\end{cases}$$

$\mu^{19}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{20}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_5\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{21}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,Y]=X_4\end{cases}$$

$\mu^{22}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,Y]=X_4+X_5\end{cases}$$

$\mu^{23}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{24}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4\\ [X_1,Y]=X_4\end{cases}$$

$\mu^{25}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,Y]=X_3\end{cases}$$

$\mu^{26}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,Y]=X_3+X_5\end{cases}$$

$\mu^{27}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_5\\ [X_1,Y]=X_3\end{cases}$$

$\mu^{28}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{29}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_4\end{cases}$$

$\mu^{30}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_4+X_5\end{cases}$$

$\mu^{31}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_4-X_5\end{cases}$$

$\mu^{32}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_5\end{cases}$$

$\mu^{33}_{(5,1,1)}$:
$$\begin{cases}[X_0,X_i]=X_{i+1}\\ [X_1,X_2]=X_4\\ [X_2,X_3]=-X_5\\ [X_1,Y]=X_4\end{cases}$$

and the decomposable algebras $\mu^{s}_{(5,1)}\oplus\mathbf{C}$, $1\le s\le 5$.

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