THEODORE K. BONI

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# ON THE EXTINCTION OF THE SOLUTION FOR AN ELLIPTIC EQUATION IN A CYLINDRICAL DOMAIN

## THEODORE K.BONI

ABSTRACT. - We obtain some conditions under which there is extinction in finite time of the solution of an elliptic equation in a cylindrical domain. We also show how the solution of our elliptic equation approaches its stationary solution.

**Keywords.** Asymptotic behavior, extinction, dead core.
**AMS classifications.** 35 B 40, 35 J 60.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the following boundary value problem for $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} + \Delta u - \lambda f(u) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$u = h(x) \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.2}$$

$$u(x,0) = u_o(x) \quad \text{in} \quad \Omega, \quad \text{with} \quad 0 \leq u_o(x) \leq 1, \tag{1.3}$$

where $h(x)$ is a nonnegative continuous function on $\partial\Omega$, $u_o(x)$ is a continuous function in $\Omega$ which satisfies the compatibility condition

$$u_o(x) = h(x) \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

For positive values of $s$, $f(s)$ is a positive and increasing function with $f(0) = 0$.
**Proposition 1.1.** *Under the above assumptions, there exists a solution $u$ of* $(1.1) - (1.3)$.
The proof of the proposition is based on the following lemma
**Comparison lemma 1.2** (maximum principle). *Let $u, v \in C(\overline{\Omega} \times [0, \infty))$ satisfying the following inequalities:*

$$\frac{\partial^2 u}{\partial t^2} + Lu - f(u) \leq \frac{\partial^2 v}{\partial t^2} + Lv - f(v) \quad \text{in} \quad \Omega \times (0, \infty),$$

$$u \geq v \quad \text{on} \quad \partial\Omega \times (0, \infty),$$

$$u(x,0) \geq v(x,0) \quad \text{in} \quad \overline{\Omega}.$$

*Then we have*
$$u(x,t) \geq v(x,t) \quad \text{in} \quad \Omega \times (0, \infty).$$

**Proof.** Assume that $w = u - v$ takes negative values and let $m = \inf w(x,t) < 0$. The function

$$m(t) = \min\{w(x,t) : x \in \overline{\Omega}\}$$

is obviously continuous and $m(0) \geq 0$, $\inf m(t) = m$. Introduce the function

$$L(t, C) = \frac{m}{2} - Ct.$$

We have

$$L(t, C) < m(t) \quad \text{in} \quad [0, \infty) \quad \text{for} \quad C \quad \text{large}$$

and the infimum of all constants $C$ with this property, call it $C_o$, has the property that

$$C_o > 0, \quad L(t, C_o) \leq m(t), \quad \text{there is} \quad t_o > 0 \quad \text{with} \quad L(t_o, C_o) = m(t_o).$$

Let $x_o$ be such that $m(t_o) = w(x_o, t_o)$. Since $w(x_o, t_o) < 0$, i.e $u(x_o, t_o) < v(x_o, t_o)$, it follows from the second inequality of the lemma that $x_o$ is in $\Omega$. Now we have

$$f(u(x_o, t_o)) < f(v(x_o, t_o)),$$

$$w(x, t_o) \geq w(x_o, t_o) \quad \text{in} \quad \Omega \Longrightarrow (Lw)(x_o, t_o) \geq 0,$$

$$w(x_o, t) \geq L(t, C_o) \quad \text{with equality for} \quad t = t_o \Longrightarrow w_{tt}(x_o, t_o) \geq 0$$

in contradiction to the assumed first inequality of the lemma.   □

**Proof of Proposition 1.1.** Since Problem $(1.1) - (1.3)$ has a comparison lemma, to prove our proposition, it suffices to show that $(1.1) - (1.3)$ has both a super-solution and a subsolution (see [10]). Let us notice that 0 is a subsolution and 1 is a supersolution, which leads us to the result.   □

By a stationary solution of $(1.1) - (1.3)$, we mean a function $s(x)$ satisfying

$$\Delta s(x) - \lambda f(s(x)) = 0 \quad \text{in} \quad \Omega, \tag{1.5}$$

$$s(x) = h(x) \quad \text{on} \quad \partial\Omega. \tag{1.6}$$

**Definition 1.3.** *We say that there is extinction in finite time for the solution* $u(x, t)$ *if there exists a finite time* $T_o$ *such that*

$$u(x, t) = 0 \quad \text{for} \quad x \in \Omega, \quad t \geq T_o.$$

In this paper, we are interested in the extinction of the solution $u$.

Elliptic equations in cylindrical domains have been the subject of research of many authors (see [1], [2], [5], [6], [7], [8], [11], [12] and the references cited therein). Concerning the extinction, Kondratiev and Veron in [8] have shown that under some assumptions, there is extinction in finite time for the solution $u$ in the case where $f(s) = s^p$ with $p < 1$ and the Dirichlet condition replaced by that of Neumann. Our aim in this paper is to generalize this result considering the problem described in $(1.1) - (1.3)$. We also obtain the asymptotic behavior as $t \to \infty$ of the solution $u$ showing how it approaches its stationary solution. We shall see in this paper that the stationary solution of $u$ plays an important

role in its asymptotic behavior as $t \to \infty$. This phenomenon is well known in the parabolic case (see, for instance [3], [4]). Our paper is organized as follows. In the next section, we give some conditions of extinction and in the last section, we study the asymptotic behavior of the solution $u$ as $t \to \infty$.

## 2. Conditions of extinction

In this section, we show that under some conditions, there is extinction in finite time for the solution $u$ of $(1.1) - (1.3)$.

Introduce the function $F(s) = \int_0^s f(\sigma)d\sigma$. Let

$$J(s) = \int_s^1 \frac{d\sigma}{\sqrt{F(\sigma)}} \tag{2.1}$$

and let $K(s)$ be the inverse function of $J(s)$. In this notation, the IVP

$$\alpha'(t) = -\sqrt{2\lambda F(\alpha(t))}, \quad \alpha(0) = 1, \tag{2.2}$$

has the solution

$$\alpha(t) = K(t\sqrt{2\lambda}) \quad \text{for} \quad t > 0 \quad \text{if} \quad \int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty, \tag{2.3}$$

$$\alpha(t) = K(t\sqrt{2\lambda}) \quad \text{for} \quad t < \frac{1}{\sqrt{2\lambda}} \int_0^1 \frac{ds}{\sqrt{F(s)}} \quad \text{if} \quad \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty, \tag{2.4}$$

$$\alpha(t) = 0 \quad \text{for} \quad t \geq \frac{1}{\sqrt{2\lambda}} \int_0^1 \frac{ds}{\sqrt{F(s)}} \quad \text{if} \quad \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty. \tag{2.5}$$

**Remark 2.1.** If $f(s) = s^p$, then $\alpha(t)$ is defined as follows:

$$\alpha(t) = [1 + t\sqrt{\lambda}\frac{\sqrt{p+1}(p-1)}{\sqrt{2}}]^{-\frac{2}{p-1}} \quad \text{if} \quad p > 1,$$

$$\alpha(t) = e^{-2\lambda t} \quad \text{if} \quad p = 1,$$

$$\alpha(t) = [1 - t\sqrt{\lambda}\frac{\sqrt{p+1}(1-p)}{\sqrt{2}}]_+^{\frac{2}{1-p}} \quad \text{if} \quad p < 1,$$

where $[x]_+ = \max\{x, 0\}$.

**Theorem 2.2.** *(a) Suppose that $h(x) = 0$.*
*If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

*then there is extinction in finite time for $u(x,t)$.*

*(b) Assume that* $\inf_{x \in \overline{\Omega}} u_o(x) = \rho > 0$. *If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

*then* $u(x,t) > 0$ *in* $\Omega \times (0, \infty)$.

*Proof.* (a) Let $\alpha(t)$ be a solution of (2.2). Then, we easily show that $\alpha(t)$ is a supersolution of $(1.1) - (1.3)$ and the maximum principle implies that

$$0 \leq u(x,t) \leq \alpha(t) \quad \text{in} \quad \Omega \times (0, T).$$

Therefore, the result follows from (2.5).

(b) Let $v(t)$ be a solution of the following differential equation

$$v'(t) = -\sqrt{2\lambda F(v(t))}, \quad v(0) = \rho. \tag{2.6}$$

Then $v(t)$ is a subsolution of $(1.1) - (1.3)$. The maximum principle implies that $u(x,t) \geq v(t)$. It follows from $\displaystyle\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty$ that $v(t) > 0$ for $t > 0$, which leads us to the result. $\square$

In the above theorem, we have obtained the extinction in finite time of the solution $u$ of $(1.1) - (1.3)$ in the case where the boundary values are zero. The question which appears is what happens if the boundary values are positive? To answer this question, we shall need the following definition:

**Definition 2.3.** *Let $\varphi(x)$ be the stationary solution of the solution $u$ of $(1.1) - (1.3)$. We say that there exists a dead core for $\varphi(x)$ if there exists a set $\Omega_o \subset \Omega$ such that $\varphi(x) = 0$ for $x \in \Omega_o$.*

The following lemma which may be found in [4] will be used later. It gives us information on the following steady-state problem:

$$\Delta\varphi(x) = \lambda f(\varphi(x)) \quad \text{in} \quad \Omega, \tag{2.7}$$

$$\varphi(x) = h(x) \quad \text{on} \quad \partial\Omega. \tag{2.8}$$

**Lemma 2.4.** *Let $h(x) > 0$. Suppose that*

$$f \in C^2(0,1), \quad f(0) = 0, \quad f(1) = 1, \quad f'(s) \geq 0 \quad on \quad (0,1),$$

$$either \quad f''(s) \geq 0 \quad on \quad (0,1) \ or \quad f''(s) \leq 0 \quad on \quad (0,1).$$

*If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

*then there is no dead core for any $\lambda$.*

*If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

*then a dead core exists for sufficiently large $\lambda$, and for any $x_o \in \Omega$, $x_o$ belongs to the dead core.*

If $\displaystyle\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty$, for $x_o \in \Omega$, define $\lambda_o$ as follows:

$$\lambda_o = \inf_\lambda \{\varphi(x_o, \lambda) = 0\}, \tag{2.9}$$

where $\varphi(x, \lambda)$ is the solution of $(2.7) - (2.8)$.

Now, introduce our result in the case where the boundary values are positive.

**Theorem 2.5.** *Assume that assumptions of Lemma 2.4 hold. For fixed $x_o \in \Omega$, choose $\lambda > \lambda_o$, where $\lambda_o$ is defined in (2.9).*

*(a) If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

*then $u(x_o, t) = 0$ for*

$$t \geq \frac{1}{\sqrt{2(\lambda - \lambda_o)}} \int_0^1 \frac{ds}{\sqrt{F(s)}}.$$

*(b) If*

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

*and $\min_{x \in \overline{\Omega}} u_o(x) > 0$, then $u(x_o, t) > 0$ for all $t$.*

*Proof. (a)* Put

$$w(x, t) = \beta(t) + \varphi(x),$$

where $\beta(t)$ is a solution of

$$\beta'(t) = -\sqrt{2(\lambda - \lambda_o)F(\beta(t))}, \quad \beta(0) = 1$$

and $\varphi(x)$ the solution of $(2.7) - (2.8)$ with $\lambda = \lambda_o$. We compute that

$$w_{tt} + \Delta w - \lambda f(w) = \beta''(t) + \Delta\varphi(x) - \lambda f(w)$$

$$= (\lambda - \lambda_o)f(\beta) + \lambda_o f(\varphi(x)) - \lambda f(w)$$

$$\leq (\lambda - \lambda_o + \lambda_o - \lambda)f(w) = 0.$$

It is easy to see that

$$w(x, t) \geq u(x, t) \quad \text{on} \quad \partial\Omega \times (0, \infty),$$

$$w(x,0) \geq u_o(x) \quad \text{in} \quad \Omega.$$

The maximum principle implies that

$$w(x,t) \geq u(x,t) \quad \text{in} \quad \Omega \times (0,\infty),$$

and the result follows from (2.5) and (2.9).

The proof of Theorem 2.5 (b) is as the proof of Theorem 2.2 (b).  □

**3. Asymptotic behavior.**

In this section, we show how the solution $u$ of $(1.1) - (1.3)$ approaches its stationary solution $\varphi$. We assume that $u_o(x) \geq \varphi(x)$ in $\Omega$. The maximum principle implies that $u(x,t) \geq \varphi(x)$ in $\Omega \times (0,\infty)$.

The following lemma will be used in the proof of the theorem in below.

**Lemma 3.1.** *Let $f(s)$ be a convex function with $f(0) = 0$ and let $a$ and $b$ be two nonnegative numbers. Then we have*

$$f(a+b) \geq f(a) + f(b).$$

*Proof.* Put $\alpha = \inf\{a,b\}$ and $\beta = \sup\{a,b\}$. Since $f(s)$ is convex with $f(0) = 0$, then using Taylor's formula, we get

$$f(a+b) = f(\beta+\alpha) \geq f(\beta)+\alpha f'(\beta) \geq f(\beta)+\alpha f'(\alpha) \geq f(\beta)+f(\alpha) = f(a)+f(b).$$

Hence the result.  □

**Theorem 3.2.** (a) *If $f$ is convex, then*

$$0 \leq u(x,t) - \varphi(x) \leq \alpha(t),$$

*where $\alpha(t)$ is a solution of (2.2).*
(b) *If $f$ is concave, then*

$$0 \leq u(x,t) - \varphi(x) \leq \zeta(t),$$

*where $\zeta(t)$ is a solution of*

$$\zeta'(t) = -\sqrt{\lambda f'(2)}\zeta(t), \quad \zeta(0) = 1.$$

*Proof.* (a) Put

$$w(x,t) = \alpha(t) + \varphi(x).$$

We compute that

$$w_{tt} + \Delta w - \lambda f(w) = \alpha''(t) + \Delta\varphi(x) - \lambda f(w)$$

$$= \lambda[f(\alpha) + f(\varphi(x)) - f(w)].$$

Since $f(s)$ is convex, it follows from Lemma 3.1 that

$$w_{tt} + \Delta w - \lambda f(w) \leq 0 \quad \text{in} \quad \Omega \times (0,\infty).$$

Obviously, we have

$$w(x,t) \geq u(x,t) \quad \text{on} \quad \partial\Omega \times (0,\infty),$$

$$w(x,0) \geq u_o(x) \quad \text{in} \quad \Omega.$$

It follows from the maximum principle that

$$w(x,t) \geq u(x,t) \quad \text{in} \quad \Omega \times (0,\infty),$$

which yields the first result.
(b) Let

$$z(x,t) = \zeta(t) + \varphi(x).$$

We compute that

$$z_{tt} + \Delta z - \lambda f(z) = \zeta''(t) + \Delta\varphi(x) - \lambda f(z)$$

$$= -\lambda f'(2)\zeta(t) + \lambda[f(\varphi(x)) - f(z)].$$

Since $\lambda f$ is concave, then using Taylor's formula, we get

$$\lambda[f(z) - f(\varphi(x))] \geq \lambda f'(z)(z - \varphi(x)) = \lambda f'(z)\zeta(t).$$

It follows from $z \leq 2$ that $\lambda[f(z) - f(\varphi(x))] \geq \lambda f'(2)\zeta(t)$, which implies that

$$z_{tt} + \Delta z - \lambda f(z) \leq 0 \quad \text{in} \quad \Omega \times (0,\infty).$$

We easily show that

$$z(x,t) \geq u(x,t) \quad \text{on} \quad \partial\Omega \times (0,\infty),$$

$$z(x,0) \geq u_o(x) \quad \text{in} \quad \Omega.$$

The maximum principle implies that

$$z(x,t) \geq u(x,t) \quad \text{in} \quad \Omega \times (0,\infty),$$

which gives the second result.  $\square$

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Théodore K. BONI
Université Paul Sabatier
UFR-MIG, MIP
118 route de Narbonne
31062 Toulouse, France.
E. Mail: boni@mip.ups-tlse.fr