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Arens algebras, associated with commutative von Neumann algebras


\(<http://www.numdam.org/item?id=AMBP_1998__5_1_1_0>\)
1. Introduction. Let $(\Omega, \Sigma, \mu)$ be a measurable space with a finite measure, $L^p(\mu) = L^p(\Omega, \Sigma, \mu)$ the Banach space of all $\mu$-measurable complex functions on $\Omega$, integrable with the degree, $p \in [1, +\infty)$. R. Arens [1] introduced and studied the set $L^w(\mu) = \bigcap_{1 \leq p < \infty} L^p(\mu)$. He showed, in particular, that $L^w(\mu)$ is a complete locally-convex metrizable algebra with respect to "$t$" topology generated by the system of norms $\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$, $p \geq 1$. Later G.R. Allan [2] observed that $(L^w(\mu), t)$ is a $GB^*$-algebra with the unit ball $B_0 = \{f \in L^\infty : \|f\|_p \leq 1\}$. Further investigation of properties of the Arens algebra $L^w(\mu)$ was made by S.J. Bhaft [3,4]. He described the ideals of the algebra $L^w(\mu)$ and considered some classes of homomorphism of this algebra. B.S. Zakirov [5] showed that $L^w(\mu)$ is an $EW^*$-algebra and gave an example of two measures, $\mu$ and $\nu$, on an atomic Boolean algebra, for which the algebras $L^w(\mu)$ and $L^w(\nu)$ are not isomorphic. It is clear that the problem of complete classification of the Arens algebras arises. Speaking more precisely, what conditions should be imposed on measures $\mu$ and $\nu$ for the corresponding Arens algebras to be isomorphic? It is natural to solve this problem in the class of equivalent measures. Therefore instead of a measurable space with a measure, one should consider a commutative von Neumann algebra $M$ with faithful normal finite traces $\mu$ and $\nu$ on $M$ and study the problem of $*$-isomorphism of $EW^*$-algebras $L^w(M; \mu) = \bigcap_{1 \leq p < \infty} L^p(M; \mu)$ and $L^w(M, \nu)$.

The present article gives the complete solution of the mentioned problem, a classification of the normalized Boolean algebras from the book by D.A. Vladimirov [6] being considerably used. All necessary notations and
results from the theory of von Neumann algebras are taken from [7] and the theory of integration on von Neumann algebras is from [8].

2. Preliminaries. Let $M$ be an arbitrary von Neumann algebra, $\mu$ a faithful normal finite trace on $M$, $P(M)$ the lattice of all projections of $M$. Let $K(M, \mu)$ be the $*$-algebra of all $\mu$-measurable operators affiliated with $M$ [8].

In the commutative case, when $M = L^\infty(\Omega, \Sigma, \mu)$ and $\mu(x) = \int x \, d\mu$, where $(\Omega, \Sigma, \mu)$ is a measurable space, the algebra $K(M, \mu)$ coincides with the algebra of all measurable complex functions on $(\Omega, \Sigma, \mu)$.

For every set $A \subset K(M, \mu)$ we shall denote by $A_+$ (respectively, by $A^+$) the set of all self-adjoint (respectively, positive self-adjoint) operators from $A$. The partial order in $K(M, \mu)$ generated by the positive cone $K_+(M, \mu)$ will be denoted by $x \leq y$.

Put $M(x) = \sup\{\mu(y) | 0 \leq y \leq x, y \in M\}$ for every $x \in K_+(M, \mu)$.

Let $p \in [1, \infty)$ and $L^p(M, \mu) = \{x \in K(M, \mu) | \mu(|x|^p) < \infty\}$, where $|x| = (x^*x)^{1/2}$. The set $L^p(M, \mu)$ is a subspace in $K(M, \mu)$ and the function $\|x\|_p = \mu(|x|^p)^{1/p}$ is a Banach norm on $L^p(M, \mu)$ [9]. Moreover,

1. $\|x\|_p = \|x^*\|_p = \|xu\|_p$ for all $x \in L^p(M, \mu)$ and a unitary element $u \in M$;
2. If $|x| \leq |y|, x \in K(M, \mu), y \in L^p(M, \mu)$, then $x \in L^p(M, \mu)$ and $\|x\|_p \leq \|y\|_p$;
3. If $x \in L^p(M, \mu), y \in L^q(M, \mu)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, 1 < p, q, r < \infty$, then $xy \in L^r(M, \mu)$ and $\|xy\|_r \leq \|x\|_p \|y\|_q$.

From these properties of the norm $\|\cdot\|_p$ it follows that the set $L^\omega(M, \mu) = \bigcap_{1 \leq p < \infty} L^p(M, \mu)$ is a $*$-subalgebra in $K(M, \mu)$, and $M \subset L^\omega(M, \mu)$. It was shown in [5] that $M = L^\omega(M, \mu)$ if and only if $\dim M < \infty$. Furthermore, since $L^\omega(M, \mu)$ is a solid $*$-subalgebra in $K(M, \mu)$ (e.g. the inequality $|x| \leq |y|, x \in K(M, \mu), y \in L^\omega(M, \mu)$ implies $x \in L^\omega(M, \mu)$), $L^\omega(M, \mu)$ is an $EW^*$-algebra, the bounded part of which coincides with $M$ [10].

Now we cite from [6] some information which will be used in the sequel.

Let $X$ be an arbitrary complete Boolean algebra, $e \in X, X_e = [0, e] = \{g \in X | g \leq e\}$. The minimal cardinality of the set which is dense in $X_e$ in the $(\omega)$-topology will be denoted $\tau(X_e)$. An infinite complete Boolean algebra $X$ is called homogeneous, if $\tau(X_e) = \tau(X_g)$ for any non-zero $e, g \in X$. The cardinality of $\tau(X) = \tau(X_1)$ where $1$ is the unit of the Boolean algebra $X$ is called a weight of a homogeneous Boolean algebra $X$. 

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Let $\mu$ be a strictly positive countably additive measure on $X$. If $\mu(\mathbf{1}) = 1$, then the pair $(X, \mu)$ is called a normalized Boolean algebra. It was shown in [6] that for any cardinal number $\tau$ there existed a complete homogeneous normalized Boolean algebra $X$ with the weight $\tau(X) = \tau$. The next theorem gives a criterion of isomorphism of two homogeneous normalized Boolean algebras.

**Theorem ([6]).** Let $(X, \mu)$ and $(Y, \nu)$ be homogeneous normalized Boolean algebras. The following conditions are equivalent:

1) $\tau(X) = \tau(Y)$;

2) There exists an isomorphism $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu(x)$ for all $x \in X$.

This theorem enables us to describe the class of von Neumann algebras for which the existence of $*$-isomorphism between the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ is equivalent to isomorphism between $M$ and $N$.

**Proposition 1.** Let $M$ and $N$ be commutative von Neumann algebras, the Boolean algebras $P(M)$ and $P(N)$ of which are homogeneous, and let $\mu$ and $\nu$ be faithful normal finite traces on $M$ and $N$, respectively. The following conditions are equivalent:

1) The Arens algebras $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ are $*$-isomorphic;

2) The von Neumann algebras $M$ and $N$ are $*$-isomorphic;

3) $\tau(P(M)) = \tau(P(N))$.

Proof. Since $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ are $EW^*$-algebras the bounded parts of which coincide with $M$ and $N$ respectively, restriction on $M$ of any $*$-isomorphism from $L^\omega(M, \mu)$ on $L^\omega(N, \nu)$ is a $*$-isomorphism from $M$ on $N$. On the other hand if the von Neumann algebras $M$ and $N$ are $*$-isomorphic, then their Boolean algebras of projectors are also isomorphic and therefore, in this case, $\tau(P(M)) = \tau(P(N))$.

Now suppose that $\tau(P(M)) = \tau(P(N))$ and assume $\mu'(x) = \mu(x)/\mu(\mathbf{1})$, $\nu'(y) = \nu(y)/\nu(\mathbf{1})$, $x \in M$, $y \in N$. According to the theorem 1, there exists an isomorphism of Boolean algebras $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu'(x)$ for all $x \in X$. This isomorphism extends to a $*$-isomorphism $\Phi : K(M, \mu) \to K(N, \nu)$ (See [11]): At the same time $\mu'(x) = \nu'(\Phi(x))$ for all $x \in L^1(M, \mu')$. Since $\mu'(|x|^p) = \nu'(\Phi(|x|^p))) = \nu'(|\Phi(x)|^p)$ we have $\Phi(L^p(M, \mu)) = \Phi(L^p(M, \mu')) = L^p(N, \nu') = L^p(N, \nu)$ for all $p \geq 1$. Hence $\Phi(L^\omega(M, \mu)) = L^\omega(N, \nu)$. 

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**Corollary.** Let $M$ and $N$ be non-atomic commutative von Neumann algebras on separable Hilbert spaces, $\mu$ and $\nu$ faithful normal finite traces on $M$ and $N$, respectively. Then the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ are $*$-isomorphic.

Proof. At first, show that if $M$ acts on a separable Hilbert space $H$, then the Banach space $(L^r(M, \mu), \| \cdot \|_r)$ is also separable. To start one should note that in this case the strong topology is metrizable on the unit ball $M_1$ of the algebra $M$ ([12] p.24). In addition, the convergence $x_\alpha \xrightarrow{s_0} 0$ in the strong topology in $M_1$ is equivalent to the convergence $\mu(x_\alpha^* x_\alpha) \to 0$ ([12] p.130).

Thus, for any sequence of $\{x_n\} \subset M$ and $x \in M$ the convergence $x_n \xrightarrow{s_0} x$ implies $\sup \|x_n\|_M < \infty$ and $\|x_n - x\|_2 \to 0$, where $\| \cdot \|_M$ is a $C^*$-norm in $M$. Hence, on any ball $M_n = \{x \in M | \|x\|_M \leq n\}$ the strong topology coincides with the topology induced from $L_2(M, \mu)$. Since $H$ is separable, there exists a countable set $X_n \subset M$ which is dense in $M_n$ in the strong topology ([13], p.568). Hence the countable set $X = \bigcup_{n=1}^{\infty} X_n$ is dense in $M$ in the topology induced from $L_2(M, \mu)$. Since $M$ is dense in $(L_2(M, \mu), \| \cdot \|_2)$, $(L_2(M, \mu), \| \cdot \|_2)$ is separable.

There is one thing left to say: the $(o)$-topology in $(P(M), \mu)$ coincides with the topology induced from $(L^2(M, \mu), \| \cdot \|_2)$. Therefore, the $P(M)$ is a non-atomic Boolean algebra which is separable in the $(o)$-topology. Hence it is homogeneous [6]. Similarly, $P(N)$ is a non-atomic Boolean algebra and $\tau(P(M)) = \tau(P(N))$. According to the proposition 1, the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(N, \nu) -$ are $*$-isomorphic.

Let $(X, \mu)$ be an arbitrary complete non-atomic normalized Boolean algebra. It was shown in [6] that there is a sequence $\{e_n\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $[0, e_n]$ are homogeneous and $\tau_n = \tau([0, e_n]) < \tau_{n+1}$, $n = 1, 2, \ldots$ This collection is determined uniquely and the matrix

$$
\begin{pmatrix}
\tau_1 & \tau_2 & \cdots \\
\mu(e_1) & \mu(e_2) & \cdots
\end{pmatrix}
$$

is called the passport of the Boolean algebra $(X, \mu)$

The following theorem will be used for investigation of isomorphisms of Arens algebra.
Theorem 2 [6]. Let \((X, \mu)\) and \((Y, \nu)\) be complete non-atomic normalized Boolean algebras. The following conditions are equivalent.

1. There exists an isomorphism \(\varphi : X \to Y\) for which \(\nu(\varphi(x)) = \mu(x)\) for all \(x \in X\).
2. The passports of the Boolean algebras \((X, \mu)\) and \((Y, \nu)\) coincide.

3. Main results. A von Neuman algebra \(M\) is called \(\sigma\)-finite if it admits at most countable family of orthogonal projections. On any \(\sigma\)-finite von Neumann algebra \(M\), there exists a normal state, in particular, if \(M\) is commutative, then its Boolean algebra of projections \(P(M)\) is a normed one. The next theorem describes the class of commutative \(\sigma\)-finite von Neumann algebras \(M\) for which the Arens algebras \(L^\omega(M, \mu)\) and \(L^\omega(M, \nu)\) are \(*\)-isomorphic for any faithful normal finite traces of \(\mu\) and \(\nu\) on \(M\).

Theorem 3. For a commutative \(\sigma\)-finite von Neumann algebra \(M\) the following conditions are equivalent:

1. The Arens algebras \(L^\omega(M, \mu)\) and \(L^\omega(M, \nu)\) are \(*\)-isomorphic for any faithful normal finite traces \(\mu\) and \(\nu\) on \(M\).
2. \(M = M_0 + \sum_{i=1}^n M_i\), where \(M_0\) is a finite-dimensional commutative von Neumann algebra, \(M_i\) is an infinite-dimensional commutative von Neumann algebra in which the lattice of projections \(P(M_i)\) is a homogeneous Boolean algebra and \(\tau_i = \tau(P(M_i)) < \tau_{i+1}\), \(i = 1, \ldots, n - 1\) (the summand \(M_0\) are \(M_i\) may be absent).

Proof. 1) \(\rightarrow\) 2). Let \(\Delta\) be the set of all atoms in \(P(M)\) and \(e = \sup \Delta\). Suppose that \(\Delta\) is a countable set. Then \(M_0 = eM\) coincides with the algebra \(\ell_\infty\) of all bounded sequences of complex numbers. Denote the atoms in \(P(\ell_\infty)\) by \(q_u = (0, \ldots, 0, 1, 0, \ldots)\). Consider two faithful normal finite traces \(\mu\) and \(\nu\) on \(M\), for which \(\mu(q_n) = n^{-2}\), \(\nu(q_n) = e^{-2n}\) and \(\mu(x) = \nu(x)\) for all \(x \in (I - e)M\). Suppose, that a \(*\)-isomorphism \(\Phi\) from \(L^\omega(M, \nu)\) on \(L^\omega(M, \mu)\) exists. Since \(\Phi(M_0) = M_0\), we have \(\Phi(L^\omega(M_0, \nu)) = L^\omega(M_0, \mu)\). Choose \(x \in K(M_0, \nu)\) such that \(xq_n = 2^n\). The series

\[
\sum_{n=1}^{\infty} \frac{2^{pn}}{e^{2^n}} = \nu(|x|^p)
\]

converges for all \(p \geq 1\). Therefore \(x \in L^\omega(M_0, \nu)\) and, so \(\Phi(x) \in L^\omega(M_0, \nu)\). Since \(M_0 = \ell_\infty\), the \(*\)-isomorphism \(\Phi\) is generated by some bijection \(\pi\) of
the set of natural numbers. It means that $\Phi(x) = \Phi(\{2^n\}) = \{2^{\pi(n)}\} = y \in L^\omega(M_0, \mu)$. In particular,

$$\nu(|y|) = \sum_{n=1}^{\infty} 2^{\pi(n)} n^{-2} < \infty$$

which is wrong. Hence, a set $\Delta$ is either finite or empty.

Now suppose that in the Boolean algebra $P((1-e)M)$ there is a countable set $\{e_n\}$ of disjoint elements, for which the algebras $X_n = P(e_nM)$ are homogeneous and $\tau_n = \tau(X_n) < \tau_{n+1}$. Choose two faithful normal finite traces $\mu$ and $\nu$ on $M$ such that $\mu(e_n) = n^{-2}$, $\nu(e_n) = e^{-2n}$ and $\mu(x) = \nu(x)$ for all $x \in M_0$. Let $\Phi$ be a $*$-isomorphisms from $L^\omega(M, \nu)$ on $L^\omega(M, \mu)$. Then $\Phi((1-e)M) = (1-e)M$ and, since weights $\tau_n$ are different, $\Phi(e_nM) = e_n(M)$ (See [6]). Choose $x \in K((1-e)M, \nu)$ such that $xe_n = 2^ne_n$. Then $x \in L^\omega((1-e)M, \nu)$, $\Phi(x) = x$ and

$$\mu(|\Phi(x)|) = \sum_{n=1}^{\infty} 2^n n^{-2} = \infty,$$

i.e. $\Phi(x)$ does not belong to $L^\omega(M, \nu)$.

The obtained contradiction implies that the set $\{e_n\}$ is at most countable.

2) $\rightarrow$ 1). Let $M = M_0 + \sum_{i=1}^{n} M_i$, where $M_0$ is finite-dimensional and $M_i$ is infinite dimensional commutative von Neumann algebra, the Boolean algebra $P(M_i)$ being homogeneous, $\tau_i < \tau_{i+1}$, $i = 1, \ldots, n-1$.

Take arbitrary faithful normal traces $\mu$ and $\nu$ on $M$. As dim $M_0 < \infty$, $L^\omega(M_0, \mu) = M_0 = L^\omega(M_0, \nu)$. According to the proposition 1 a $*$-isomorphism $\Phi_i$ from $L^\omega(M, \mu)$ on $L^\omega(M_i, \nu)$ exists. Each element $x$ from $L^\omega(M, \mu)$ is represented as $x = x_0 + \sum_{i=1}^{n} x_i$, where $x_0 \in M_0 = L^\omega(M_0, \mu)$, $x_i \in L^\omega(M_i, \mu)$, $i = 1, \ldots, n$. It is obvious that $\Phi(x) = x_0 + \sum_{i=1}^{n} \Phi_i(x_i)$ is a $*$-isomorphism from $L^\omega(M, \mu)$ on $L^\omega(M, \nu)$. The theorem is proved.

Using theorem 3, it is easy to construct an example of a non-atomic commutative von Neumann algebra $M$ with traces $\mu$ and $\nu$, such that the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are isomorphic, while there is no $*$-isomorphism $\varphi$ from $M$ on $M$, for which $\nu \circ \varphi = \mu$. Indeed, assume that
$M = M_1 + M_2$, where $M_1, M_2$ are non-atomic commutative $\sigma$-finite von Neumann algebras in which the lattice of projections form homogeneous Boolean algebras and $\tau(P(M_1)) < \tau(P(M_2))$. Identify $M_1$ with the subalgebra $e_1 M_1$ and $M_2$ with $(I - e_1)M_1$, $e_1 \in P(M)$. Let $\mu$ be an arbitrary faithful normal finite trace on $M$, $\mu(I) = 1$. Assume that

$$\nu(x) = p(\mu(e_1)^{-1}\mu(x e_1) + q(\mu(I - e_1))^{-1}\mu(x(I - e_1))),$$

$x \in M$, $p, q > 0$, $p + q = 1$. It is evident that $\nu$ is a faithful normal finite trace on $M$. Choose $p$ and $q$ such that $\mu(e_1) \neq \nu(e_1) = p$, $\mu(I - e_1) \neq \nu(I - e_1) = q$. According to the theorem 2, there is no *-isomorphism $\varphi : M \to M$ for which $\nu \circ \varphi = \mu$. At the same time, according to the theorem 3, the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are *-isomorphic.

Now, let us find out when the Arens algebras coincide for different traces. Let $\mu$ and $\nu$ be two faithful normal finite traces on a commutative von Neumann algebra $M$. Denote by $h = \frac{\partial \mu}{\partial \nu}$ the Radon-Nikodim derivate of the trace $\mu$ relative $\nu$, i.e. $h$ is the element from $L^1_+(M, \nu)$ for which $\mu(x) = \nu(hx)$ for all $x \in M$.

It is clear that the element $x$ from $K(M, \mu)$ belongs to $L^1(M, \mu)$ if and only if $hx \in L^1(M, \nu)$. In this case the equality $\mu(x) = \nu(hx)$ holds.

**Proposition 2.** $L^\omega(M, \nu) \subset L^\omega(M, \mu)$ if only if

$$h \in \bigcup_{1 < p \leq \infty} L^p(M, \nu),$$

where $L^\infty(M, \nu)$ is identified with $M$.

Proof. Let $L^\omega(M, \nu) \subset L^\omega(M, \mu) \subset L^1(M, \mu)$. Then $\mu(x) = \nu(hx)$ for all $x \in L^\omega(M, \nu)$, and $\mu$ is a positive linear functional on $L^\omega(M, \nu)$. Since $L^\omega(M, \nu)$ is a complete metrizable locally-convex algebra with respect to the $t$-topology generated by the system of norms $\left\{\|x\|_p = (\nu(|x|^p))^{1/p}\right\}_{p \geq 1}$ (see[3]) and involution in $L^\omega(M, \nu)$ is continuous in this topology, $\mu$ is continuous [14]. It was shown in [3] that the dual space of $(L^\omega(M, \nu)')$ may be identified with $\bigcup_{1 < p \leq \infty} L^p(M, \nu)$. Hence one can find such $y \in L^p(M, \nu)$ for some $p \in (1, \infty]$ that $\nu(hx) = \mu(x) = \nu(yx)$ for all $x \in L^\omega(M, \nu)$. It means that $h = y$ and $h \in \bigcup_{1 < p \leq \infty} L^p(M, \nu)$. 

Conversely, if $h \in L^p(M, \nu)$ for some $p \in (1, \infty]$, then $\nu(hx)$ is a $t$-continuous linear functional on $L^\omega(M, \nu)$ (See[3]) and therefore $\mu(|x|^q) = \nu(h|x|^q) < \infty$ for any $x \in L^p(M, \nu)$ and $q \geq 1$; we recall that $|x|^q \in L^\omega(M, \nu)$ for all $x \in L^\omega(M, \nu)$ and $q \geq 1$. Thus,

$$L^\omega(M, \nu) \subset \bigcap_{q \geq 1} L^q(M, \mu) = L^\omega(M, \mu)$$

The following criterion of coincidence of the algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ arises from the proposition 2.

**Theorem 4.** Let $\mu, \nu$ be faithful normal finite traces on a commutative von Neumann algebra $M$. Then $L^\omega(M, \mu) = L^\omega(M, \nu)$ if only if

$$\frac{d\mu}{d\nu} \in \bigcup_{1 < p \leq \infty} L^p(M, \nu) \text{ and } \frac{d\nu}{d\mu} \in \bigcup_{1 < p \leq \infty} L^p(M, \mu).$$

**Remarks.**
1. In the example constructed after theorem 3 $L^\omega(M, \mu) = L^\omega(M, \nu)$ since

$$\frac{d\mu}{d\nu} = \mu(e_1)p^{-1}e_1 + \mu(1 - e_1)q^{-1}(1 - e_1)).$$

Now everything is ready to obtain the criterion of *-isomorphism of the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$. Let $M$ be an arbitrary non-atomic commutative $\sigma$-finite von Neumann algebra. According to [6] the Boolean algebra $P(M)$ of projections $M$ possesses uniquely determined collection $\{e_r\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $X_n = \{e \in P(M) : e \leq e_n\}$ are homogeneous and $\tau(X_n) < \tau(X_{n+1})$. Assume that the collection $\{e_n\}$ is infinite otherwise all Arens algebras $L^\omega(M, \mu)$ are *-isomorphic (see theorem 3).

**Theorem 5.** Let $\mu$ and $\nu$ be faithful normal finite traces on a non-atomic commutative $\sigma$-finite von Neumann algebra $M$. The following conditions are equivalent:
1) The Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are *-isomorphic;
2) There are such $p, q \in (1, \infty]$ that

$$\sum_{n=1}^\infty \mu_n^p \nu_n^{1-p} < \infty, \quad \sum_{n=1}^\infty \nu_n^q \mu_n^{1-q} < \infty$$
in the case \( p \neq \infty, q \neq \infty, \) and \( \sup_{n \geq 1} |\mu_n \nu_n^{-1}| < \infty \) if \( p = \infty, \) \( \sup_{n \geq 1} |\nu_n \mu_n^{-1}| < \infty \) if \( q = \infty. \)

Proof. 1) \( \rightarrow \) 2). Let \( \Phi \) be a \(*\)-isomorphism from \( L^\omega(M, \mu) \) on \( L^\omega(M, \nu). \) Since all \( \tau(x_n) \) are different, \( \Phi(e_n \mu) = e_n \mu. \)

Denote by \( N \) the atomic von Neumann subalgebra of all elements \( x \) from \( M, \) for which \( xe_n = \lambda_n \) for some complex numbers \( \lambda_n, \) \( n = 1, \ldots. \) It is evident that \( N \) is identified with the algebra \( l_\infty \) of all bounded sequences of complex numbers. Since \( \Phi(e_n) = e_n, \) \( n = 1, 2, \ldots, \) it follows that \( \Phi(z) = z \) for all \( z \in N. \) If \( z \in L^\omega(N, \mu) \cap K(N, \mu) = L^\omega(N, \mu), \) \( z \geq 0, \) then \( z = \sup_{m \geq 1} z \sum_{n=1}^{m} e_n, \) and \( (z \sum_{n=1}^{m} e_n) \in N_+. \) Therefore,

\[
\Phi(z) = \sup_{m \geq 1} \Phi(z \sum_{n=1}^{m} e_n) = \sup_{m \geq 1} z \sum_{n=1}^{m} e_n = z.
\]

Thus the restriction of \( \Phi \) on \( L^\omega(N, \mu) \) coincides with the identity mapping. It means that \( L^\omega(N, \nu) = \Phi(L^\omega(N, \mu)) = L^\omega(N, \mu). \)

Therefore, according to the theorem 4 \( h \in \bigcup_{1 < p \leq \infty} L^p(N, \nu), \) and \( h^{-1} \in \bigcup_{1 < p \leq \infty} L^p(N, \mu), \) where \( h \) is the Radon-Nikodym's derivative of the trace \( \mu \) relative the trace \( \nu, \) being considered in \( N. \) So using the equality \( he_n = \mu_n \nu_n^{-1}e_n, \) \( n = 1, 2, \ldots, \) the required inequalities follow from the condition 2).

2) \( \rightarrow \) 1). Let the inequalities from the condition 2) hold. Consider the faithful normal finite trace on \( M \) given by the equality

\[
\lambda(x) = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} \mu(e_n x), \ x \in M.
\]

Since \( x_n \) is a homogeneous Boolean algebra and \( \lambda(e_n) = \nu_n = \nu(e_n), \) using the proof of proposition 1, construct a \(*\)-isomorphism \( \Phi_n : K(e_n M, \nu) \rightarrow K(e_n M, \lambda) \) for which \( \nu(y) = \lambda(\Phi_n(y)) \) for all \( y \in L^1(e_n M, \nu). \) For each \( x \in K(M, \nu) \) denote by \( \psi(\lambda) \) such an element from \( K(M, \lambda) \) for which \( e_n \psi(x) = \Phi_n(e_n x). \) It is evident that \( \psi \) is a \(*\)-isomorphism from \( K(M, \nu) \) on \( K(M, \lambda). \) At the same time, if \( x \in L^1_+(M, \nu), \) then

\[
\nu(x) = \sum_{n=1}^{\infty} \nu(e_n x) = \sum_{n=1}^{\infty} \lambda(\Phi_n(e_n x)) =
\]
\[
\sum_{n=1}^{\infty} \lambda(e_n \psi(x)) = \lambda(\psi(x)),
\]

therefore \(\psi(L^\omega(M, \nu)) = L^\omega(M, \lambda)\).

Let's show that \(L^\omega(M, \lambda) = L^\omega(M, \mu)\). Let \(h\) be such an element from \(K(M, \mu)\) that \(he_n = \mu_n \nu_n^{-1}e_n\). For every \(x \in M\) we have

\[
\lambda(hx) = \sum_{n=1}^{\infty} \lambda(he_n x) = \sum_{n=1}^{\infty} \mu_n \nu_n^{-1} \lambda(e_n x) =
\]

\[
= \sum_{n=1}^{\infty} \mu(e_n x) = \mu(x),
\]

therefore \(h = \frac{d\mu}{d\lambda}\). According to the inequalities from the condition 2, we obtain that

\[
\text{If } \sup_{n \geq 1} (\mu_n \nu_n^{-1}) < \infty, \text{ then } h \in M.
\]

Suppose that \(\sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty\) for some \(p \in (1, \infty)\). Then

\[
\lambda(h^p) = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} \mu(e_n h^p) = \sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty.
\]

Thus,

\[
h \in \bigcup_{1<p\leq \infty} L^p(M, \lambda)
\]

and, using the theorem 4, we get \(L^\omega(M, \lambda) = L^\omega(M, \mu)\).

Therefore \(\psi(L^\omega(M, \nu)) = L^\omega(M, \mu)\).

Remarks 2. Repeating the argument from the proof of the theorem 5, it is easy to obtain the following criterion of \(*\)-isomorphism of the Arens algebras \(L^\omega(l_\infty, \mu)\) and \(L^\omega(l_\infty, \nu)\):

Let \(\mu\) and \(\nu\) be faithful normal finite traces on a infinite dimensional atomic commutative von Neumann algebra \(N\), \(\{q_n\}_{n=1}^{\infty}\) – the set of all atoms in \(P(N)\), \(\mu_n = \mu(q_n)\), \(\nu_n = \nu(q_n)\), \(n = 1, 2, \ldots\). Then, the Arens algebras
and v) are *-isomorphic only in the case when there are such $p, q \in (1, \infty)$ and permutation $\pi$ of a set of natural numbers, that

$$\sum_{n=1}^{\infty} \mu_n^{p-1} p^{1-p} < \infty, \quad \sum_{n=1}^{\infty} \nu_n^{q} \mu_n^{1-q} < \infty,$$

in the case $p, q \in (1, \infty)$

and $\sup_{n \geq 1} |\mu_n \nu_n^{-1}| < \infty$ if $p = \infty$, $\sup_{n \geq 1} |\nu_n \mu_n^{-1}| < \infty$ if $q = \infty$.

3. Any von Neumann algebra $M$ is represented as $M = M_1 + M_2$, where $M$ is an atomic von Neumann algebra and $M_2$ is a non-atomic von Neumann algebra. Moreover, if $\Phi$ is a *-automorphism of $M$, then $\Phi(M_1) = M_1$ and $\Phi(M_2) = M_2$. Therefore theorem 5 and Remark 2 give criterion of isomorphism of Arens algebras for arbitrary commutative $\sigma$-finite von Neumann algebras

References