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Transformation of gaussian measures

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Transformation of Gaussian measures

Introduction

We shall be, in our lecture, mainly concerned by some particular cases of the following problem :

Let $(X, \mathcal{F}, \mu)$ be a measure space and $T : X \to X$ measurable. We denote by $T(\mu)$ or $\mu \circ T^{-1}$ the image of μ by T :

$$T(\mu)\ (A) = \mu \circ T^{-1}\ (A) = \mu\ (T^{-1}A), \quad \forall A \in \mathcal{F}.$$

When does $T(\mu) \ll \mu$ and how to compute the density?

Example 1 : Let $X = \mathbb{R}^n$, $\mu = \lambda_n$ (the Lebesgue measure) and $T : X \to X$ a diffeomorphism. Then from the formula

$$\int f(T(x))|\det T'(x)|dx = \int f(y)dy,$$

we conclude that $T(\lambda_n)$ is absolutely continuous with respect to λ_n and

$$T(\lambda_n)\ (dy) = |\det T'(T^{-1}y)|^{-1}dy = |\det\ (T^{-1})'(y)|dy.$$

Example 2 : Let $(\Omega, \mathcal{F}, P)$ be the classical Wiener space, $\Omega = \mathcal{C}_0([0,1]), \mathcal{F}$ the Borel σ-field, P the Wiener measure. Let $u : [0,1] \times \Omega \to \mathbb{R}$ be a measurable and ***adapted*** stochastic process such that $\int_0^1 u_t^2(\omega)dt < \infty$ almost surely, and let $T : \Omega \to \Omega$ be defined by :

$$(T\omega)_t = \omega_t + \int_0^t u_s(\omega)\ ds.$$

Girsanov has proven that

$$T(P) \ll P.$$

On the other hand, let

$$\xi = \exp\Big\{-\int_0^1 u_t d\omega_t - \frac{1}{2}\int_0^1 u_t^2(\omega)\ dt\Big\}$$

then, if $\mathbb{E}(\xi) = 1$. $(T\omega)_t$ is a Brownian motion with respect to $(\Omega, \mathcal{F}, Q)$, where $\dfrac{dQ}{dP} = \xi$. That is $Q \circ T^{-1} = P$.
(This fact was first proven by means of the Itô-calculus, but as we shall see, we can obtain this by analytic methods).

This has an application in Statistical Communication Theory :

Suppose we are receiving a signal corrupted by noise, and we wish to determine if there is indeed a signal or if we are just receiving noise.
If $x(t)$ is the received signal, $\xi(t)$ the noise and $s(t)$ the emitted signal :

$$x(t) = s(t) + \xi(t) \tag{A}$$

In general, we make an hypothesis on the noise : it is a ***white noise***.

The "integrated" version of (A) is

$$X(t) = \int_0^t s(u)\, du + W_t = S_t + W_t \tag{A'}$$

(W is the standard Wiener process, $X(t) = \displaystyle\int_0^t x(s)\, ds$ is the cumulative received signal).

Now we ask the question : is there a signal corrupted by noise, or is there just a noise ($s(t) = 0, \quad \forall t$)?

The hypotheses are :

$$H_0 : X_t = W_t$$

$$H_1 : X_t = \int_0^t s(u)\, du + W_t.$$

We consider the likelihood ratio

$$\frac{d\mu_W}{d\mu_X} = \exp\Big(-\int_0^1 s(t)\, dW_t - \frac{1}{2}\int_0^1 s(t)^2\, dt\Big)$$

and we fix a threshold level for the type 1-error :

if : $\dfrac{d\mu_W}{d\mu_X}(\omega) \leq \lambda$ we reject (H_0)

if : $\dfrac{d\mu_W}{d\mu_X}(\omega) \geq \lambda$ we accept (H_0).

Some general considerations and examples.

$$\text{If } P \ll Q, \quad \text{then} \quad T(P) \ll T(Q). \tag{a}$$

Therefore, we do not lose very much if we suppose that P and Q are probabilities.

In the case where Q is a probability, we can have an expression of $\dfrac{dT(P)}{dT(Q)}$ as conditional mathematical expectation.

Remark : From (a) we see that, if there exists a probability Q such that

$$P \ll Q \quad \text{and} \quad T(Q) = P, \quad \text{then} \quad T(P) \ll P.$$

The converse is true if moreover $\dfrac{dT(P)}{dP} > 0$. (The measures are equivalent). Therefore the following properties are equivalent :

(i) : $T(P) \sim P$,
(ii) : $\exists Q \sim P$ such that $T(Q) = P$.

Let us now consider an example which allows us to guess the situation in infinite dimensional space.

Let $\Omega = \mathbb{R}^n$ and $P = \gamma_n$ the canonical Gaussian measure with density :

$$\frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\|x\|^2}{2}\right)$$

and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a diffeomorphism, then

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(y) T(\gamma_n)(dy) = \int_{\mathbb{R}^n} f(Tx)\, \gamma_n(dx) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx)\, \exp\left(-\frac{\|x\|^2}{2}\right) dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx)\, \exp\left(-\frac{1}{2}\, \|T^{-1}Tx\|^2\right) dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{f(y)}{|\det T'(T^{-1}y)|} \exp\left(\frac{1}{2}\, \|y\|^2 - \frac{1}{2}\, \|T^{-1}y\|^2\right) \exp\left(-\frac{1}{2}\|y\|^2\right) dy.
\end{aligned}$$

Therefore :

$$\frac{dT(\gamma_n)}{d\gamma_n}(y) = \frac{1}{|\det T'(T^{-1}y)|}\exp\left(\frac{1}{2}\|y\|^2 - \frac{1}{2}\|T^{-1}y\|^2\right)$$
$$= |\det (T^{-1})'(y)| \exp\left(\frac{1}{2}\|y\|^2 - \frac{1}{2}\|T^{-1}y\|^2\right).$$

Now if we write :

$$T^{-1} = (I+S) \text{ with } S \text{ self adjoint,}$$

then :

$$(T^{-1})'(y) = I + S'(y)$$

and we obtain :

$$\frac{d(I+S)^{-1}(\gamma_n)}{d\gamma_n}(y) = |\det (I+S'(y))| \exp\{-(Sy,y)_{\mathbb{R}^n} - \frac{1}{2}\|S(y)\|^2\}. \qquad (B)$$

This can be written as :

$$|\det (I+S'(y)| \exp(-\text{Trace } S'(y)) \exp\left\{-(Sy,y)_{\mathbb{R}^n} + \text{Trace } S'(y) - \frac{1}{2}\|S(y)\|^2\right\},$$

where $|\det (I+S'(y)| \exp(-\text{Trace } S'(y))$ is the Carleman determinant.

General remark : If $T = Id(\Omega)$, it is clear that $TP = P$ for every P. The idea is to perturb the identity operator.

The problem is :

"what does the word *perturbation* mean ?"

CHAPTER ONE

Anticipative stochastic integral

1 - Gaussian measures on Banach spaces

Let E be a (real) separable Banach space, E' its dual. A (Borelian) probability μ on E is said to be ***"Gaussian centered"*** if for every $x' \in E'$, $\langle \bullet, x' \rangle_{E,E'} = x'(\bullet)$ is a Gaussian centered (real) variable (eventually degenerated) under μ. All what we shall say is true whatever be the dimension of E (finite or infinite).

If $x' \in E'$ we define $A : E' \to E$ by

$$Ax' = \int_E \langle x, x' \rangle_{E,E'} \; x \; d\mu(x),$$

(Bochner integral of a vector function). It is the ***barycenter*** of the measure $\langle \bullet, x' \rangle d\mu$.

A is injective if $\operatorname{Supp} \mu = E$.

Let $x \in A(E')$ so $x = A(u')$ and let $y \in A(E')$ so $y = A(v')$, we shall put on $A(E') \subset E$ the following scalar product :

$$(x, y) \rightsquigarrow (x, y)_\mu := \int_E \langle u', z \rangle \; \langle v', z \rangle \; d\mu(z)$$

(it does not depend on u' and v').

$A : E' \to E$ is continuous. (Since $\displaystyle\int_E \|x\|^2 d\mu(x) < \infty$ by Fernique's theorem).

Therefore, if i denotes the canonical injection of $A(E')$ into E :

$$i \; : \; (A(E')), \|\bullet\|_\mu) \to (E, \|\bullet\|) \quad \text{is continuous.}$$

Actually :

$$\begin{aligned}
\|Ax'\|_E &= \sup_{\|y'\| \le 1} \left| \int_E \langle x', x \rangle \; \langle y', x \rangle \; d\mu(x) \right| \\
&\le \sup_{\|y'\| \le 1} \left(\int_E |\langle x', x \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_E |\langle y', x \rangle|^2 \; d\mu(x) \right)^{\frac{1}{2}} \\
&\le \left(\int |\langle x', x \rangle|^2 \; d\mu(x) \right)^{\frac{1}{2}} \left(\int \|x\|^2 \; d\mu(x) g \right)^{\frac{1}{2}};
\end{aligned}$$

hence,

$$\|Ax'\|_E \leq C \ \|Ax'\|_\mu \quad \text{(where } C \text{ is a constant).}$$

Let H_μ be the completion of $A(E')$ with respect to $\|\bullet\|_\mu$. We have $\hat{\imath} : H_\mu \to E$. I say that $\hat{\imath}$ is injective (it will allow us to consider H_μ as a subspace of E).

H_μ is called the ***"reproducing kernel Hilbert space"*** (r.k.H.s.) of μ.

Example 1 : Finite dimension

$E = \mathbb{R}^n, \quad \operatorname{Supp} \mu = \mathbb{R}^n$:

$$Ax' = \int_E \langle x\,,\, x'\rangle \; x d\mu(x),$$

or :

$$\langle Ax'\,,\, y'\rangle = \int_E \langle x\,,\, x'\rangle \; \langle x\,,\, y'\rangle \; d\mu(x).$$

A is the covariance, it is invertible and

$$(x,y)_\mu = \int_E \langle A^{-1}x, z\rangle \; \langle A^{-1}y, z\rangle \; d\mu(z) = \langle x, A^{-1}y\rangle,$$

and therefore :

$$H_\mu = \mathbb{R}^n\,.$$

Example 2 : Brownian motion, Wiener space.

Let $T > 0$ and $\Omega = E = \mathcal{C}([0,T],\mathbb{R})$ be the space of real continuous functions on $[0,T]$.

There exists an unique centered measure μ such that :

a) the support of μ is $\mathcal{C}_0([0,T],\mathbb{R})$, the space of the continuous functions vanishing at 0,
b) $\forall t \in [0,T]$: $\quad \omega \rightsquigarrow \omega_t$ has the variance t,
c) let $0 \leq t_1 < t_2 < \ldots < t_n \leq T$, then : $\omega_{t_1},\ \omega_{t_2} - \omega_{t_1},\ \ldots,\ \omega_{t_n} - \omega_{t_{n-1}}$ are independent.

We shall call μ the Wiener measure on $\mathcal{C}([0,T],\mathbb{R})$; then E' is the space of signed measures ν on $[0,T]$. We shall also denote :

$$\omega_t = B(t,\omega)$$

and call $t \rightsquigarrow B(t,\bullet)$: the ***"Brownian motion"*** on $[0,T]$.

For $\nu_1, \nu_2 \in E'$ let :

$$B(\nu_1, \nu_2) = E\ [\langle \nu_1\,,\, B\rangle\ \langle \nu_2\,,\, B\rangle]$$
$$= \int_\Omega \langle \nu_1, \omega\rangle\ \langle \nu_2, \omega\rangle\ d\mu(\omega)\,.$$

We have for $\nu \in E'$

$$\langle \nu, B\rangle = \int_{[0,T]} B(t,\omega)\ d\nu(t) = \int_0^T \nu\ ([t,T])dB(t)\ \text{(stochastic integral)}.$$

This fact can be verified as follows :

- it is true for $\nu = \delta_s$ (by definition of Brownian motion) ,
- by linearity this remains true if $\nu = \sum\ \alpha_i \delta_{t_i}$,
- then we apply a continuity argument.

Therefore

$$B(\nu_1, \nu_2) = \int_{[0,T]} \nu_1([t,T])\ \nu_2([t,T])dt\,.$$

Now let ν_1 be a measure on $[0,T]$. We want to find the barycenter m_{ν_1} of the random variable on Ω : $\omega \rightsquigarrow \langle \omega, \nu_1\rangle$. ($m_{\nu_1}$ is an element of $\Omega = \mathcal{C}([0,T])$. It is defined by

$$\nu \rightsquigarrow \langle m_{\nu_1}, \nu\rangle = \int_{[0,T]} m_{\nu_1}(t)\ \nu(dt) = B(\nu,\nu_1) = \int_{[0,T]} \nu_1([t,T])\ \nu([t,T])\ dt\,.$$

By the generalized integration by parts this is equal to :

$$\int_{[0,T]} J(\nu_1)(t)\ d\nu(t)$$

where

$$J(\nu_1)(t) = \int_0^t \nu_1([u,T])\ du\,.$$

$J(\nu_1)$ is then absolutely continuous. On the space

$$\Big\{J(\nu_1),\ \nu_1 \in \mathcal{M}([0\,,\,T])\Big\}$$

we put the norm

$$J(\nu_1) \rightsquigarrow \int_0^T \nu_1([t,T])^2 dt.$$

Its completion is the space of functions from $[0,T]$ into $\mathbb{R}$ absolutely continuous, null at zero, whose derivative belongs to $L^2([0,T],dt)$. It is the Cameron-Martin space.

Then the Cameron-Martin space is the reproducing kernel Hilbert space of the Wiener measure μ.

Definition : *We call an* ***"abstract Wiener space"*** *a triple (H,E,μ) where :*

- *E is a separable Banach space, and μ is a centered Gaussian measure on E, whose topological support is E.*
- *H is the r.k.H.s. associated to μ.*

Actually H ***is dense in*** E. This can be proven as follows :

Let $i : H \longrightarrow E$ be the canonical injection and $i^* : E' \rightarrow H$ its transpose (we identify H to its dual).
Suppose that $\langle x', i(x)\rangle_{E,E'} = 0$ for every $x \in H$. This is equivalent in saying that :

$$(x \mid i^*(x'))_H = 0\,, \text{ for every } x \in H\,.$$

Therefore

$$i^*(x') = 0\,.$$

This means that

$$\|i^*(x')\|_H^2 = \int_E |\langle x', y\rangle_{E,E'}|^2 \; d\mu(y) = 0.$$

Therefore

$$\langle x', y\rangle = 0 \quad \text{almost surely,}$$

so this holds for all y $\in E$ since $\operatorname{Supp}\mu = E$ and x' is continuous.
The transpose i^* from $i : H \rightarrow E$ is therefore injective and dense and we have :

$$E' \xrightarrow{i^*} H \xrightarrow{i} E \quad (\, i \text{ is the canonical injection})\,.$$

Every $x' \in E'$, defines a Gaussian centered random variable on E', whose variance is

$$\|i^*(x')\|_H^2\,.$$

Now we give without proof some properties of an abstract Wiener space :

1) H is separable, as a Hilbert space. Therefore it is a borelian subset of E,
2) $\mu(H) = 0$ or 1 and $\mu(H) = 0 \Leftrightarrow dim H = +\infty$ (therefore $\mu(H) = 1 \Leftrightarrow dim H < \infty$),
3) H is the intersection of the family of measurable subspaces of E, whose probability is equal to one,
4) the canonical injection $i : H \longrightarrow E$ is compact,
5) for every Hilbert space K and $u : E \longrightarrow K$ linear continuous, $u \circ i : H \longrightarrow K$ is Hilbert-Schmidt,
6) for every Hilbert space K and $v : K \longrightarrow E'$ linear continuous, $i^* \circ v : K \longrightarrow H$ is Hilbert-Schmidt.

As a consequence of 5) and 6) we have :

7) let K_1, K_2 two Hilbert spaces ; $u_1 : K_1 \longrightarrow E'$ and $u_2 : E \longrightarrow K_2$ linear continuous then

$$K_1 \xrightarrow{u_1} E' \xrightarrow{i^*} H \xrightarrow{i} E \xrightarrow{u_2} K_2 ,$$

the composition $u_2 \circ i \circ i^* \circ u_1$ is nuclear (i.e. it possesses a trace).

2 - L^2-functionals on an abstract Wiener space

Let (H, E, μ) be an abstract Wiener space.

Suppose $(e_j)_{j\geq 1}$ is a sequence of elements of E' such that $\big(i^*(e_j)\big)_{j\geq 1}$ is an orthonormal basis in H. A function $f : E \longrightarrow \mathbb{R}$ is said to be a polynomial in the (e_j) if there exists an integer n and a polynomial function P on $\mathbb{R}^n$ such that

$$f(x) = P\big(e_1\ (x), ..., e_n(x)\big), \qquad \forall x \in E.$$

We denote deg $f :\equiv$ deg P (P is not defined uniquely but the degree of f is independent of the choice of P).

We denote by $\mathcal{P}((e_j))$ the set of polynomials and by $\mathcal{P}^n((e_j))$ the set of polynomials of degree $\leq n$. It is easy to see that $\mathcal{P}((e_j))$ is contained in each $\mathcal{L}^p(E, \mu)$ $0 \leq p < \infty$ (but clearly not in $L^\infty(E, \mu)$). Moreover, $\mathcal{P}((e_j))$ is dense in $L^p(E, \mu)$ for these p. Therefore, $\overline{\mathcal{P}((e_j))}_{L^p}$ is independent of the chosen orthonormal family (e_j). The same is true for each $\mathcal{P}^n((e_j))$.

Example : If $n = 1$, $\mathcal{P}^1((e_j))$ is the family of affine continuous functions : an element of $\mathcal{P}^1((e_j))$ is a linear continuous function on E plus a constant.

We have :

$$\overline{\mathcal{P}^1}_{L^2(E,\mu)} \equiv H \oplus \mathbb{R} \quad \text{(see infra).}$$

We call $\overline{\mathcal{P}^n}_{L^2}$ the set of ***generalized*** polynomials of degree at most n ; $\overline{\mathcal{P}^n}_{L^2}$ is a Hilbert space.

Let us now introduce the ***"Wiener chaos decomposition"*** (or ***"Wiener-Itô decomposition"***). Let $\mathcal{C}_0 = \overline{\mathcal{P}^0}_{L^2}$ the vector space of (μ-equivalence classes of) constants. We define $\mathcal{C}_n$ inductively as follows :

$\mathcal{C}_n$ is the orthogonal complement in $\overline{\mathcal{P}^n}_{L^2}$ of $(\mathcal{C}_0 \oplus ... \oplus \mathcal{C}_{n-1})$.
(In other words $\mathcal{C}_n$ is the set of generalized polynomials of degree n, orthogonal to all generalized polynomials of degree less than n).
It is clear that for every n :

$$\overline{\mathcal{P}^n}_{L^2} = \mathcal{C}_0 \oplus ... \oplus \mathcal{C}_n$$

and moreover

$$L^2(E,\mu) = \bigoplus_{n=0}^{\infty} \mathcal{C}_n \, .$$

The $\mathcal{C}_n$ are called the ***"nth chaos"*** (or ***"chaos of order n"***). $\mathcal{C}_1$ is isomorphic to H. We have a description of elements of $\mathcal{C}_n$ in term of Hermite polynomials.

We recall that the Hermite polynomials in one variable are defined by :

$$H_n(t) = \frac{(-1)^n}{n!} \exp\{\frac{t^2}{2}\} \frac{d^n}{dt^n} \left(\exp\{-\frac{t^2}{2}\}\right), \qquad n \in \mathbb{N}.$$

Then they satisfy :

- $\displaystyle\sum_{n=0}^{\infty} \lambda^n \, H_n(t) = \exp\{-\frac{\lambda^2}{2} + \lambda t\}$
- $\displaystyle\frac{d}{dt} \, H_n(t) = H_{n-1}(t)$
- $\displaystyle\int_{\mathbb{R}} H_m(t) \, H_n(t) \, \frac{1}{\sqrt{2\pi}} \, \exp\{-\frac{t^2}{2}\} \, dt = \frac{1}{n!} \, \delta_{nm}.$

Let $\alpha = (\alpha_1, \alpha_2, ...,) \in \mathbb{N}^{\mathbb{N}}$ such that $|\alpha| := \sum_{i=1}^{\infty} \alpha_i < \infty$. We set $\alpha! := \prod_{i=1}^{\infty} \alpha_i!$.

Now let $(e_n)_{n\geq 1}$ be a sequence of elements of E' which is an orthonormal basis in H. If $\alpha \in \mathbb{N}^{\mathbb{N}}$ let

$$H_\alpha(x) := \prod_{i=1}^{\infty} H_{\alpha_i}\ (e_i(x))$$

(This product is well defined). Then :

$$\{\sqrt{\alpha!}\ H_\alpha(x), \quad \alpha \in \mathbb{N}^{\mathbb{N}} \text{ and } |\alpha| < +\infty\} \text{ is an orthonormal basis in } L^2(E,\mu) \text{ and :}$$
$$\{\sqrt{\alpha!}\ H_\alpha(x), \quad |\alpha| = n\} \quad \text{is an orthonormal basis in } \mathcal{C}_n.$$

In the case of the Wiener measure associated to Brownian motion, we have the following characterization of $\mathcal{C}_n$ in terms of multiple stochastic integrals :

$F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ belongs to $L^2(P)$ where P is the Wiener measure if and only if for each n there exists $f_n \in L^2(\Delta_n, dt)$ where $\Delta_n = \{t \in \mathbb{R}^n, \quad 0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq T\}$ such that

$$F = \sum_n \int_{\Delta_n} f_n(t_1,...,t_n)\ dB\ (t_1)...dB\ (t_n) = \sum_n F_n\,.$$

Here

$$F_0 = \mathbb{E}(F) \in \mathcal{C}_0 \text{ and } F_n \in \mathcal{C}_n\,.$$

3 - Measurable linear functionals and linear measurable operators

Let (H,E,μ) be an abstract Wiener space. Without loss of generality, we shall identify H as a subspace of E (i.e., $i(x) = x$).

A linear mapping $f : E \to \mathbb{R}$ is said to be a ***"linear measurable functional"*** if there exists a sequence of linear continuous functionals on E, converging to f, μ-almost surely.

If $x \in H$, it defines a linear measurable functional $\widetilde{x}(\bullet)$. Actually, if x_n is a sequence of elements of $E' \subset H$ such that $x_n \longrightarrow x$ in H, then $x_n(\bullet)$ converges to the random variable $\widetilde{x}$ defined by x, in $L^2(E,\mu)$. Therefore,there exists a subsequence converging almost surely to $\widetilde{x}$. Moreover,

$$\int_E |\widetilde{x}(x)|^2\ d\mu(x) < \infty\,.$$

The converse is true, shown by the following proposition .

If $h \in H$, the random variable $\widetilde{h}$ on E will be denoted by

$$x \rightsquigarrow (x,h)_H .$$

Proposition : *Every linear measurable functional, f, has a restriction to H which is continuous (for the Hilbertian topology). If we denote by f_0 this restriction we have*

$$\|f\|_{L^2(E,\mu)} = \|f_0\|_H .$$

The converse is true.

Proof :

We have already noticed that the converse is true. Let $(x_n) \subset E' \subset H$ such that

$$x_n(x) \longrightarrow f(x) \quad \forall x \in A, \text{ where } \mu(A) = 1.$$

Take $\mathcal{E}$ the linear subspace generated by A, we see that the above convergence holds for all $x \in \mathcal{E}$. Since $\mu(\mathcal{E}) = 1$, then $H \subset \mathcal{E}$ and therefore

$$x_n(x) \longrightarrow f(x), \qquad \forall x \in H.$$

Therefore the restriction of f to H is uniquely defined.
Now,

$$\int_E \exp\,\{i(x_n - x_m)(x)\}\ \mu(dx) = \exp\,\{-\frac{1}{2}\|x_n - x_m\|_H^2\} \longrightarrow 1.$$

Therefore, (x_n) converges in H, and

$$\int_E |x_n(x) - x_m(x)|^2\ \mu(dx) = \|x_n - x_m\|_H^2 \quad \xrightarrow[m,n\to\infty]{} 0\,.$$

Therefore $\big(x_n(\bullet)\big)$ converges in $L^2(\mu)$. The limit is equal to f almost surely, as we can see immediately.

— Q.E.D.—

Now let K be a Hilbert space. As before we define a linear measurable function from E to K, as the almost sure limit of a sequence of linear continuous functions from E to K.

And, as before, if A is a linear measurable function from E into K, its restriction to H is well defined and continuous from H to K.

Let us remark that if A is a linear measurable function from E to K, we can define its transpose as a linear function from K to H since, for every $\varphi \in K$, $x \rightsquigarrow \langle Ax, \varphi\rangle_K$ is a linear measurable functional on E therefore defined by an element of H. We have

$$\begin{aligned}\langle Ax\,,\,\varphi\rangle_K &= (\widetilde{A^*\varphi})(x), \qquad \text{almost surely}\\ &= (x, A^*\varphi)_H\end{aligned}$$

where A^* is the conjugate of the restriction of A to H.

Now we can prove the following result :

THEOREM : *If A is a linear measurable function from E to K such that $\int \|Ax\|_K^2 \, d\mu(x) < \infty$, then its restriction to H is a Hilbert-Schmidt mapping B from H to K. Conversely if B is a Hilbert-Schmidt mapping from H to K, (we shall note $B \in \mathcal{L}^2(H,K)$ or $B \in \mathcal{L}_2(H,K)$), it possesses a linear measurable continuation on E, denoted by A.*

Moreover, we have :

$$\int_E \|Ax\|_K^2 \, d\mu(x) = \|B\|_{H\text{-}S}^2 .$$

Proof :

Let (φ_j) be an orthonormal basis of K.
We have :

$$\|Ax\|_K^2 = \sum_j (Ax\,,\,\varphi_j)_K^2 \overset{a.s}{=} \sum_j (x\,,\,A^*\varphi_j)_H^2.$$

If we integrate term by term these equalities, we obtain :

$$\begin{aligned}\int_E \|Ax\|_K^2 \, d\mu(x) &= \sum_j \int_E (x, A^*\varphi_j)_H^2 \, d\mu(x) \\ &= \sum_j \|A^*\varphi_j\|_H^2 = \sum_j \|B^*\varphi_j\|_H^2 = \|B^*\|_{H\text{-}S}^2 .\end{aligned}$$

Conversely let $B \in \mathcal{L}_2(H,K)$. We have for $x \in H$:

$$\begin{aligned}Bx &= \sum_j (Bx, \varphi_j)_K \, \varphi_j \\ &= \sum_j (x, B^*\varphi_j)_H \, \varphi_j .\end{aligned}$$

Now each term in the right-hand member possesses a linear measurable continuation to E, and the series converges in $\mathcal{L}_2(E, \mu, K)$.

We have then defined a linear measurable extension of A to E.

— *Q.E.D.*—

4 - Derivatives of functionals on a Wiener space

Let (E, H, μ) be an abstract Wiener space and let K be another Hilbert space. Let $f : E \to K$ be a function.

We say that f possesses a Fréchet derivative in the direction of H, at the point $x_0 \in E$ if there exists an element denoted $f'(x_0)$ or $Df(x_0)$ or $\nabla f(x_0) \in \mathcal{L}(H, K)$ such that $f(x_0 + h) - f(x_0) = f'(x_0) \bullet h + o(\|h\|_H)$, $\forall h \in H$.

Inductively we can define derivatives of all orders.

Example : Let $x_0 \in H \setminus i^*(E')$ and let f be a measurable continuation of $h \rightsquigarrow (x_0, h)_H$ to E. (f is not continuous).

Then f is derivable at every x, and $f'(x_0) \in H$.

This example shows that a discontinuous function may have Fréchet derivatives in the direction of H.

Definition 1 : *Let us denote by $\mathcal{C}^{2,1}(E, K)$ the set of functions $f : E \to K$ possessing the following properties :*

- f possesses H-derivatives at every point $x \in E$ and $f'(x)$ is Hilbert-Schmidt for every x,

- f and f' are continuous from H to K and to $\mathcal{L}_2(H, K)$ respectively,

- $|||f|||_{2,1}^2 := \int_E \left[\|f(x)\|_K^2 + \|f'(x)\|_{\mathcal{L}^2(H,K)}^2\right] \mu(dx) < \infty$.

Then $\mathcal{C}^{2,1}(E, K)$ is a vector space and $|||\bullet|||_{2,1}$ is a Hilbertian norm on this space.

Definition 2 : *Let $\mathbb{D}^{2,1}(E, K)$ be the completion of $\mathcal{C}^{2,1}(E, K)$ for the preceding norm; $\mathbb{D}^{2,1}(E, K)$ is then a Hilbert space.*

Clearly the elements of $\mathbb{D}^{2,1}(E, K)$ are μ-equivalence classes of functions.

Convention : Often we shall write $\mathbb{D}^{2,1}(K)$ instead of $\mathbb{D}^{2,1}(E, K)$. In the same manner we shall write $\mathbb{D}^{2,1}$ instead of $\mathbb{D}^{2,1}(E, \mathbb{R})$ or $\mathbb{D}^{2,1}(\mathbb{R})$.

Now the map $f \rightsquigarrow f'$ from $\mathcal{C}^{2,1}(E, K)$ into $L^2(E, \mu, \mathcal{L}_2(H, K))$ is clearly continuous ; therefore it possesses a unique continuous extension from $\mathbb{D}^{2,1}(H, K)$ into $L^2(E, \mu, \mathcal{L}_2(H, K))$. This extension is again denoted by f', or Df, or ∇f.

Example 1 : Let f be a polynomial function on E, with values in $\mathbb{R}$:

$$f(x) = P(\langle f_1, x\rangle_{E',E}, ..., \langle f_n, x\rangle_{E',E}), \quad f_1, ..., f_n \in E'.$$

Then $f \in \mathcal{C}^{2,1}$ and

$$f'(x) = \sum_{j=1}^{n} \frac{\partial P}{\partial y_j} \left(\langle f_1, x\rangle_{E',E}, \dots, \langle f_n, x\rangle_{E',E} \right) \, i^*(f_j).$$

The same result is true if P is a $\mathcal{C}^1(\mathbb{R}^n)$-function such that P and the partial derivatives $\dfrac{\partial P}{\partial y_j}$ have polynomial growth.

In the same manner if f is defined (μ-almost everywhere) as

$$f(\bullet) = P\left(\widetilde{h}_1(\bullet), \; \dots, \widetilde{h}_n(\bullet)\right), \quad h_j \in H$$

with P a polynomial function (or a $\mathcal{C}^1(\mathbb{R}^n)$-function with polynomial growth together with its derivatives),

$$\nabla f = \sum_{j=1}^{n} \frac{\partial P}{\partial y_j} \left(\widetilde{h}_1(\bullet), \; \dots, \widetilde{h}_n(\bullet)\right) \, h_j.$$

Example 2 : Let $\mu = \gamma_n$ the canonical Gaussian measure on $\mathbb{R}^n$, $\mathbb{D}^{2,1}$ is the Sobolev space $W^{2,1}(\gamma_n)$ of the distributions in $\mathbb{R}^n$ such that :

- $f \in L^2(\mathbb{R}^n, \gamma_n)$,
- the distribution derivatives of f belong to $L^2(\mathbb{R}^n, \gamma_n)$. The norm of $\mathbb{D}^{2,1}$ is the usual Hilbertian norm :

$$f \rightsquigarrow \left(\int_{\mathbb{R}^n} \left[|f(x)|^2 + \sum_{j=1}^{n} \left| \frac{\partial f}{\partial x_j}(x) \right|^2 \right] d\gamma_n(x) \right)^{\frac{1}{2}}.$$

Example 3 : If f is a polynomial function with values in K :

$$f(x) = \sum_{j=1}^{m} P_j\left(\langle f_1, x\rangle_{E',E}, \dots, \langle f_n, x\rangle_{E',E} \right) k_j$$

$$(k_j \in K, \quad f_1, \dots, f_n \in E').$$

$$\nabla f(x) = \sum_{j} \sum_{i} \frac{\partial P_j}{\partial y_i} \left(\langle f_1, x\rangle_{E',E}, \dots, \langle f_n, x\rangle_{E',E} \right) \, f_i \otimes k_j.$$

(Analogous assertion for generalized polynomials, or "moderate" regular functions P_j).

Example 4 : Characterization of the elements of $\mathbb{D}^{2,1}$ in the case of the Wiener measure.

If $E = C_0([0,T],\mathbb{R})$ and μ is the Wiener measure, we have seen that an element of $L^2(\mu)$ can be written as a series

$$F = \sum_{n=0}^{\infty} \sqrt{n}! \int_{\Delta_n} f_n(t_1, t_2, ..., t_n)\; dB_{t_1}, ..., dB_{t_n}$$

with

$$\sum_{n=0}^{\infty} n!\; \|f_n\|^2_{L^2(\Delta_n)} < \infty.$$

Then F belongs to $\mathbb{D}^{2,1}$ if and only if

$$\sum_{n=1}^{\infty} nn!\; \|f_n\|^2_{L^2(\Delta_n)} < \infty$$

and in this case

$$\nabla F = \sum_{n=1}^{\infty} nJ\big(I_{n-1}(f_n^t)\big)$$

where f_n^t is the function defined on $\Delta_{n-1}^t = \{0 \le t_1 < t_2 < ... < t_{n-1} < t\}$ by

$$f_n^t(t_1, t_2, ..., t_{n-1}) = f_n^{SYM}(t_1, t_2, ..., t_{n-1}, t)\,,$$

f_n^{SYM} being the symetrisation of f_n .

The formula needs an explanation :

In the right member

$$(t,\omega) \rightsquigarrow I_{n-1}\; (f_n^t)(\omega) = g(t,\omega)$$

belongs to

$$L^2\big([0,T] \times \Omega, dt \otimes dP\big)\,,$$

therefore for almost ω,

$$t \rightsquigarrow g(t,\omega) \text{ is a } L^2([0,T], dt) \text{ function .}$$

$J\big(I_{n-1}(f_n^t)\big)(\omega)$ is the indefinite integral null at zero of $I_{n-1}(f_n^t)(\omega)$:

$$J\big(I_{n-1}(f_n^t)\big) = \int_0^t I_{n-1}(f_n^s)\; ds\,.$$

Therefore $\nabla F(\omega)$ is an element of the Cameron-Martin space.

We now give several useful properties of $\mathbb{D}^{2,1}(E, K)$:

- The set of polynomial functions on E, with values in K is dense in $\mathbb{D}^{2,1}(K)$.
- Therefore the algebraic sum of chaos $\sum C_n$ is dense in $\mathbb{D}^{2,1}$.
- The set of ***smooth functions*** on E is dense in $C^{2,1}$ (a function is said to be ***"smooth"*** if it has the form :

$$x \rightsquigarrow f(\langle f_1, x\rangle_{E',E}, ..., \langle f_n, x\rangle_{E',E})$$

with f belonging to $C_b^\infty(\mathbb{R}^n)$; f and its derivatives are bounded).

- Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function in $C_b^1(\mathbb{R}^n)$ and let $F^1, ..., F^n \in \mathbb{D}^{2,1}$. Then $\varphi(F^1, ..., F^n)$ is in $\mathbb{D}^{2,1}$ and

$$\nabla\big(\varphi(F^1, ..., F^n)\big) = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial y_i}(F^1, ..., F^n)\ \nabla F^i.$$

This result is false if the above hypothesis is not satisfied. For instance on $\mathbb{R}$,

$$f = g = e^x \in \mathbb{D}^{2,1}.\ \text{but}\ f \circ g \notin L^2(\mathbb{R}^n\,,\,\gamma_n)\,.$$

Remark : The operator ∇, called the ***"stochastic"*** gradient, or ***"stochastic"*** derivative, is very close to the ordinary gradient as we can see. The usual gradient at the point x_0 is an element of E' (if the function takes its values in $\mathbb{R}$). The stochastic gradient is the composite of the ordinary gradient by the application i^* from E' to H.

In an analogous manner if $f : E \to K$ has an ordinary gradient, this gradient is a linear mapping of E into K ; $f' : E \to K$.

The transpose ${}^t f'$ is a linear continuous mapping from K into E'. Then the stochastic gradient is equal to $i^*({}^t f') \in \mathcal{L}(K, H)$.

In his lectures at the EIPES in 1989, D. Nualart, in the case of usual Wiener space defined the stochastic derivative of the functional of the form :

$$F = f(W_{t_1}, ..., W_{t_n}), \qquad f \in C_b^\infty(\mathbb{R}^n) \quad (\text{or } f \text{ polynomial})$$

by

$$DF = \sum_{j=1}^{n} \frac{\partial F}{\partial y_j}\ (W_{t_1}, ..., W_{t_n})\ 1_{[0,t_j]}.$$

This definition is actually equivalent to ours, up to the notations.

Actually, let $h_j(t) = \int_0^t 1_{[0,t_j]}(s)\, ds$, h_j belongs to the Cameron-Martin space and

$$W_{t_j} = \widetilde{h}_j = \langle h_j, \bullet \rangle_{C\text{-}M}$$

The stochastic derivate of F in our notations is therefore

$$\sum_{j=1}^{n} \frac{\partial F}{\partial y_j} (\widetilde{h}_1, ..., \widetilde{h}_n)\, h_j \,.$$

There are actually equivalent since the Cameron-Martin space is isomorphic as Hilbert space to $L^2([0,T], dt)$. We shall have to consider ∇ as an operator (densely defined) from $L^2(E,\mu,K)$ into $L^2(E,\mu,\mathcal{L}_2(H,K))$. It is a closed operator, naturally not continuous.

5 - Anticipative stochastic integral

Definition : *The transpose of the operator ∇ is called the **"Skorokhod integral"**, or the **"divergence operator"**.*

The definition needs an explanation : on $L^2(E,\mu,K)$ (K : Hilbert space) we have defined the scalar product

$$(f,g) \rightsquigarrow \int_E \langle f(x), g(x) \rangle_K \, d\mu(x)$$

and on $L^2(E,\mu,\mathcal{L}_2(H,K))$ we have the pairing :

$$\begin{aligned}(F,G) \rightsquigarrow &\int_E \langle F(x), G(x) \rangle_{\mathcal{L}_2(H,K)} d\mu(x) \\ = &\int_E \text{Trace}\, \big(G^*(x) \circ F(x)\big) d\mu(x)\,.\end{aligned}$$

Then $G \in L^2(E,\mu,\mathcal{L}_2(H,K))$ belongs to dom(δ) if and only if the linear form on $\mathbb{D}^{2,1}(K)$: $F \rightsquigarrow \int_E \langle DF, G \rangle_{\mathcal{L}_2(H,K)}(x)\, d\mu(x)$ is continuous for the topology induced by $L^2(E,\mu,K)$.

We denote δ the Skorokhod integral and we have by definition, for every $F \in \mathbb{D}^{2,1}(K)$,

$$\int_E \langle F, \delta G \rangle_K d\mu = \int_E \langle \nabla F, G \rangle_{\mathcal{L}^2(H,K)} d\mu \quad \text{if } \delta(G) \text{ is defined}\,.$$

Example 1 : Let $a \in H$, and $\varphi \in \mathbb{D}^{2,1}(K)$. Then $G := \varphi \otimes a$ is Skorokhod integrable and

$$\delta(a \otimes \varphi) = \widetilde{a}(\bullet)\ \varphi - \langle \nabla \varphi, a \rangle .$$

In particular, if $G : E \to H$ is such that $G(x) = a, \forall x$:

$$\delta G = \widetilde{a}(\bullet).$$

Example 2 : $E = \mathbb{R}^n, \mu = \gamma_n,\quad G : \mathbb{R}^n \to \mathbb{R}^n$.
Then

$$\begin{aligned} \delta G(x) &= \langle x, G(x) \rangle_{\mathbb{R}^n} - \sum_{j=1}^{n} \frac{\partial G_j}{\partial x_j}\ (x) \\ &= \langle x, G \rangle - \operatorname{div}\ G(x). \end{aligned}$$

This formula can be written in another manner :

$$\delta G = \langle \bullet, G \rangle - \text{Trace } (\nabla G).$$

Example 3 : If $G \in \mathbb{D}^{2,1}(E, \mu, \mathcal{L}^2(H, K))$, then it is δ-integrable, and δ is continuous from $\mathbb{D}^{2,1}(\mathcal{L}_2(H, K))$ in $L^2(E, \mu, K)$.

Example 4 : Let $F \in L^2(E, \mu, H)$ such that for every $h \in H$: $\nabla\ (\langle F, h \rangle_H)$ exists. Then for every linear continuous operator $A : H \to H$ with ***finite rank*** ,$A(F)$ is Skorokhod integrable.

More precisely, if $A = \sum_{j=1}^{n} \langle \bullet, a_j \rangle_H\ e_j$ (with a_j and e_j in H, (e_j) being orthonormal) we have :

$$\begin{aligned} A(F) &= \sum_{j=1}^{n} \langle F, a_j \rangle_H\ e_j \\ \delta(A(F)) &= \sum_{j=1}^{n} \left[\langle F, a_j \rangle\ \widetilde{e}_j - \nabla_{e_j}\ \left(\langle F, a_j \rangle \right) \right]. \end{aligned}$$

(see example 1).

This can be written in another manner :

Let A^* be the transpose of A : $A^* = \sum_{j=1}^{n} \langle \bullet, e_j \rangle_H \; a_j$ and let $\widetilde{A}^*$ defined as :

$$\widetilde{A}^* = \sum_{j=1}^{n} a_j \, \widetilde{e}_j \, .$$

Then

$$\delta\big(A(F)\big) = \langle F, \widetilde{A}^* \rangle_H - \sum_{j=1}^{n} \nabla_{e_j} \; \big(\langle F, a_j \rangle\big).$$

If we now suppose that DF exists, we have :

$$\sum_{j=1}^{n} \nabla_{e_j} \; \big(\langle F, a_j \rangle\big) = \text{Trace } (A \circ DF) \, .$$

Therefore, we have :

$$\delta\big(A(F)\big) = \langle F(\bullet), \widetilde{A}^*(\bullet) \rangle_H - \text{Trace } (A \circ DF) \, .$$

Example 5 : The Skorokhod integral coincides with the ordinary Itô-Integral for adapted processes (see the above mentioned Nualart's Lecture Notes for a precise statement of this fact).

Now we give some properties of the Skorokhod integral :

a) Let $A : K \to K'$ be a linear continuous operator (K and K' Hilbert spaces) and let $F \in L^2\big(E, \mu, \mathcal{L}_2(H, K)\big)$. If F is Skorokhod-integrable so is $A \circ F$ and we have

$$\delta(A \circ F) = A(\delta F) \, .$$

As a consequence we have :

- Let $F \in L^2\big(E, \mu, \mathcal{L}_2(H, K)\big)$ such that $\delta(F)$ exists, then for every k in K we have $\langle \delta(F), k \rangle = \delta\big(F^*(k)\big)$.

- Let $F \in L^2\Big(E, \mu, \mathcal{L}_2\big(H, \mathcal{L}_2(H,K)\big)\Big)$ such that $\delta(F)$ exists, then

$$\text{for every } h \in H,\ \delta\ \big(\overset{\vee}{F}(\bullet)(h)\big) \text{ exists}$$

and

$$\delta\ (F)(h) = \delta\ \big(\overset{\vee}{F}(\bullet)(h)\big)\,.$$

If $F \in \mathcal{L}^2\big(H, \mathcal{L}_2(H,K)\big)$, $\overset{\vee}{F}$ denotes the operator of $\mathcal{L}^2\big(H\,,\ \mathcal{L}_2(H,K)\big)$ such that :

$$\overset{\vee}{F}(h)(h') = F(h')(h), \qquad h, h' \in H\,.$$

b) Let $\varphi \in \mathbb{D}^{2,1}$, $F \in \mathcal{L}^2(E,\mu,H)$ such that F is Skorokhod integrable. Suppose that $\varphi F \in L^2(E,\mu,H)$ and that $\delta(F)\varphi - \langle F, D\varphi\rangle_H$ belongs to $L^2(E,\mu)$, then φF is Skorokhod integrable and

$$\delta(\varphi F) = \delta(F)\varphi - \langle F, D\varphi\rangle_H\,.$$

c) Let $A_n : H \to H$ a sequence of linear continuous operators such that $A_n \longrightarrow Id_H$ in the simple convergence.

Let $F \in \mathbb{D}^{2,1}\ (\mathcal{L}_2(H,K))$, then $\delta(F \bullet A_n) \longrightarrow \delta(F)$ in $L^2(E,\mu,K)$. In particular, if (e_n) is an orthonormal basis of H, the sequence

$$\Big(\sum_{i=1}^{n} \widetilde{e}_i\ F(e_i) - \nabla_{e_i} F(e_i)\Big)$$

converges to $\delta(F)$.

d) Let F, G in $\mathbb{D}^{2,1}(H)$ we have :

$$\begin{aligned}\mathbb{E}(\delta(F)\delta(G)) &= \mathbb{E}\{\langle F, G\rangle_H\} + \mathbb{E}\{\langle DF, (DG)^*\rangle_{\mathcal{L}_2(H,H)}\}\\ &= \mathbb{E}\{\langle F, G\rangle_H\} + \mathbb{E}\{\text{Trace } DG(\bullet) \circ DF(\bullet)\}\,.\end{aligned}$$

More generally, if F and G belong to $\mathbb{D}^{2,1}\big(\mathcal{L}_2(H,K)\big)$ we have :

$$\mathbb{E}\{\langle \delta F, \delta G\rangle_K\} = \mathbb{E}\{\langle F, G\rangle_{\mathcal{L}^2(H,K)}\} + \mathbb{E}\{\langle DF, \overset{\vee}{DG}\rangle_{\mathcal{L}_2\left(H, \mathcal{L}_2(H,K)\right)}\}.$$

e) The operator δ, as an operator densely defined from $L^2\big(E, \mu, \mathcal{L}_2(H,K)\big)$ into $L^2(\Omega, \mu, K)$ is ***closed.***

We now briefly introduce the Ogawa integral.

Let $P : H \to H$ be an orthogonal projector with finite rank : $P(h) = \sum_{j=1}^{n} \langle h, e_j \rangle_H \, e_j$.

We denote $\widetilde{P}$ the random variable with values in H :

$$\widetilde{P}(\bullet) := \sum_{j=1}^{n} \widetilde{e}_j(\bullet) \, e_j .$$

Now let $F \in L^0(E, \mu. H)$ be a random variable with values in H. We shall say that F is ***"Ogawa integrable"***, if there exists $G \in L^0(E, \mu)$ such that, for every increasing sequence (P_n) of orthogonal projectors converging simply to Id_H, the sequence of real random variables $(\langle F. \widetilde{P}_n \rangle_H)_n$ converges to G in probability.

We shall denote by $\overset{\circ}{\delta}(F)$ the Ogawa integral G of F.

If $F \in L^2(E. \mu, H)$ is such that, for every $a \in H$:

$$\langle F, a \rangle_H \, \widetilde{a}(\bullet) \text{ belongs to } L^2(E, \mu),$$

we shall say that F is ***"2-Ogawa integrable"*** when there exists $G \in L^2(E, \mu)$ such that

$$\langle F, \widetilde{P}_n \rangle_H \longrightarrow G \text{ in quadratic mean.}$$

(The P_n being as above).

Example : $(E, \mu) = (\mathbb{R}^n, \gamma_n)$. The Ogawa integral is equal to $\langle \bullet, F(\bullet) \rangle_{\mathbb{R}^n}$.
In this case , we have :

$$\overset{\circ}{\delta}(F) = \delta(F) + \text{Trace } (\nabla F).$$

Remark : There exists elements of $\mathbb{D}^{2,1}(H)$ which do not possess an Ogawa integral (Rosinski).

For instance, in the case of the Brownian motion, the function : $\omega \rightsquigarrow J\big(B(T - \bullet)(\omega)\big)$ where J denotes the indefinite integral null at zero, belongs to $\mathbb{D}^{2,1}(H)$ but is not Ogawa integrable.

Next we give a necessary and sufficient condition for Ogawa integrability :

Let $F \in \mathbb{D}^{2,1}(H)$; F is Ogawa integrable if and only if, for almost every x :

$$DF \in \mathcal{L}_1(H,H) \quad (\Longleftrightarrow DF \quad \textit{is nuclear})$$

and we have :

$$\overset{\circ}{\delta}(F) = \delta(F) + \text{Trace } (DF).$$

Sketch of the proof :

Suppose $P : H \to H$ is an orthogonal projector with finite rank. We know that :

$$\delta(PF) = \langle F, \widetilde{P} \rangle - \text{Trace } \big(D(PF)\big).$$

Let $P_n \uparrow Id$. We know that

$$\delta(P_n F) \longrightarrow \delta(F).$$

It is trivial that :

$$\langle F, \widetilde{P}_n \rangle \longrightarrow \overset{\circ}{\delta}(F)$$

(if $\overset{\circ}{\delta}(F)$ exists) and

$$\text{Trace}\big(D(P_n F)\big) \longrightarrow \text{Trace } (DF)$$

— *Q.E.D.*—

6 - Extensions and remarks - Localization

Now we shall consider the case where (E, H, μ) is the Wiener space. If $F \in \mathbb{D}^{2,1}$, then ∇F is a random variable with values in the Cameron-Martin space. Therefore, if $t \in [0,T]$ we can speak of the value of $\nabla F(\omega)$ at t, denoted $\nabla_t F(\omega)$. Analogously, time derivative of $\nabla F(\omega)$ at time t (***defined for almost every*** t) makes sense. We shall denote it : $\dot{\nabla}_t \, F(\omega)$. We have the equality :

$$\|\nabla F(\cdot)\|^2_{L^2(H)} = \mathbb{E}\Big(\int_0^t |\dot{\nabla}_t \, F(\omega)|^2 dt\Big).$$

Lemma 1 : *Let $F \in \mathbb{D}^{2,1}$. Then $1_{\{F=0\}} \dot{\nabla}_t F = 0$ almost everywhere on $[0,T] \times \Omega$.*

For the proof see Nualart-Pardoux.
This results in a localization theorem : if F is null (almost everywhere) on a set, so is its derivative. The derivation is a ***"local operator"***.

Definition 1 : *A random variable F will be said to belong to $\mathbb{D}^{2,1}_{loc}$ if there exist*

- *a sequence of measurable sets of E, $E_k \uparrow E$*

and

- *a sequence $(F_k) \subset \mathbb{D}^{2,1}$ such that $F_{|E_k} = F_{k|E_k}$ a.s. $\forall k \in \mathbb{N}$.*

Thanks to the preceding lemma we can define the derivation operator for an element of $\mathbb{D}^{2,1}_{\text{loc}}$.

Definition 2 : *Let $F \in \mathbb{D}^{2,1}_{\text{loc}}$ localized by the sequence (E_k, F_k). DF is the unique equivalence class of $dt \times dP$ a.e equal processes such that*

$$DF_{|E_k} = DF_{k|E_k}, \quad \text{for all } k \text{ in } \mathbb{N}.$$

This generalized derivative has the usual properties of composition :

let $\varphi : \mathbb{R}^m \to \mathbb{R}$ of the class C^1; suppose $F = (F^1, ..., F^m)$ is a random vector whose components belong to $\mathbb{D}^{2,1}_{\text{loc}}$; then

$$\varphi(F) \in \mathbb{D}^{2,1}_{\text{loc}}$$

and

$$\nabla\varphi(F) = \sum_{i=1}^{m} \frac{\partial\varphi}{\partial x_i}(F) \,.\, DF^i.$$

In the same manner we define $(\text{Dom }\delta)_{loc}$ as follows :
$F : E \longrightarrow H$ belongs to $(\text{Dom }\delta)_{\text{loc}}$ if there exists a sequence $E_k \uparrow E$, and a sequence $F_k : E \longrightarrow H$ such that $F_k \in (\text{Dom }\delta)$ for every k, such that

- *$F = F_k$ on E_k*
- *$\delta(F_k) = \delta(F_\ell)_{|F_k}$ a.s if $k < \ell$;*

*we shall say that F is **"localized"** by (E_k, F_k).*

For sufficiently reasonable integrands on $(\text{Dom }\delta)$ Nualart-Pardoux have shown that δ is local.

Definition 3 : *Let $F \in (\text{Dom }\delta)_{\text{loc}}$ localized by (E_k, F_k), $\delta(F)$ is defined as the unique equivalence class on random variables on E such that*

$$\delta(F)_{|E_k} := \delta(F_k)_{|E_k}, \quad \text{for all } k \text{ in } \mathbb{N}.$$

(Note that $\delta(F)$ may depend on the localizing sequence).

We shall need another notion of stochastic derivatives and Skorokhod integrals for some functions not necessarily belonging to $\mathbb{D}^{2,1}$, nor Skorokhod integrable, introduced by Buckdahn :

Let $T : E \to E$ be a measurable mapping of the form :

$$x \rightsquigarrow x + Fx \quad \text{where } F \in \mathbb{D}^{2,1}(H).$$

Let $\xi \in \mathbb{D}^{2,1}$ and suppose that for every sequence of smooth random variables $(\xi_n) \in \mathbb{D}^{2,1}$ converging to ξ in $\mathbb{D}^{2,1}$, the following limit exists and is independent of the approximating sequence chosen :

$$\lim_{n \longrightarrow \infty} \nabla\ (\xi_n \circ T)$$

where the limit is taken in probability.

Let us remark that $\xi_n \circ T$ belongs to $\mathbb{D}^{2,1}$ since the ξ_n are ***smooth***.

The common limit of the above sequences is denoted by $\widetilde{\nabla}\ (\xi \circ T)$.

Lemma 2 : *Suppose that $T(\mu) \ll \mu$, then the limit exists and we have, μ-almost surely :*

$$\widetilde{\nabla}\ (\xi \circ T) = \big(I_H + (\nabla F)^*\big)\big((\nabla \xi) \circ T\big) = (I_H + \nabla F)^*\big((\nabla \xi) \circ T\big)$$

(where $(\quad)^*$ denotes the adjoint of the bounded operator).

Moreover, if $\xi \circ T \in \mathbb{D}^{2,1}$: $\quad \widetilde{\nabla}\ (\xi \circ T) = \nabla(\xi \circ T)$.

Proof :

We have, ***since the*** (ξ_n) ***are smooth*** :

$$\nabla(\xi_n \circ T) = (I_H + \nabla F)^*\big((\nabla \xi_n) \circ T\big).$$

Moreover, $\nabla \xi_n$ converges in probability, and since $T(\mu)$ is absolutely continuous with respect to μ, $(\nabla \xi_n) \circ T$ converges in probability, so does $\nabla(\xi_n \circ T)$.

It now remains to prove that the limit does not depend upon the approximating sequence (ξ_n).

Let $\xi_n \longrightarrow \xi$ and $\eta_n \longrightarrow \xi$ in $\mathbb{D}^{2,1}$. Since the operator ∇ is closed we have :

$$\lim_n \nabla(\xi_n \circ T) = \lim_n \nabla(\eta_n \circ T).$$

Therefore, $\widetilde{\nabla}$ is well defined by what precedes. It is obvious that :

$$\widetilde{\nabla} = \nabla \quad \text{if} \quad \xi \circ T \in \mathbb{D}^{2,1}.$$

By duality, we can define a generalized Skorokhod integral of $\xi \circ T$, for $\xi \in D^{2,1}(H)$:

— *Lemma 2 is proven.*—

Definition : *Let $(e_i)_{i\in\mathbb{N}}$ be a fixed orthonormal basis of H. We define*

$$\widetilde{\delta}(\xi\circ T) := \sum_i (\langle \xi\circ T, e_i\rangle_H \, \widetilde{e}_i - \widetilde{\nabla}_{e_i}(\langle \xi\circ T, e_i\rangle_H),$$

if the limit of the right member is taken in probability.
($\widetilde{\nabla}_{e_i}$ denotes the generalized derivative in the e_i-direction introduced just above).

Lemma 3 : *Suppose $T = I + F$ as above is such that $T(\mu) \ll \mu$. Then $\widetilde{\delta}(\xi\circ T)$ exists and satisfies the following identity :*

$$\big(\delta(\xi)\big)\circ T = \widetilde{\delta}(\xi\circ T) + \langle \xi\circ T, F\rangle_H + \text{Trace}\,\big((\nabla\xi)\circ T \bullet \nabla F\big) \quad \mu\text{-almost surely}.$$

Proof :

Let $\xi^N = \sum_{i=1}^{N} \langle \xi, e_i\rangle_H e_i$, then

$$\widetilde{\delta}(\xi^N\circ T) = \sum_{i=1}^{N} \langle \xi\circ T, e_i\rangle_H \widetilde{e}_i - \sum_{i=1}^{N} \widetilde{\nabla}_{e_i}\ (\langle \xi\circ T, e_i\rangle_H).$$

But

$$\widetilde{e}_i\circ T = \widetilde{e}_i + \langle F, e_i\rangle_H,$$

therefore :

$$\delta(\xi^N\circ T) = \sum_{i=1}^{N} \Big\{\langle \xi\circ T, e_i\rangle_H\ [\widetilde{e}_i\circ T - \langle F, e_i\rangle_H] - \langle (I_H+\nabla F)^*\big(\nabla(\langle \xi, e_i\rangle_H)\big)\circ T, e_i\rangle_H$$

(by the preceding lemma)

$$= \sum_{i=1}^{N} \Big\{\langle \xi\widetilde{e}_i, e_i\rangle_H\circ T - \langle \xi\circ T, e_i\rangle_H\langle F, e_i\rangle_H - \langle (I_H+\nabla F)^*\big(\nabla(\langle \xi, e_i\rangle_H)\big)\circ T, e_i\rangle_H$$

$$= \sum_{i=1}^{N} [\langle \xi, e_i\rangle_H\ \widetilde{e}_i - \langle \nabla_{e_i}\xi, e_i\rangle_H]\circ T - \langle \xi^N\circ T, F\rangle_H - \text{Trace}\,\big(\nabla F^*, (\nabla\xi^N)\circ T\big).$$

Now $\xi^N \longrightarrow \xi$ in $\mathbb{D}^{2,1}(H)$; then the right member of this last equality converges in $L^0(E,\mu)$. Hence the sum is convergent in $L^0(E,\mu)$ and

$$\sum_{i=1}^{\infty} \langle \xi\circ T, e_i\rangle_H\ \widetilde{e}_i - \widetilde{\nabla}_{e_i}\ (\langle \xi\circ T, e_i\rangle_H) \quad \text{is convergent in } L^0(E,\mu).$$

— *Lemma 3 is proven.*—

CHAPTER TWO

Transformation of a Gaussian measure

Given an abstract Wiener space (H, E, μ) and $T : E \to E$ of the form :

$$Tx = x + F(x), \quad F : E \to H.$$

We shall examine when $T(\mu) \ll \mu$. We shall consider the following cases :

- F is linear continuous from E into H,
- F is regular (i.e., possesses stochastic derivatives).

We shall give some expressions for the Radon-Nikodym density $\dfrac{dT(\mu)}{d\mu}$.

In the following chapter we shall study a family of flows : $T_t = I + F_t$ where $F_t : E \to H$, $(t \in [0,1])$ and shall study the work of Cruzeiro, Buckdahn and Ustunel-Zakai on this subject. We shall only give the statements of the results and from time to time sketch of the proofs.

1 - Preliminary results on equivalence and orthogonality of product measures

Let $(E_k, \mathcal{B}_k)_{k \in \mathbb{N}^*}$ be a sequence of measurable spaces and for every k, let μ_k and ν_k be two probabilities on $(E_k, \mathcal{B}_k)$ such that $\mu_k \ll \nu_k$. Let us set $\rho_k = \dfrac{d\mu_k}{d\nu_k}$.

Let us consider the product measures :

$$\mu = \prod_{k=1}^{\infty} \mu_k$$

and

$$\nu = \prod_{k=1}^{\infty} \nu_k$$

and let

$$\alpha_k = \int_{E_k} \sqrt{\rho_k(x_k)}\ \nu_k\ (dx_k).$$

These notations having been fixed we have the following result of Kakutani :

THEOREM 1 : *We have the dichotomy :*

$$\mu \ll \nu \quad \text{or} \quad \mu \perp \nu.$$

a) $\mu \ll \nu \Longleftrightarrow \prod \alpha_k$ *converges ; and in this case the density is equal to* $\rho(x) = \prod_1^\infty \rho_k(x_k)$ *(convergence in mean).*

b) $\mu \perp \nu \Longleftrightarrow \prod \alpha_n$ *diverges to zero. (We cannot have divergence to infinity since* $\alpha_k^2 \leq 1$*).*

Applications : $E_k = \mathbb{R}$ for every k

$$\nu_k(dx_k) = \frac{1}{\sigma_k\sqrt{2\pi}} \exp\Big\{-\frac{(x_k-\gamma_k)^2}{2\sigma_k^2}\Big\}\, dx_k$$

$$\mu_k(dx_k) = \frac{1}{\lambda_k\sqrt{2\pi}} \exp\Big\{-\frac{(x_k-\beta_k)^2}{2\lambda_k^2}\Big\}\, dx_k\,.$$

Then

$$\rho_k(x_k) = \frac{\sigma_k}{\lambda_k} \exp\Big\{-\frac{1}{2\sigma_k^2\lambda_k^2}\Big[(x_k-\beta_k)^2\,\sigma_k^2 - (x_k-\gamma_k)^2\lambda_k^2\Big]\Big\}$$

and

$$\alpha_k = \int_{\mathbb{R}} \sqrt{\rho_k(x_k)}\, d\nu_k(x_k) = \sqrt{\frac{2\lambda_k\sigma_k}{\lambda_k^2+\sigma_k^2}} \exp\Big\{-\frac{(\beta_k-\gamma_k)^2}{4(\lambda_k^2+\sigma_k^2)}\Big\}.$$

We now give some particular cases :

- **Same covariance** ($\lambda_k = \sigma_k$ for every k). μ and ν are equivalent if and only if

$$\sum_k \frac{\left(\beta_k-\gamma_k^2\right)^2}{\sigma_k^2} < \infty$$

and the density is then equal to

$$\exp\Big\{\sum_{k=1}^\infty \frac{x_k(\beta_k-\gamma_k)}{\sigma_k^2} - \frac{\beta_k^2-\gamma_k^2}{2\sigma_k^2}\Big\}.$$

Otherwise, we have orthogonality of measures.

- **Same mean** $\beta_k = \gamma_k = 0$ for every k.
  μ and ν are equivalent if and only if :

$$\sum_{k=1}^{\infty} \frac{(\lambda_k - \sigma_k)^2}{\lambda_k \sigma_k} < \infty$$

and in this case the density is equal to :

$$\frac{d\mu}{d\nu}(x) = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\sigma_k}{\lambda_k} \exp\Big\{-\frac{x_k^2}{2}\Big(\frac{\sigma_k^2 - \lambda_k^2}{\sigma_k^2\lambda_k^2}\Big)\Big\}.$$

If this condition is not satisfied we have orthogonality.

2 - Affine transformations of Gaussian measures

Now let (E, H, μ) be an abstract Wiener space. If (e_n) is an orthonormal basis of H, the random variables $\widetilde{e}_n$ are independent Gaussian variables on E, with mean zero and variance one. The law of the sequence $(\widetilde{e}_n)$ is therefore a product measure on $\mathbb{R}^{\mathbb{N}}$:

$$\gamma_{\mathbb{N}} = \bigotimes_{n=0}^{\infty} \gamma_n$$

where $\gamma_n = \gamma$ (Gaussian measure on $\mathbb{R}$) for every n.
Now we have a measurable (defined almost everywhere) map θ of E into $\mathbb{R}^{\mathbb{N}}$:

$$x \rightsquigarrow \big(\widetilde{e}_n(x)\big)_n.$$

If the e_n belong to E', the $\widetilde{e}_n$ are everywhere defined and θ is continuous from E into $\mathbb{R}^{\mathbb{N}}$.

It is clear now that the image of μ under θ is equal to $\gamma_{\mathbb{N}}$. We have $\theta(H) = \ell^2$ as we can see immediately (the $\widetilde{e}_n(x)$ are defined in a unique way on H).

Proposition 1 : *Let $a \in E$ and $\tau_a(\mu)$ be the translate of μ by a. Then we have the dichotomy :*

$\tau_a(\mu) \sim \mu$ or $\tau_a(\mu) \perp \mu$,

$\tau_a(\mu) \sim \mu$ if and only if $a \in H$ and the density is equal to $\exp\{\widetilde{a}(\cdot) - \frac{1}{2}\,\|a\|_H^2\}$.

Proof :

$\tau_a(\mu)$ is a Gaussian (non centered if $a \neq 0$) measure with the same covariance than μ.

Let $(e_n) \subset E'$ (orthonormal in H). It suffices to prove the same result for $\theta(\mu)$ and $\theta(\tau_a(\mu))$. But $\theta(\tau_a(\mu))$ is the product of Gaussian measures on $\mathbb{R}$ with variances one and mean $e_n(a)$. Therefore it suffices to apply the result of the previous paragraph.

— *Q.E.D.*—

Now let $T = I + F$ be a linear continuous transform of E into E. Let us suppose that $F(E) \subset H$. In this case F is continuous for the topology of H by closed graph theorem.

Suppose moreover, that $T_{|H} = Id_H + F_{|H}$ is an ***invertible operator***. Then $T : E \to E$ is also invertible and

$$T^{-1} = I - (T_{|H})^{-1} \circ F.$$

Proposition 2 : *Suppose $T = I + F$ with the above properties and that $F_{|H}$ is nuclear. Then $T^{-1}(\mu)$ and μ are equivalent and*

$$\frac{dT^{-1}(\mu)}{d\mu}(x) = \exp\Big\{-(Fx, x)_H - \frac{1}{2}\|Fx\|_H^2\Big\} \, |\det\, T| .$$

Proof :

Let us explain what this formula means. Indeed, $F_{|H}$ being nuclear, admits the decomposition : $F_{|H}(x) = \sum_n \lambda_n \, (x, e_n)_H f_n$, $(e_n, f_n$ orthonormal in $H)$ and we can define $\langle F(x), x\rangle_H$ on E by $\sum_n \lambda_n \, \widetilde{e}_n(x) \, \widetilde{f}_n(x)$, we set : $\det\,(I + F) = \prod_n (1 + \lambda_n)$. (This has sense since $\sum_n |\lambda_n| < \infty$).

- **Let us suppose first that F is symmetrical** :

$$F(x) = \sum_n \lambda_n (x, e_n)_H \, e_n$$

where e_n is an orthonormal basis composed of eigenvectors of F.

Let $\theta : E \to \mathbb{R}^{\mathbb{N}}$ associated to these e_n. We have seen that : $\theta(\mu) = \gamma_{\mathbb{N}}$ (product measure).

Now $\theta((I+F)^{-1}\mu)$ is the product of measures with densities :

$$\frac{1}{\sqrt{2\pi}}(1+\lambda_n)\ \exp\Big\{-\frac{1}{2}\ (1+\lambda_n)^2x_n^2\Big\}.$$

We have

$$\frac{d((1+\lambda_n)^{-1}\ \tilde{e}_n(\mu))}{d(\tilde{e}_n(\mu)))}\ (x_n)=(1+\lambda_n)\ \exp\Big\{-\lambda_n\ x_n^2-\frac{1}{2}\ \lambda_n^2x_n^2\Big\}$$

$$\frac{d\Big(\theta((I+F^{-1})(\mu))\Big)}{d\theta(\mu)}\ (x)=\prod\ (1+\lambda_n)\ \exp\Big\{-(Fx,x)_H-\frac{1}{2}\ \|Fx\|_H^2\Big\}.$$

• **Now let us consider the general case (F non necessarily symmetrical)**

$$H\ \overset{i}{\to}\ E\ \overset{I+F}{\to}\ H\ \overset{i}{\to}\ E$$

$(I+F)\circ i$ is an operator from H into H. There exists a unitary operator $U: H \to H$ ***"diagonalizing"*** $F\circ i$, therefore $(I+F)\circ i$. Let $\widetilde{U}$ its extension to $E\to E$. We apply the result for $\widetilde{U}(I+F)\ \widetilde{U}^{-1}$.

— *Q.E.D.*—

Now we shall consider the case where $F_{|H}$ is not nuclear.

We know that in any case $F_{|H}$ is **Hilbert-Schmidt.**

• **Suppose at first that rank (F) is finite.**

Then the formula of Proposition 2 gives :

$$\prod_{i=1}^{n}\ (1+\lambda_i)\ \exp\Big\{-\ \sum_{i=1}^{n}\ \lambda_i x_i^2-\frac{1}{2}\sum_{i=1}^{n}\ \lambda_i^2x_i^2\Big\}$$

$$=\prod_{i=1}^{n}\ (1+\lambda_i)\ e^{-\lambda_i}\ \exp\Big\{-(\sum_{i=1}^{n}\ \lambda_i x_i^2-\sum_{i=1}^{n}\lambda_i-\frac{1}{2}\ \|Fx\|_H^2\Big\}.$$

• **Now suppose F Hilbert-Schmidt with infinite rank :**

$$\prod_i\ (1+\lambda_i)\ e^{-\lambda_i}\ \text{converges since}\ \sum_i\ |\lambda_i|^2<\infty.$$

The limit is called the ***"Carleman determinant"***.

Now we can prove that

$$\lim_{n \to \infty} \exp\Big\{-\Big(\sum_{i=1}^{n} \lambda_i\, x_i^2 - \sum_{i=1}^{n} \lambda_i\Big) - \frac{1}{2}\, \|Fx\|_H^2\Big\} \text{ exists in } L^1(\mu) \text{ if } F \text{ is } H\text{-}S.$$

We denote it by :

$$\exp\Big\{-\big[\text{“}(Fx,x)_H - \text{Trace } F\text{”}\big] - \frac{1}{2}\, \|Fx\|_H^2\Big\}.$$

Therefore we have the following theorem :

THEOREM 2 : *Let $T : E \to E$ linear continuous, such that $Tx = x+Fx$ with $F(E) \subset H$. Then $F_{|_H}$ defines a Hilbert-Schmidt operator from H into H. Suppose that $T_{|H}$ is invertible then $T : E \to E$ is invertible. Moreover, $T^{-1}(\mu)$ is absolutely continuous with respect to μ and we have*

$$\frac{d\big(T^{-1}(\mu)\big)}{d\mu}\,(x) = \widetilde{\Delta}(I+F)\, \exp\Big\{-\big[\text{“}(Fx,x)_H - \text{Trace } F\text{”}\big] - \frac{1}{2}\, \|Fx\|_H^2\Big\}$$

with

$$\widetilde{\Delta}(I+F) = \prod_{1}^{\infty} (1+\lambda_i)\, e^{-\lambda_i},$$

the λ_i being the eigenvalues of F.

We have seen the affine case.

Now we may give the result for the general case announced in the beginning.

THEOREM 3 : *Let $F \in \mathbb{D}^{2,1}(H)$. Suppose that $(I+F)$ is invertible and that for every $x \in E$, the operator $I_H + \nabla F(x)$ from H to H is invertible, then $(I+F)^{-1}(\mu)$ is absolutely continuous with respect to μ and we have :*

$$\frac{d\big((I+F)^{-1}\mu\big)}{d\mu}(x) = \widetilde{\Delta}\,\big(I_H + \nabla F(x)\big)\, \exp\Big\{-\delta(F)(x) - \frac{1}{2}\, \|Fx\|_H^2\Big\}.$$

CHAPTER THREE

Transformation of Gaussian measures under anticipative flows

Let (Ω, H, P) be an abstract Wiener space and let T be an invertible transformation of Ω into Ω $\big($the only interesting case will be of the form : $T := Id + F$ with $F \in \mathbb{D}^{2,1}(H)\big)$.

Definition : *A family of tranformations $(T_t)_{t\in[0,1]}$ from Ω to Ω will be called an* ***"interpolation"*** *of the invertible transformation T if*

a) $T_0 = Id, \quad T_1 = T$,

b) each T_t is invertible,

c) for each ω, $\quad t \rightsquigarrow T_t\omega$ and $t \rightsquigarrow T_t^{-1}\omega$ are strongly continuous.

Moreover, if

d) for each ω, $\quad t \rightsquigarrow T_t\omega$ and $t \rightsquigarrow T_t^{-1}\omega$ are strongly continuously differentiable, the interpolation will be said to be ***"smooth"***.

Example 1 : $T_t(\omega) = \omega + tA(\omega)$ where A is a function from Ω to H, such that

$$\omega \rightsquigarrow \omega + tA(\omega) \text{ is invertible for every } t.$$

Example 2 : Suppose $A : \Omega \to H$ is continuous and suppose that we have defined a family of transformations (T_t) from Ω into Ω by :

$$T_t\omega = \omega + \int_0^t A(T_s\omega)\, ds \quad \text{(time homogeneous case)}$$

$$i.e. \quad \left| \begin{array}{ll} \dfrac{dT_t}{dt}(\omega) & = A(T_t\omega) \\ T_0(\omega) & = \omega \end{array} \right.$$

we have then :

$$\frac{dT_t}{dt}\left(T_t^{-1}(\omega)\right) = A(\omega).$$

Example 3 : $T_t(\omega) = \omega + \int_0^t \sum(s, T_s(\omega))\, ds.$

If $\sum(r,\omega)$ is **continuous** on $[0,1]\times\Omega$ into Ω or into H and satisfies a global Lipschitz condition :

$$|\sum(t,\omega_1) - \sum(t,\omega_2)| \leq L\|\omega_1 - \omega_2\|_\Omega$$

We can consider $T_t(\omega)$ as the solution of the ordinary differential equation

$$\begin{cases} \dfrac{dT_t}{dt}\,(\omega) &= \sum\,(t, T_t(\omega)) \\ T_0\,(\omega) &= \omega \end{cases}$$

on the Banach space Ω.

If for every $t \in [0,1]$, $\sum\,(t,\bullet)$ is Fréchet differentiable, with Fréchet differential denoted by $\partial\sum\,(t,\omega)$, and if we assume that $\partial\sum\,(t,\omega)$ is bounded continuous on $[0,1]\times\Omega$, then the equation

$$T_t\omega = \omega + \int_0^t \sum\,\left(r, T_r(\omega)\right)\, dr$$

has a unique solution.
Moreover, $\omega \rightsquigarrow T_t(\omega)$ is Fréchet differentiable and $\partial T_t(\omega)$ is continuous, invertible on $[0,1]\times\Omega$, and satisfies the differential equation :

$$\frac{d}{dt}\,(\partial T_t\omega) = \left(\partial\sum(t,\bullet)\circ T_t(\omega)\right) \bullet \partial T_t(\omega)\,.$$

Its inverse $\partial^{-1}T_t\omega$ satisfies :

$$\frac{d}{dt}\,(\partial^{-1}T_t\omega) = -\partial^{-1}T_t(\omega) \bullet \left(\partial\sum(t,\bullet)\circ T_t(\omega)\right).$$

Consequently, by the global inverse theorem, $T_t(\omega)$ is a C_1-diffeomorphism. Therefore, we have an interpolation of T defined by

$$T(\omega) = \omega + \int_0^1 \sum\,(r, T_r\omega)\, dr.$$

Later on we shall come back to this example. Now let us return to the general situation.

THEOREM 1 : *Let T be a transformation from Ω to Ω and $(T_t, t \in [0,1])$ be an interpolation of T. Let us assume moreover that*

(a) $T_t(P) \ll P, \quad \forall t \in [0,1]$ *and let* $X_t(\omega) = \dfrac{dT_t(P)}{dP}(\omega)$,

(b) $G_t = T_t^{-1} - I \in \mathbb{D}^{2,1}(H)$ *and* $\dfrac{dT_t^{-1}}{dt} \in H$,

(c) $\dfrac{dT_t^{-1}}{dt}$ *as a function from $[0,1] \times \Omega$ into H is almost surely continuous in (t,ω) (for $dt \otimes dP$) and $\nabla T_t^{-1}(\omega)$ will be assumed to possess a continuous extension $[0,1] \times \Omega$,*

(d) $\dfrac{dT_s^{-1}}{ds} \circ T_s \in \mathbb{D}^{2,1}(H)$.

Then

$$X_t(\omega) = \exp\Big\{ - \int_0^t \Big(\delta \, [\frac{dT_s^{-1}}{ds} \circ T_s]\Big) \circ T_s^{-1}(\omega) \, ds \Big\} \tag{1}$$

This implies that the measures $T_t(P), T_t^{-1}(P)$ and P are equivalent.
Moreover

$$\begin{aligned} X_t = \exp\Big\{ &- \int_0^t \tilde{\delta} \, [\frac{dG_s}{ds}] \, ds \\ &- \frac{1}{2} \langle G_t, G_t \rangle_H \\ &- \int_0^t \text{Trace} \Big[\Big(\nabla [\frac{dG_s}{ds} \circ T_s] \circ T_s^{-1} \Big) \bullet \nabla G_s \Big] \, ds \Big\} \end{aligned} \tag{2}$$

where $\tilde{\delta}$ was defined precedently by :

$$\tilde{\delta} \, (\xi \circ T) = (\delta \xi) \circ T - \langle \xi \circ T, F \rangle_H - \text{Trace} \, \big((\nabla \xi) \circ T \bullet \nabla F \big).$$

Moreover, if $\dfrac{dG_s}{ds}$ and G_s are in $\mathbb{D}^{2,1}(H)$, then the formula (**2**) *becomes :*

$$\begin{aligned} X_t = \exp\Big\{ &- \delta(G_t) - \frac{1}{2} \langle G_t, G_t \rangle_H \\ &- \int_0^t \text{Trace} \Big[\Big(\nabla [\frac{dG_s}{ds} \circ T_s] \circ T_s^{-1} \Big) \bullet \nabla G_s \Big] \, ds \Big\}. \end{aligned} \tag{3}$$

Proof of (1) :

We have :

$$
\begin{aligned}
0 &= \frac{1}{\varepsilon}\left[T_{t+\varepsilon}^{-1} \circ\ T_{t+\varepsilon} - T_t^{-1} \circ T_t\right] \\
&= \frac{1}{\varepsilon}\left[T_{t+\varepsilon}^{-1} \circ T_{t+\varepsilon} - T_{t+\varepsilon}^{-1} \circ T_t\right] + \frac{1}{\varepsilon}\left[T_{t+\varepsilon}^{-1} \circ T_t - T_t^{-1} \circ T_t\right].
\end{aligned}
$$

Therefore by (c)

$$
\left[(\nabla T_t^{-1}) \circ T_t(\omega)\right] . \frac{dT_t}{dt}(\omega) + \frac{dT_t^{-1}}{dt} \circ T_t\omega = 0 \tag{4}
$$

Let now $a : \Omega \to \mathbb{R}$ smooth and let $h \in H$. By (d) we have :

$$
\begin{aligned}
\langle(\nabla a) \circ T_t(\omega), h\rangle_H &= \lim_{\varepsilon \longrightarrow 0} \frac{\partial}{\partial \varepsilon}\ a\ (T_t\omega + \varepsilon h) \\
&= \lim_{\varepsilon \longrightarrow 0} \frac{\partial}{\partial \varepsilon}\left[(a \circ T_t)\ (T_t^{-1}(T_t\omega + \varepsilon h))\right] \\
&= \lim_{\varepsilon \longrightarrow 0} \frac{\partial}{\partial \varepsilon}\left[(a \circ T_t)\ (\omega + \varepsilon(\nabla T_t^{-1}) \circ (T_t\omega)\ .\ h + o(\varepsilon)\right] \\
&= \langle \nabla(a \circ T_t),\ (\nabla T_t^{-1}) \circ T_t(\omega)\ .\ h\rangle_H .
\end{aligned}
$$

Now if we set $h = \frac{d}{dt}\ T_t(\omega)$, comparing with (4), we obtain :

$$
\langle(\nabla\ a) \circ T_t\omega, \frac{d}{dt}\ T_t\omega\rangle_H = -\langle\nabla\ (a \circ T_t)(\omega), \frac{dT_t^{-1}}{dt} \circ T_t(\omega)\rangle_H .
$$

But the left-hand member of this equality is equal to $\frac{d}{dt}\ (a \circ T_t)(\omega)$. Therefore we obtain :

$$
\begin{aligned}
\mathbb{E}\{a \circ T_t\omega - a(\omega)\} &= \mathbb{E}\left(\int_0^t \frac{d}{ds}\ (a \circ T_s\omega)\ ds\right) \\
&= -\mathbb{E}\left(\int_0^t \langle\nabla\ (a \circ T_s)\ (\omega), \frac{dT_s^{-1}}{ds} \circ T_s\omega\rangle\ ds\right).
\end{aligned}
$$

But from condition (d) , $\left(\frac{dT_s^{-1}}{ds} \circ T_s \in \mathbb{D}^{2,1}(H)\right)$, and integrating by parts we obtain :

$$
\mathbb{E}\{a \circ T_t(\omega) - a(\omega)\} = -\int_0^t \mathbb{E}\left\{(a \circ T_s\omega)\ \delta\left[\frac{dT_s^{-1}}{ds} \circ T_s\right](\omega)\right\}\ ds
$$

and

$$\mathbb{E}\ \{a(\omega).(X_t(\omega)-1)\} = -\mathbb{E}\left(\int_0^t a(\omega)\ X_s(\omega)\Big(\delta\big[\frac{dT_s^{-1}}{ds}\circ T_s\big]\Big)\circ T_s^{-1}\omega\ ds\right).$$

Since this last inequality is true for smooth functions we have :

$$X_t(\omega) = 1 - \int_0^t X_s(\omega)\Big(\delta\big[\frac{dT_s^{-1}}{ds}\circ T_s\big]\Big)\circ T_s^{-1}\omega\ ds\,.$$

Finally, since X_t is P-almost surely positive, T_tP and P are equivalent.

On the other hand, if $a : \Omega \to \mathbb{R}$ is smooth, then :

$$\mathbb{E}\ \{a\circ T_t^{-1}\ X_t\} = \mathbb{E}a.$$

Hence if B is a Borelian subset of Ω, then

$$P(B) = 0 \Longleftrightarrow \mathbb{E}\{1_B\circ T_t^{-1}\ X_t\} = 0 \Longleftrightarrow 1_B\circ T_t^{-1} = 0,\ \text{a.s.}$$

Therefore, $T_t^{-1}\ (P)$ and P are equivalent.

Proof of (2) :

We start from

$$(\delta\xi)\circ T = \tilde{\delta}\ (\xi\circ T) + \langle\xi\circ T, F\rangle_H + \text{Trace}\ \big((\nabla\xi)\circ T\ \bullet\ \nabla F\big)$$

with

$$\xi = \frac{dT_s^{-1}}{ds}\circ T_s, \quad T = T_s^{-1}, \quad F = T - Id = G_s$$

and

$$\frac{dG_s}{ds} = \frac{dT_s^{-1}}{ds}.$$

Then

$$\delta\big[\frac{dT_s^{-1}}{ds}\circ T_s\big]\circ T_s^{-1} = \tilde{\delta}\Big(\frac{dG_s}{ds}\Big) + \langle\frac{dG_s}{ds}, G_s\rangle + \text{Trace}\ \left(\Big(\nabla\ \big[\frac{dG_s}{ds}\circ T_s\big]\Big)\circ T_s^{-1}\ \bullet\ \nabla G_s\right)$$

and we integrate from 0 to t.

Proof of (3) :

It is immediate from (2) since $\tilde{\delta} = \delta$ under this hypothesis.

We have expressed the density X_s in terms of $\frac{dT_s^{-1}}{dt}$. (The next result will give an expression of X_t in terms of $\frac{dT_s}{ds}$).

— *Q.E.D.*—

Corollary : *Under the assumptions and conditions of the theorem 1 let us replace* T, T_t, T_s *and* X_t *by* T^{-1}, T_t^{-1}, T_s^{-1}, $\frac{dT_t^{-1}(P)}{dP} = Y_t$. *Then we have :*

$$X_t(\omega) = \frac{dT_t(P)}{dP}\ (\omega)$$
$$= \exp\Big\{ \int_0^t \Big(\delta\big[\frac{dT_s}{ds} \circ T_s^{-1}(\bullet)\big]\Big) \circ T_s T_t^{-1}(\omega)\ ds \Big\}$$

and

$$X_t(\omega) = \exp\Big\{ -\delta(G_t)(\omega) - \frac{1}{2}\ \langle G_t, G_t\rangle_H(\omega)$$
$$+ \int_0^t \text{Trace}\ \Big[\Big(\nabla\big[\frac{dT_s}{ds} \circ T_s^{-1}\big] \circ T_s T_t^{-1}(\omega)\Big) \bullet \nabla\Big(G_t - G_s\ (T_s T_t^{-1})\Big)(\omega)\Big]\ ds\Big\}.$$

Proof :

By Theorem 1:

$$Y_t(\omega) = \exp\Big\{ - \int_0^t \Big(\delta\ \big[\frac{dT_s}{ds} \circ T_s^{-1}\big]\Big) \circ T_s(\omega)\ ds\Big\}. \qquad (A)$$

On the other hand, if a is a smooth functional :

$$\mathbb{E}\{a(\omega)\ Y_t^{-1}\ (T_t^{-1}\omega)\} = \mathbb{E}\{a(T_t T_t^{-1}\omega)\ Y_t^{-1}\ (T_t^{-1}(\omega))\}$$
$$= \mathbb{E}\{a(T_t(\omega))\ Y_t^{-1}(\omega)\ Y_t(\omega)\}$$
$$= \mathbb{E}\{a(\omega)\ X_t(\omega)\}.$$

Therefore :

$$X_t(\omega) = Y_t^{-1}\ (T_t^{-1}(\omega)) = \exp\Big\{ \int_0^t \Big(\delta\big[\ \frac{dT_s}{ds} \circ T_s^{-1}(\bullet)\big]\Big) \circ T_s \circ T_t^{-1}(\omega)\ ds\Big\},$$

— *which proves the first formula.*—

To prove the second formula let us start from

$$T_s\omega = \omega + F_s(\omega)$$

which implies

$$T_s T_t^{-1}\omega = T_t^{-1}\omega + F_s\ (T_t^{-1}\omega),$$

and if $s = t$

$$\omega = T_t^{-1}\omega + F_t\ (T_t^{-1}\omega)\,.$$

Therefore

$$T_s T_t^{-1}\omega = \omega + F_s\ (T_t^{-1}\omega) - F_t\ (T_t^{-1}\omega).$$

Now

$$G_t(\omega) = T_t^{-1}(\omega) - \omega = -F_t\ (T_t^{-1}\omega).$$

Therefore :

$$T_s T_t^{-1}\omega = \omega + G_t(\omega) - G_s\ (T_s T_t^{-1}\omega).$$

In the formula

$$X_t(\omega) = \exp\left\{\int_0^t \left(\delta\left[\frac{dT_s}{ds}\circ T_s^{-1}\right]\right)\circ T_s T_t^{-1}\omega\ ds\right\},$$

let us apply the formula given δ in terms of $\tilde{\delta}$. We obtain :

$$\begin{aligned} X_t(\omega) = \exp\Bigg\{\int_0^t \Bigg(&\tilde{\delta}\ \left[\frac{dT_s}{ds}\circ T_t^{-1}\right](\omega) \\ &+ \langle\frac{dT_s}{ds}\circ T_t^{-1}(\omega),\ G_t(\omega) - G_s(T_s T_t^{-1}\omega)\rangle_H \\ &+ \text{Trace}\ \left[\left(\nabla\left[\frac{dT_s}{ds}\circ T_s^{-1}\right]\circ T_s T_t^{-1}(\omega)\right)\bullet\nabla\Big(G_t - G_s(T_s T_t^{-1})\Big)(\omega)\right]\Bigg)\ ds\Bigg\} \end{aligned}$$

Now we integrate with respect to s, by using :

$$\frac{d}{ds}(T_s\circ T_t^{-1}(\omega)) = -\frac{d}{ds}(G_s(T_s T_t^{-1}\omega)) = \frac{d}{ds}\big(G_t(\omega) - G_s(T_s T_t^{-1}\omega)\big).$$

— *We obtain the second formula.*—

Now we give an integral equation satisfied by X_t.

THEOREM 2 : *Let $T : \Omega \to \Omega$ and $T_t : \Omega \to \Omega$ $(t \in [0,1])$ be an interpolation of T. Assume that for each $t \in [0,1]$, $T_t(P) \ll P$ and that $X_s\ [\frac{dT_s}{ds} \circ T_s^{-1}] \in \mathbb{D}^{2,1}_{loc}(H)$ (this condition is satisfied if $\frac{dT_s}{ds} \circ T_s^{-1} \in \mathbb{D}^{2,1}(H)$ and $X_s \in \mathbb{D}^{2,1}_{loc}$), then X_t satisfies :*

$$X_t = 1 + \int_0^t \delta\big[X_s\, \frac{dT_s}{ds} \circ T_s^{-1}\big]\, ds .$$

Proof :

Let a be a smooth functional. Then

$$\begin{aligned}
\mathbb{E}\{X_t(\omega)a(\omega)\} &= \mathbb{E}\Big\{a(T_t(\omega))\Big\} \\
&= \mathbb{E}\Big\{a(\omega) + \int_0^t \frac{da(T_s(\omega)}{ds}\, ds\Big\} \\
&= \mathbb{E}\Big\{a(\omega) + \int_0^t \langle(\nabla a)\circ T_s\omega,\ \frac{d}{ds}\, T_s(\omega)\rangle\, ds\Big\} \\
&= \mathbb{E}\{a(\omega)\} + \int_0^t \mathbb{E}\Big\{X_s(\omega)\langle\nabla(a)(\omega)\,,\ [\frac{dT_s}{ds}\circ T_s^{-1}(\omega)]\rangle\Big\}\, ds \\
&= \mathbb{E}\{a(\omega)\} + \int_0^t \mathbb{E}\Big\{a(\omega)\ \delta\big[X_s\, \frac{dT_s}{ds}\circ T_s^{-1}\big](\omega)\Big\}\, ds
\end{aligned}$$

— *Q.E.D.*—

Applications of these formulas.

- **In the example (1)** : $T_t(\omega) = \omega + t\, A(\omega)$,

$$X_t(\omega) = \exp\Big\{\int_0^t \Big(\delta\big[A\big(T_s^{-1}(\bullet)\big)\big]\Big) \circ T_s T_t^{-1}(\omega)\, ds\Big\}$$

(this result was obtained by Bell).

- **In the example (2)** : $T_t(\omega) = \omega + \displaystyle\int_0^t A\big(T_s(\omega)\big)\, ds$

$$\frac{dT_s}{ds}\ \big(T_s^{-1}(\omega)\big) = A(\omega)$$

and

$$X_t(\omega) = \exp\Big\{\int_0^t \Big(\delta(A)\Big) \circ T_s T_t^{-1}(\omega)\ ds\Big\}.$$

• **We shall now study the example three** :

$$T_t(\omega) = \omega + \int_0^t \sum\big(r, T_r(\omega)\big)\ dr. \tag{B}$$

We have given some hypotheses insuring that $T_t\omega$ is a solution of the ODE with values in the Banach space Ω

$$\left| \begin{array}{ll} \dfrac{dT_t}{dt}(\omega) & = \sum\big(t, T_t(\omega)\big) \\ T_0(\omega) & = \omega \end{array} \right.$$

and that $\omega \rightsquigarrow T_t(\omega)$ and $\omega \rightsquigarrow T_t^{-1}(\omega)$ are Fréchet differentiable (in ω). Then :

$$I_H + \nabla \int_0^t \sum(s, T_s\omega)\ ds$$

is invertible and satisfies the hypotheses of Ramer's theorem

As a consequence the probabilities

$$T_t P,\ P \text{ and } T_t^{-1} P \text{ are equivalent.}$$

Now in (B) we replace ω by $T_s^{-1}\omega$:

$$T_t\, T_s^{-1}(\omega) = T_s^{-1}(\omega) + \int_0^t \sum\big(r, T_r T_s^{-1}(\omega)\big)\ dr\,.$$

Setting : $T_t T_s^{-1}(\omega) = \varphi_{s,t}(\omega)$ and $T_s T_t^{-1}(\omega) = \psi_{s,t}(\omega)$, $t \geq s$, we have :

$$\psi_{s,t} \circ \varphi_{s,t} = \varphi_{s,t} \circ \psi_{s,t} = Id$$

and :

$$\varphi_{s,t}(\omega) = \omega + \int_s^t \sum\big(r, \varphi_{s,r}(\omega)\big)\ dr$$

$$\psi_{s,t}(\omega) = \omega - \int_s^t \sum\big(r, \psi_{r,t}(\omega)\big)\ dr\,.$$

Note that $\varphi_{(1-s)t,t}$, $s \in [0,1]$ is, for t fixed, an interpolation of T_t and naturally $(T_t)_{t\in[0,1]}$ is an interpolation of T_1 : $\varphi_{s,t}$ is a ***"two-parameter"*** interpolation of T.

• **Now we shall specialize the example in the case** $\Omega = \mathcal{C}_0[0,1]$, with the Wiener measure and we shall use the following notations in this case :

If U, U_1 and U_2 are random functions with values in H ; if H is the Cameron-Martin space, then

$$U(\omega)\ (\bullet) = \int_0^\bullet \overset{\bullet}{u}(\theta,\omega)\ d\theta$$

$$\delta(U) = \int_0^1 \overset{\bullet}{u}\ (\theta,\omega)\ \delta_\theta(W)$$

$$\langle U_1, U_2\rangle_H = \int_0^1 \overset{\bullet}{u}_1\ (\theta,\omega)\ \overset{\bullet}{u}_2\ (\theta,\omega)\ d\theta\,.$$

But if H is the $L^2[0,1]$ space

$$U(\omega)\ (\bullet) = u(\bullet,\omega)$$

$$\delta U = \int_0^1 u(\theta,\omega)\ \delta_\theta(W)$$

$$\langle U_1, U_2\rangle_H = \int_0^1 u_1(\theta,\omega)\ u_2(\theta,\omega)\ d\theta$$

$$(T_t\omega)\ (\bullet) = \omega(\bullet) + \int_0^t \rho(r,\bullet)\ \sigma(r, T_r\omega)\ dr \tag{C}$$

where ρ is a smooth function on $[0,1]^2$ and $\sigma : [0,1]\times\Omega \to \mathbb{R}$ is assumed to satisfy Lipschitzian and differentiability conditions.

In terms of $\varphi_{s,t}$ and $\psi_{s,t}$, $(s \le t)$ we have :

$$\varphi_{s,t}(\omega)\ (\bullet) = \omega(\bullet) + \int_s^t \rho(r,\bullet)\ \sigma\big(r, \varphi_{s,r}(\omega)\big)\ dr$$

$$\psi_{s,t}(\omega)\ (\bullet) = \omega(\bullet) - \int_s^t \rho(r,\bullet)\ \sigma\big(r, \psi_{r,t}(\omega)\big)\ dr.$$

We consider these equations as ODE in Banach space (the first in t with s fixed ; the second in s for t fixed), we have existence and unicity of solutions with

$$\varphi_{s,s}(\omega) = \omega\,,\ \ \psi_{t,t}(\omega) = \omega \quad \text{and} \quad \varphi_{s,t}\circ\psi_{s,t}(\omega) = \omega.$$

Then $\psi_{s,t}(\omega)$ and $\varphi_{s,t}(\omega)$ are Fréchet differentiable in $\omega \in \mathcal{C}_0([0,1])$.

Consequently, $\partial\varphi_{s,t}$ and $\partial\psi_{s,t}$ restricted to H are invertible, and by Ramer's theorem: $\varphi_{s,t}(P)$, $\psi_{s,t}(P)$ and P are equivalent.
Set

$$L_{s,t}(\omega) = \frac{d\varphi_{s,t}(P)}{dP}$$

and

$$\Lambda_{s,t} = \frac{d\psi_{s,t}(P)}{dP}.$$

Now let us fix t in the equation :

$$T_t\omega\ (\bullet) = \omega(\bullet) + \int_0^t \rho(r,\bullet)\ \sigma(r, T_r\omega)\ dr.$$

Let $s = t - \lambda$ and $\lambda \in [0,t]$ be the interpolation parameters.
Now let us recall that $\big($cf (3)$\big)$

$$X_t = \exp\Big\{ -\delta(G_t) - \frac{1}{2}\ \langle G_t, G_t\rangle_H - \int_0^t \text{Trace}\ \Big(\nabla\big[\frac{dG_s}{ds} \circ T_s\big] \circ T_s^{-1} \bullet \nabla G_s\Big)\ ds\Big\} \qquad (D)$$

where $G_t = T_t^{-1} - Id$, and apply the result for T_t satisfying the relation :

$$T_t\omega\ (\bullet) = \omega(\bullet) + \int_0^t \rho(r,\bullet)\ \sigma(r, T_r\omega)\ dr.$$

Then we obtain an expression for X_t :

$$\begin{aligned} X_t = \exp\Big\{ &\int_0^1 \Big[\int_0^t \frac{\partial\rho}{\partial\theta}\ (r,\theta)\ \sigma(r,\psi_{0,r})dr\Big]\delta_\theta(W) \\ &- \frac{1}{2}\int_0^1 \Big[\int_0^t \frac{\partial\rho(r,\theta)}{\partial\theta}\ \sigma(r,\psi_{0,r})\ dr\Big]^2\ d\theta \\ &- \int_0^t\int_0^t\int_0^t \Big[\int_0^\lambda \frac{\partial\rho(r,\eta)}{\partial\eta}\ D_\theta\ \sigma(r,\psi_{0,r})\ dr\Big] \circ \frac{\partial\rho(\lambda,\theta)}{\partial\theta}\ \big(D_\eta\ \sigma(\lambda,\bullet)\big) \circ \psi_{0,\lambda}\ d\lambda\ d\theta\ d\eta\Big\}. \end{aligned}$$

We can obtain another formula for the Radon-Nikodym density using the relation :

$$\delta(aU) = a\delta U - \langle\nabla a, U\rangle_H$$

in the expression :

$$X_t(\omega) = \exp\left\{\int_0^t \left(\delta\left[\frac{dT_s}{ds} \circ T_s^{-1}\right]\right) \circ T_s T_t^{-1}(\omega)\, ds\right\}.$$

We then obtain :

$$L_{s,t} = \exp\left\{ \int_s^t \sigma(r, \psi_{r,t})\left[\delta\rho(r, \bullet) - \int_s^r \sigma(u, \psi_{u,t})\ \langle\rho(r,\bullet),\ \rho(u,\bullet)\rangle_H\ du\right] dr \right.$$
$$\left. - \int_s^t \langle(\nabla\sigma)(r, \psi_{r,t}), \rho(r, \bullet)\rangle_H\ dr\right\}.$$

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