ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 3, nº 2 (1996), p. 67-109
<http://www.numdam.org/item?id=AMBP_1996__3_2_67_0>

Ann. Math. Blaise Pascal, Vol. 3, N° 2, 1996, pp.67-109

ON THE HYPERGROUPS WITH FOUR PROPER PAIRS AND WITHOUT SCALARS [1]

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1. INTRODUCTION.-

We remember that a **hypergroup** H is a non-empty set equipped with a hyperoperation such that the following two conditions are satisfied:

(1.1) $\forall (x,y,z) \in H^3$, $(xy)z = x(yz)$ (**associativity**);
(1.2) $\forall x \in H$, $Hx = xH = H$ (**reproducibility**).

Given a hypegroup H, we say that a pair $(x,y) \in H^2$ is **proper**, if the hyperproduct $x \cdot y$ is not a singleton. Moreover an element $x \in H$ is said to be a **left scalar** (respect. **right scalar**), if $\forall\ y \in H$, (x,y) (respect. (y,x)) is not a proper pair. An element is called **scalar** if it is at the same time left scalar and right scalar.

In [3], [4], [5], Freni, Gutan C., Gutan M. and Sureau Y. have determined all the hypergroups which have at the most three proper pairs.

In the present paper the authors succeed in finding all the hypergroups with exactly **four proper pairs** and **without right scalars**. The corresponding left case can be obtained for symmetry. (We remember that if the set of scalars is non-empty, then the set of left scalars is equal to the set of right scalars (see [3]). This case is handled in other papers from the same authors).

Obviously we have that $|H| \in \{2, 3, 4\}$, otherwise there would exist at least one scalar.

In the rest H denotes a hypergroup without right scalars, P(H) the set of proper

[1] This work is produced by support of the Italian M.U R.S.T. (quota 40%). No version of this paper will be published elsewhere.

pairs of H and S the set of left scalars of H. Besides $M = H\setminus S$ indicates the complement of S in H and U means the set of left scalar identities (the elements $a \in H$ such that $\forall\, x \in H$, $a \cdot x = x$).

If $|H| = 2$, we obtain only the total hypergroup of size two.

If $|H| \in \{3,4\}$, then we have to consider two cases :

(I) $S = \emptyset$; (II) $S \neq \emptyset$.

2. SOME PRELIMINARY RESULTS.-

In this section, H will denote a hypergroup (finite or infinite).
Every element $a \in S$ defines a map $\hat{a}: H \to H$ such that $\forall\, x \in H$, $\hat{a}(x) = a \cdot x$. This map is clearly surjective, in consequence of reproducibility.

We prove now :

(2.1) PROPOSITION.- (i) $\forall\, a \in S$, $a \cdot S \subseteq S$ and $a \cdot M \supseteq M$;
(ii) If $\hat{a}$ is injective, then $a \cdot S = S$ and $a \cdot M = M$.

Proof.- (i) Let $b \in S$. We have $\forall\, x \in H$, $|a \cdot b \cdot x| = 1$ and so $a \cdot b \in S$. This proves that $a \cdot S \subseteq S$ and consequently $a \cdot M \supseteq M$.
(ii) Let $\hat{a}$ injective. Now, $\forall b \in M$, $\exists\, x \in H$ such that $|b \cdot x| > 1$, whence $|a \cdot b \cdot x| > 1$. It follows that $a \cdot b \in M$ and so $a \cdot M \subseteq M$. For (i), we may write $a \cdot M = M$ and thus $a \cdot S = S$

□

As an immediate consequence we obtain the

(2.2) COROLLARY.- S is a subsemigroup of H.

We demonstrate now the

(2.3) PROPOSITION.- $\forall\, a \in H$, $a \in U$ if and only if $a \in S$ and $\exists\, b \in S$ such that $a \cdot b = b$.

Proof.- If $a \in U$, the implication is trivial. Conversely, $\forall\, x \in H$ and $\forall\, b \in S$, $\exists\, y \in H$ such that $x = b \cdot y$. It follows that $a \cdot x = a \cdot b \cdot y = b \cdot y = x$, $\forall\, x \in H$. This completes the proof.

□

We go on with the :

(2.4) PROPOSITION.- If S is finite, then $\forall\, c \in S$, the map $\hat{c}$ defines a permutation of H of finite order.

Proof.- Certainly there exist an integer $n \geq 1$ and an element $b \in S$ such that $c^n \cdot b = b$ and so , for (2.3), $c^n \in U$ and consequently $\hat{c}$ is injective.

□

From (ii) of (2.1) and (2.4), it follows that :

(2.5) COROLLARY.- If S or M is finite, then $\forall a \in S$, $a \cdot S = S$ and $a \cdot M = M$.

(2.6) COROLLARY.- If S is finite, then $\forall c \in S$, $\exists\, n \geq 1$ such that $c^n \in U$.

Now, we prove the following :

(2.7) PROPOSITION.- Let $a \in U$, $b \in U$, $c \in H$ such that $c \cdot a = b$, then $c \in U$. This means that $\forall\, c \in S$, $c \in U \Leftrightarrow U \cap c \cdot U \neq \emptyset$.

Proof.- In fact we have $\forall\, x \in H$, $c \cdot x = c \cdot a \cdot x = b \cdot x = x$

□

(2.8) COROLLARY.- If U is finite and $|U| > |S \setminus U|$ then $S = U$.

Proof.-For (2.4), taking $c \in S$, the map $\hat{c}$ permutes the element of S, whence for the hypothesis, one deduces that $c \cdot U \cap U \neq \emptyset$, so that, for (2.7), $c \in U$.

□

(2.9) PROPOSITION.- If $a \in U$, $b \in M$ and $b \cdot a = c$, then $\hat{c} = \hat{b}$ and in particular $c \cdot a = c \in M$.

Proof.- $\forall\, x \in H$, we have $c \cdot x = b \cdot a \cdot x = b \cdot x$ and thus $\hat{c} = \hat{b}$. Now $c \cdot a = b \cdot a = c$, whence $\exists\, x$ such that (b,x) is proper. Therefore also (c,x) is proper and $c \in M$.

□

Now, we give two useful results whose proofs can be trivially obtained :

(2.10) PROPOSITION.- If $x \in H$ and $x \cdot x$ is a singleton, then $x \cdot x \in \{y \in H / x \cdot y = y \cdot x\}$.

(2.11) PROPOSITION.- Let $x \in H$. If K is a subset of H with at least two elements,

such that $K \cdot x$ is a singleton and a, b are the only two elements such that $a \cdot x$ and $b \cdot x$ are singletons, then $K = \{a, b\}$ and $a \cdot x = b \cdot x$.

A hypergroup H is **thin to right**, if $\forall\ y \in H$ **there exists an unique element $x \in H$ such that the pair (x,y) is proper.**
Obviously if H is thin to right, then there are no right scalars and $|P(H)| = |H|$.

We conclude this section with the following :

(2.12) PROPOSITION.- If H is thin to right and $b \in M$, then :

(j) $a \in S \Rightarrow a \cdot b = b$;

(jj) $a \in U \Rightarrow b \in b \cdot a$.

Proof.- (j). For reproducibility, there exists an element $x \in H$ such that $b = a \cdot x$. Moreover, there exists $y \in H$ such that the pair (b, y) is proper and consequently $b \cdot y = a \cdot x \cdot y$ implies that (x, y) is also proper. But H is thin to right and thus $b = x$. This proves the implication.

(jj). There exist x and y such that $b \in x \cdot a$ and (b, y) is proper. Now $b \cdot y \subseteq x \cdot a \cdot y = x \cdot y$ and so (x, y) is also proper, and thus $b = x$. Whence the thesis.

□

(2.13) PROPOSITION.- Let H be thin to right and $b \in M$ such that (b,s) is a proper pair $\forall s \in S$, then :

(i) provided M is finite, there exists at most an element $m_0 \in M$ such that (b, m_0) is a proper pair ;

(ii) $m \cdot u = m$, $\forall (m,u) \in (M - \{b\}) \times U$;

(iii) $b \cdot s \subseteq \{b\} \cup S$, $\forall s \in S$;

(iv) $m \in b \cdot m$, $\forall m \in M$. Particularly if (b,m) is not a proper pair then $m = b \cdot m$;

Proof.- Statement (i) is immediate while (ii) follows directly from (jj) of (2.12) being H thin to right.

(iii) Let $z \in b \cdot s$ $(z \neq b)$, then $z \cdot m \subseteq (b \cdot s) \cdot m = b \cdot m$. Being H thin to right it results $|z \cdot m| = 1$, $\forall m \in M$ and $|z \cdot s| = 1$, $\forall s \in S$. Whence $z \in S$.

(iv) We have $b \cdot m = b \cdot (s \cdot m) = (b \cdot s) \cdot m$. Being $b \cdot s \subseteq \{b\} \cup S$, there exists $s_0 \in S$ such that $s_0 \in b \cdot s$ and so $m = s_0 \cdot m \in b \cdot m$.

□

(2.14) COROLLARY.- **In the same hypothesis of (2.13), if $S = \{a\}$ then :**

(i) $x \cdot a = x$, $\forall x \in H - \{b\}$;

(ii) $b \cdot a = \{a, b\}$;

(iii) **provided M is finite,** $b \cdot x = x$, $\forall x \neq a$ **with the exception at most of one element** $m_0 \neq a$ **for which** $m_0 \in b \cdot m_0$ **and** (b, m_0) **is a proper pair.**

Moreover supposing $H = \{a, b, c, d\}$ **one obtains:**

(iv) **if** $x \in \{c, d\}$ **and** $|x \cdot b| = 1$, **then** $x \cdot b = x$;

(v) **let** $b \cdot m = m$, $\forall m \neq a$ **and** $(x, y) \in \{c, d\}^2$. **If** $|x \cdot y| = 1$ **and** (x, b) **is a proper pair, then** $x \cdot y = y$.

Proof.- (i), (ii) and (iii) are trivial.

(iv) Suppose for contradiction $x \cdot b \neq x$. Then from $(x \cdot b) \cdot a = x \cdot (b \cdot a)$ we deduce $(x \cdot b) \cdot a = \{x\} \cup (x \cdot b)$ and so $|(x \cdot b) \cdot a| \geq 2$. Being H thin to right, it follows $x \cdot b = b$ and so $\{a, b\} = \{x, b\}$, which is a contradiction.

(v) From $(x \cdot b) \cdot y = x \cdot (b \cdot y) = x \cdot y$, we obtain $(x \cdot b) \cap \{a, b\} \neq \emptyset$, otherwise $x \cdot b = \{c, d\}$ and so $c \cdot y = d \cdot y$, which is absurd because H is thin to right. Being $a \cdot y = b \cdot y = y$, the statement follows .

□

Given a hypergroup $(H, \circ)$, it is possible to consider the hypergroup $(H, *)$ equipped with the hyperoperation $*$ such that $\forall (x,y) \in H^2$, $x * y = y \circ x$. $(H, *)$ will be called the **symmetric hypergroup of** $(H, \circ)$.

Later on, H will denote always a hypergroup without scalars and with four proper pairs.

In order to simplify the writing, when a hyperproduct $x \cdot y$ is a singleton $\{\alpha\}$, we can omit the brackets and write $x \cdot y = \alpha$ instead of $x \cdot y = \{\alpha\}$. As regards the multiplicative table of H, we agree to denote with capital letters the hyperproducts which correspond to the proper pairs .

3. THE CASE $S=\emptyset$; $|H|=3$.

We have to find, up to isomorphisms, all the hypergroups $H=\{a,b,c\}$, having exactly four proper pairs and such that $\forall j \in H$ there exists at least an element $i \in H$ for which the pair (i,j) is proper.

We can suppose that the four proper pairs are of the following type: (a,m), (a,n), (b,p) and (c,q). We have, up to isomorphisms, only the following eight types, which are listed in the lexicographic ordering:

(3_1) P(H)={(a,a),(a,b),(b,a),(c,c)};
(3_2) P(H)={(a,a),(a,b),(b,b),(c,c)};
(3_3) P(H)={(a,a),(a,b),(b,c),(c,a)};
(3_4) P(H)={(a,a),(a,b),(b,c),(c,b)};
(3_5) P(H)={(a,a),(a,b),(b,c),(c,c)};
(3_6) P(H)={(a,b),(a,c),(b,a),(c,a)};
(3_7) P(H)={(a,b),(a,c),(b,a),(c,b)};
(3_8) P(H)={(a,b),(a,c),(b,a),(c,c)}.

We will study separately the eight possible types. As regards the types (3_2), (3_3), (3_5), one does not obtain any hypergroup. For everyone of the types (3_4), (3_7), (3_8), one finds only one hypergroup. The type (3_1) gives rise to four hypergroups. The richest type of all is the type (3_6), which we will study lastly.

The next table summarizes the results:

type	hypergroups	type	hypergroups
(3_1)	4	(3_5)	0
(3_2)	0	(3_6)	12
(3_3)	0	(3_7)	1
(3_4)	1	(3_8)	1

In the following, it will be convenient to put:

am=M, an=N, bp=P, cq=Q,

and to denote the remaining five products, in the order, with **r, s, t, u, v**. Moreover, we agree to indicate the relation of associativity **(xy)z=x(yz)**, simply by **(xyz)**.

TYPE (3_1).

	a	b	c
a	M	N	r
b	P	s	t
c	u	v	Q

$|aQ| > 1$

(acc) rc=aQ and so r=c.

(aac) Mc=ar=ac=c thus M={a,b} and t=c. For symmetry, u=v=c.

(abc) Nc=at=ac=c hence N={a,b} and, for symmetry, P=N.

(bbc) sc=bt=bc=c and so $s \in \{a,b\}$.

(acc) Q=aQ : if $b \in Q$ then $ab=N \subseteq Q$,
if $a \in Q$ then $aa=M \subseteq Q$; consequently $\{a,b\} \subseteq Q$

Therefore, there are the following **four** solutions:

	a	**b**	**c**
a	**a,b**	**a,b**	**c**
b	**a,b**	**s**	**c**
c	**c**	**c**	**Q**

$s \in \{a,b\}$, $Q \in \{H, H \setminus \{c\}\}$.

One verifies that all four are hypergroups.

TYPE (3_2).

	a	b	c
a	M	N	r
b	s	P	t
c	u	v	Q

$|aQ| > 1$
$|Qb| > 1$

(acc) rc=aQ then r=c.

(ccb) Qb=cv then v=c.

(aac) Mc=ar=ac=c then M={a,b}.

(cbb) vb=cP, that is cP=cb=v=c and so P={a,b}.

(baa) sa=bM $\supseteq$ bb=P thus s=a.

(bba) Pa=bs=ba=a whence M=aa $\subseteq$ Pa=a, absurd.

There are no solutions.

TYPE (3_3).

	a	b	c
a	M	N	r
b	s	t	P
c	Q	u	v

|aQ| > 1

bb=t=b , cc=v=c (lemma 2.10)

(aca) ra=aQ then r $\in$ {a,c}.

(bba) s=bs then s $\in$ {a,b}.

(ccb) u=cu then u $\in$ {b,c}.

(bac) sc=br. If r=a then sc=ba=s thus s $\neq$ b whence s=a.

(acb) au=rb=ab=N then u=b.

(cba) s=cs=ca=Q , absurd.

If r=c then sc=bc=P and so s=b.

(acb) au=rb=cb=u then u=c.

(cba) Q=cb=c, absurd.

There are no solutions.

TYPE (3_4).

	a	b	c
a	M	N	r
b	s	t	P
c	u	Q	v

|aQ| > 1

|Qb| > 1

(cbb) Qb=ct then t=b.

(acb) rb=aQ then r $\in$ {a,c}.

(acc) rc=av : if r=a then a=av and so v=c; if r=c then v=av and thus v=c;

in every case, it is **v=c**.

(bba) ba=bs , that is s=bs then $s \in \{a,b\}$.

If s=b then:

(bac) P=br then r=c;

(aac) Mc=ar=ac=r=c then M={a,c};

(baa) sa=bM, that is $b=ba \cup bc \supseteq P$, impossible.

Therefore **s=a.**

(cba) Qa=cs=ca=u then **Q={b,c}** and ba=ca=u , whence **u=a.**

(aca) ra=au=aa=M then **r=a.**

(bca) Pa=bu=ba=s=a then **P={b,c}.**

(aba) Na=aa=M then $a \in M \cap N$.

(baa) M=bM.

(caa) M=cM.

If $b \in M$ then $cb \subseteq M$ and so $\{b,c\} \subseteq M$; if $c \in M$ then $bc \subseteq M$ and so $\{b,c\} \subseteq M$; in any case we have **M=H.**

Analogously from (bab) and (cab), one obtains **N=H.**

There is only **one** solution, which is the following hypergroup:

	a	b	c
a	H	H	a
b	a	b	b,c
c	a	b,c	c

TYPE (3_5).

	a	b	c
a	M	N	r
b	s	t	P
c	u	v	Q

|Mc| > 1

|aQ| > 1

t=bb=b (lemma 2.10)

(aac) Mc=ar then $r \in \{a,b\}$.

(acc) rc=aQ then $r \in \{b,c\}$.

Therefore r=b.

(bac) sc=br=bb=t=b then s=a.

(aca) ra=au, that is a=au whence u=c and b=r=ac=au=a, impossible.

There are no solutions.

TYPE (3_7).

	a	b	c
a	r	M	N
b	P	s	t
c	u	Q	v

$|aN|>1$

$|Qb|>1$

v=cc=c (lemma 2.10)

(aac) rc=aN and so **r=a.**

(cbb) Qb=cs thus s=b.

(bbc) t=bt then $t\neq a$.

(cca) u=cu then $u\neq b$.

If t=b :

(bcb) b=bQ and so Q={b,c}; if t=c : (cbc) Qc=c then Q={b,c}; in every case it results Q={b,c}.

If u=a :

(aca) Na=a then N={a,c}; if u=c : (cac) c=cN hence N={a,c}; it follows that

N={a,c}.

(bca) ta=bu then ($t=b \Leftrightarrow u=a$) and ($t=c \Leftrightarrow u=c$).

Suppose t=c and then u=c;

(cba) Qa=cP thus $ba\cup ca=cP\subseteq\{b,c\}$, that is $P\cup\{c\}\subseteq\{b,c\}$, and then P={b,c};

(bac) Pc=bN , then $c=P\cup\{c\}$, impossible.

Therefore t=b and u=a.

(acb) Nb=aQ , thus $ab\cup cb=ab\cup ac$ hence $M\cup\{b,c\}=M\cup\{a,c\}$ and so $M\supseteq\{a,b\}$.

(cab) M=cM then $c\in M$ and M=H.

(baa) Pa=P then $a\in P$.

(bac) $Pc=bN=P\cup\{b\}$ then $c\in P$ whence P=H.

One obtains an **unique solution**, which is a hypergroup.

	a	b	c
a	a	H	a,c
b	H	b	b
c	a	b,c	c

TYPE (3_8).

	a	b	c
a	r	M	N
b	P	s	t
c	u	v	Q

By the cyclic permutation (abc), we obtain:

	c	a	b
c	Q	u	v
a	N	r	M
b	t	P	s

the symmetric of the type (3_4). Therefore we have the following **unique solution:**

Q=N=H , P=M={a,b} , r=a , s=b , t=u=v=c ,

that is the hypergroup below:

	a	b	c
a	a	a,b	H
b	a,b	b	c
c	c	c	H

TYPE (3_6).

	a	b	c
a	r	M	N
b	P	s	t
c	Q	u	v

We begin to observe that:

1. **r=a.**
2. **s=a ⇔ t=a ⇔ v=a ⇔ u=a.**
3. **If s=a then t=u=v=a and M=N=P=Q={b,c}, unique solution in this case which characterizes a hypergroup.**
4. **If v=b then s=b and t=u.**

Proof.-

1. (baa) Pa=br then r=a.
2. (bbc) sc=bt. If s=a then N=bt thus t=a and N=P.
   (bcc) tc=bv. If t=a then N=bv then v=a.
   (ccb) vb=cu. If v=a then M=cu then u=a and M=Q.
   (cbb) ub=cs. If u=a then M=cs then s=a.
3. Let s=a. Therefore, for 2. , t=u=v=a , and M=Q and N=P for what is preceding.
   (bcb) tb=bu then ab=ba that is M=P.
   (bba) sa=bP then a=bP whence P={b,c}.
4. Suppose v=b. Then, in particular, $u \neq a$.
   (ccb) vb=cu then s=cu whence, for $u \in \{b,c\}$, one finds s=b.
   (ccc) vc=cv then bc=cb that is t=u.

□

In case of s=a, in accordance with the 3. above, we obtain the **unique** following hypergroup:

	a	b	c
a	a	b,c	b,c
b	b,c	a	a
c	b,c	a	a

As regards the remaining cases s=b, s=c, one observes that by exchanging b and c we obtain an automorphism for the type 3_6 and, by using the 2. and 4. above, we can limit ourselves to study only the following five cases, where s=b. The table summarizes in advance the results:

	v	t	u	semi-hypergroups	hypergroups
C1	b	b	b	5	4
C2	b	c	c	4	1
C3	c	b	b	5	4
C4	c	b	c	2	1
C5	c	c	b	2	1

The case (c,c,c) is isomorphic to C3. The last two cases C4 and C5 are symmetric one another.

Case C1

	a	b	c
a	a	M	N
b	P	b	b
c	Q	b	b

One obtains the following five solutions. Only the first one is not a hypergroup.

M=N=P=Q={a,b};

M=P=H and N,Q $\in$ {{a,c},H}.

Proof.-

(acc) Nc=M then $a \in N$, $b \in M$ and $N=ac \subseteq Nc=M$. Therefore $a \in N \subseteq M$ and $\{a,b\} \subseteq M$.

Symmetrically, $a \in Q \subseteq P$ and $\{a.b\} \subseteq P$.

(aac) N=aN. If $b \in N$ then $N \supseteq ab=M$ then M=N.

We have M=P; in fact:

(bab) $Pb=bM=P \supseteq \{b\}=P$, but $a \in P$ and so $M=ab \subseteq bM=P$, and symmetrically,

$P \subseteq M$.

Finally, if $M=\{a,b\}$ then $M=N=P=Q$. If $M=H$ then $P=H$, and $b \in N \Leftrightarrow N=H$.

□

Case C2.-

	a	b	c
a	a	M	N
b	P	b	c
c	Q	c	b

We are going to show that $M=N \in \{\{b,c\},H\}$, $P=Q \in \{\{b,c\},H\}$, and so there are four solutions, only one of which is a hypergroup.

Proof.-

(abc) $Mc=N$. If $b \notin N$ then $N=\{a,c\}$ and $M=\{a,b\}$.

(acb) $Nb=N$ then $M=ab \subseteq Nb=N$, absurd. Thus $b \in N$.

(aac) $N=aN$ then $M=ab \subseteq aN=N$.

(acc) $Nc=M$ therefore $c=bc \subseteq Nc=M$ and so $c \in M \subseteq N$, and $b=cc \subseteq Nc=M$ then $b \in M$.

Consequently $\{b,c\} \subseteq M \subseteq N$.

If $a \in N$ then $N=ac \subseteq Nc=M$ and so $M=N$.

Thus $M=N \in \{\{b,c\},H\}$. Symmetrically, $P=Q \in \{\{b,c\},H\}$.

□

Case C3.-

	a	b	c
a	a	M	N
b	P	b	b
c	Q	b	c

One obtains the following five solutions:

$M=N=P=Q=\{a,b\}$;

$M=P=H$ and $N,Q \in \{\{a,c\},H\}$.

Proof.-

(acb) $Nb=M$ and so $a \in N$ and $b \in M$; symmetrically $a \in Q$ and $b \in N$.

(aab) $M=aM$. If $c \in M$ then $a \in N=ac \subseteq aM=M$ thus $M=H$. That is $M \in \{\{a,b\},H\}$.

Analogously for P.

(bab) $Pb=bM$. If $M=\{a,b\}$ then $Pb=ba \cup bb=P \cup \{b\}=P \subseteq M$ then $M=P$. Otherwise, we have $M=H$, $Pb=P$ and so $a \in P$ and $H=M=ab \subseteq P$. In every case, one has $M=P$. Symmetrically $N=Q$.

(aac) $N=aN$. If $M=H$ then $(b \in N \Leftrightarrow N=H)$ whence $(N=\{a,c\} \Leftrightarrow N \neq H)$. If $M \neq H$ then $M=N=P=Q=\{a,b\}$; in fact:

(bac) $Pc=bN$ then $ac \cup bc=bN$, that is $N \cup \{b\}=bN \subseteq P \cup \{b\}=P$ whence $N=P$ and symmetrically $M=Q$.

□

Case C4.-

	a	b	c
a	a	M	N
b	P	b	b
c	Q	c	c

We will prove that there are two solutions, unless isomorphisms:

$M=N=P=\{a,b\}$ and **$Q=H$** (isomorphic to **$M=N=Q=\{a,c\}$** and **$P=H$**);

$M=N=P=Q=H$ which is a hypergroup.

Proof.-

(bca) $P=bQ$. $\{b,c\} \cap Q \neq \emptyset$ and so $b \in P$ and $a \in Q$.

(cba) $Q=cP$. $\{b,c\} \cap P \neq \emptyset$ and so $c \in Q$ and $a \in P$.

Therefore $\{a,b\} \subseteq P$ and $\{a,c\} \subseteq Q$.

If $P=\{a,b\}$ then $M=N=P$ and $Q=H$.

In fact:

(bab) $Pb=bN$ that is $M \cup \{b\}=bM \subseteq P$ then $M=P$.

(bac) $Pc=bN$ that is $N \cup \{b\}=bN \subseteq P$ then $N=P$.

(cac) $Qc=cN=ca \cup cb=Q \cup \{c\}=Q$: $b \in N=ac \subseteq Qc=Q$ thus $Q=H$.

Analogously, if $Q=\{a,c\}$ then $M=N=Q$ and $P=H$.

□

	a	b	c
a	a	a,b	a,b
b	a,b	b	b
c	H	c	c

	a	b	c
a	a	H	H
b	H	b	b
c	H	c	c

In all, there exist 19 hypergroups, such that $|H|=3$, $S_I(H)=S_J(H)=\emptyset$.

4. THE CASE $S=\emptyset$; $|H|=4$.

Suppose $H=\{a,b,c,d\}$.

Up to isomorphism, it suffices to study the following five cases (in lexicographical ordering) :

(4_1) $P(H)=\{(a,a),(b,b),(c,c),(d,d)\}$;

(4_2) $P(H)=\{(a,a),(b,b),(c,d),(d,c)\}$;

(4_3) $P(H)=\{(a,a),(b,c),(c,d),(d,b)\}$;

(4_4) $P(H)=\{(a,b),(b,a),(c,d),(d,c)\}$;

(4_5) $P(H)=\{(a,b),(b,c),(c,d),(d,a)\}$.

In this section, for convenience, we shall put $am=M$, $bn=N$, $cp=P$ and $dq=Q$.

CASE (4_1).

In this case the table is of the following type:

	a	b	c	d
a	M			
b		N		
c			P	
d				Q

Let x,y be two distinct elements of H such that $y \in xx$, then

(xxy) $|x(xy)| > 1$ and then $xy = x$.

(yxx) $|(yx)x| > 1$ and then $yx = x$.

Without loss of generality, we can suppose $b \in M$ and thus $ab = ba = a$.

(aab) we obtain $M \cdot b = M$ and so $N \subseteq M$.

(bba) we obtain $N \cdot a = \{a\}$ and so $a \notin N$.

We can suppose $c \in N$ and thus $cb = bc = b$.

(bbc) we obtain $N \cdot c = N$ and so $P \subseteq N$.

(ccb) we obtain $P \cdot b = \{b\}$ and so $b \notin P$ and then $P = \{c,d\}$ and $cd = dc = c$.

(ccd) we obtain $P \cdot d = P$ and so $Q \subseteq P$.

(ddc) we obtain $Q \cdot c = \{c\}$ and so $c \notin Q$ that implies $Q = \{d\}$, absurd.

Then **there don't exist hypergroups** .

CASE (4_2).

In this case the table is of the following symmetric type:

	a	b	c	d
a	M			
b		N		
c				P
d			Q	

If we suppose that $M = N = \{a,b\}$, then (aab) and (bba) give respectively $ab = a$ and $ba = b$. Taking in account the symmetry of H we obtain a contradiction. Then, without loss of generality, we can suppose $c \in M$.

(aad) $|a(ad)| > 1$ and so $ad = a$ and by symmetry of H $da = a$.

(adc) $|a(dc)| = 1$ and so $a \notin Q$. By symmetry of H, we have $a \notin P$.

Now we can prove that :

1. $dd = d$.
2. $c \in P \cap Q$.
3. $bd = db = b$.
4. $bc = cb = b$ and $P = Q = \{c,d\}$.
5. $cc = c$ and $ac = ca = a$
6. $\{c,d\} \subseteq M$ and $ba \in \{a,b\}$.

Proof.-

1. (add) gives $dd \neq a$.
   If $dd = b$, (ddb) gives $db = c$ and so from (ddc) we obtain $|bc| > 1$ which is absurd.
   If $dd = c$, (ddc) gives $|dQ| = 1$ and so $Q = \{b,d\}$ and $db = dd = c$. This is impossible because (ddb) gives $|cb| > 1$. So $dd = d$.
2. (bdd) gives $bd \neq c$ and by (1.2) $c \in P$. By symmetry of H, we have $c \in Q$.
3. (cdd) and (cda) give $P = Pd$ and $ca = Pa$.
   The proof of 2. gives $bd \in \{a,b,d\}$.
   If $bd = a$, by (1.2) $b \in P$ and so $a \in P$. This is impossible.
   If $bd = d$, by (1.2) $b \in P$. $P = Pd$ gives $d \in P$ and so $P = \{b,c,d\}$. From $ca = Pa$, we obtain $ba = ca = da = a$.
   (bbd) implies $\{d\} = Nd$ and so $N = \{b,d\}$. By (1.2) $bc = c$ then , by (bbc) we have $\{c\} \cup Q = \{c\}$. This is absurd, therefore $bd = b$ and by symmetry of H $db = b$.
4. (bdc) $bQ = bc$ that implies $b \notin Q$ and so $Q = \{c,d\}$ and $bc = bd = b$. By symmetry of H we have $P = \{c,d\}$ and $cb = b$.
5. (dcc) $Qc = cc \cup Q$ and so $|d(cc)| > 1$ gives $cc = c$
   (adc) $aQ = ac$ and so $ad = ac = a$. By symmetry of H we have $ca = a$
6. (aad) $Md = M$ and so, being $c \in M$, $P = \{c,d\} \subseteq M$.
   (baa) $bM = (ba)a$. Being $b = bc \subseteq bM$ we obtain $ba \in \{a,b\}$.

□

Now we observe that by (1.2), $c \in N$ and so, taking in account that the permutation (ab)(c)(d) is an isomorphism, we can suppose $ba = a$.

(bba) gives a=Na and so $a \in M$. By (1.2), we have **ab=a** and $M \supseteq \{b,c,d\}$.

We obtain the following **four hypergroups** :

	a	b	c	d
a	M	a	a	a
b	a	N	b	b
c	a	b	c	c,d
d	a	b	c,d	d

with $M \in \{H\text{-}\{a\}\ ;\ H\}$ **and** $N \in \{H\text{-}\{a,b\}\ ;\ H\text{-}\{a\}\}$

CASE (4_3).

The table is of the following type:

	a	b	c	d
a	M			
b			N	
c				P
d		Q		

We observe that the permutations (a)(b,c,d) and (a)(b,d,c) give tables of the same kind of case (4_3) and so, we can suppose that $d \in M$.

(aab) $M \cdot b = a(ab)$. Since $|M \cdot b| > 1$, we have $ab = a$.

(aba) $ba = a$.

In the same manner, (caa) and (aca) give $ca = ac = a$.

Now, (dac) , (dab) and (bad) give $da \notin \{b,d\}$ and $ad \neq c$ and so. by (1.2), $M \supseteq \{b,c,d\}$.

Finally (daa) and (aad) give $da = ad = a$.

It is easy to see that $H' = \{b,c,d\}$ is a sub-hypergroup of H and this is absurd because in [3] one has showed that there are no hypergroups on H' of this kind.

Then **there don't exist hypergroups** .

CASE (4_4).

We have a table of the following symmetric type:

	a	b	c	d
a		M		
b	N			
c				P
d			Q	

We observe that the permutations (a)(b)(c,d); (a,b)(c,d); (a,b)(c)(d); (a,c)(b,d) and (a,d)(b,c) give tables of the same kind.

Suppose that $|xy| > 1$ implies $xy = \{x,y\}$. In this case $M = N = \{a,b\}$ and $P = Q = \{c,d\}$.

(aab) , (bba) , (ccd) = c(cd) and (ddc) give zz=z, $\forall z \in H$.

By (1.2), $ac \neq bc$. (abc) gives $ac \cup bc = a(bc)$ and so $bc = b$.

This is absurd, because (1.2) is not satisfied, then there are x,y such that $|xy| > 1$ and $xy \neq \{x,y\}$. Without loss of generality we can suppose $c \in ab = M$.

(abd) and (dab) give $\mathbf{bd = b}$, $\mathbf{da = a}$, $\mathbf{P \subseteq M}$, $\mathbf{Q \subseteq M}$.

Now we can prove the following :

1. $dd = d$.
2. $c \in N$.
3. $db = b$, $ad = a$, $P = Q = \{c,d\}$, $cb = bc = b$, $ca = ac = a$, $cc = c$.
4. **If aa = a then M = N = H and bb = b .**
5. **If aa = b then bb = a and M = N = {c,d}.**

Proof.-

1. (dda) and (bdd) give $dd \in \{c,d\}$.
   If $dd = c$, then $ca = a$ and $bc = b$. From (bbd) it follows $bb \neq c$, and so, by (1.2), $c \in N$. (bad) and (dba) give $ad = a$ and $db = b$. Now, (cdd) gives $Pd = cc$ and it is absurd because $|P \cdot d| > 1$. Then $\mathbf{dd = d}$.
2. Let $c \notin N$. From (bbd) we have $bb \neq c$ and so, by (1.2), $bc = c$.
   Now, (bdc) gives $c = b \cdot Q$, which is absurd, because $|b \cdot Q| > 1$.
3. We have $c \in N$ and so we can use the symmetry of H which gives $\mathbf{db = b}$ and $\mathbf{ad = a}$.
   From (cda) and (bdc) we obtain $|P \cdot a| = 1$ and $|b \cdot Q| = 1$ and so $b \notin P$ and $a \notin Q$. By symmetry of H, we have $\mathbf{P = Q = \{c,d\}}$.

Now, from (cdb) and (cda) we obtain $cb = b$ and $ca = a$. By symmetry of H, we have also $bc = b$ and $ac = a$. Finally, from (dcc) we obtain $d(cc) = cc \cup \{c,d\}$ and so, $|d(cc)| > 1$, whence $cc = c$.

4. From (aab) we obtain $M = aM$ and so $a \in M$. By (1.2) and by symmetry of H, we obtain $M = N = H$. Since $(ab)b = H$, we obtain $bb = b$.
5. From (aab)we obtain $bb = aM$ and so $bb = a$ and $M = \{c,d\}$. By symmetry of H, it follows $N = \{c,d\}$.

□

Thus we obtain the following **two hypergroups** :

	a	b	c	d
a	a	H	a	a
b	H	b	b	b
c	a	b	c	c,d
d	a	b	c,d	d

	a	b	c	d
a	b	c,d	a	a
b	c,d	a	b	b
c	a	b	c	c,d
d	a	b	c,d	d

CASE (4_5).

	a	b	c	d
a		M		
b			N	
c				P
d	Q			

We observe that the permutations (a,b,c,d); (a,c)(b,d); (a,d,c,b) give tables of the same kind of (4_5).

Let $(x,y) \in P(H)$ such that $xy = \{x,y\}$. From (xxy) we obtain $(xx)y = xx \cup xy$ and so, $|(xx)y| \geq 2$ which gives $xx = x$. By (1.2), there exist x_0 and y_0 such that $|x_0y_0| > 1$ and $x_0y_0 = \{x_0,y_0\}$. Without loss of generality we can suppose $M = ab \neq \{a,b\}$.

Now we prove that :

1. $c \notin M$.
2. $d \notin M$.

Proof.-

1. Suppose $c \in M$. From (abd) we obtain $bd = b$. Analogously from (bab) we obtain $ba = a$ and $N \subseteq M$.
   From (bdc) and (dba), we have $dc = c$ and $db = d$.
   Now (aba) gives $|M \cdot a| = 1$ whence $d \notin M$ and so, $d \notin N$.
   By (1.2), it follows $bb = d$. It is absurd because (bba) gives $Q = \{a\}$.
2. Suppose $d \in M$. From (aba) we obtain $ba = b$, $Q \subseteq M$ (and $c \notin Q$) . Analogously from (cab) we obtain $ca = a$ and by (1.2), $aa = c$.
   This is impossible because (baa) gives $N = \{b\}$.

□

In consequence of 1. and 2. , we can say that **there don't exist hypergroups.**

5. THE CASE $S \neq \emptyset$; $|H|=3$.

If $S \neq \emptyset$, then $|H| \in \{3,4\}$.

We begin to study the case $|H|=3$. Since $|P(H)|=4$, necessarily the set of left scalars is a singleton.

We put $H=\{a,b,c\}$, $S=\{a\}$.

For (2.2), (2.3), a is left scalar identity.

Up to isomorphisms, we have the following **7 cases** selected so that there are at least two proper pairs (c,x) and (c,y) and listed in lexicographical ordering :

(5_1) $P(H) = \{(b,a),(b,b),(c,a),(c,c)\}$;
(5_2) $P(H) = \{(b,a),(b,b),(c,b),(c,c)\}$;
(5_3) $P(H) = \{(b,a),(b,c),(c,a),(c,b)\}$;
(5_4) $P(H) = \{(b,a),(b,c),(c,b),(c,c)\}$;
(5_5) $P(H) = \{(b,a),(c,a),(c,b),(c,c)\}$;
(5_6) $P(H) = \{(b,b),(c,a),(c,b),(c,c)\}$;
(5_7) $P(H) = \{(b,c),(c,a),(c,b),(c,c)\}$.

In consequence of (2.9) and (2.6), (2.11) we have respectively the following statements:

(5.1) .- If $|H|=3$, a is left scalar identity and $b \notin S_l(H)$, then: $|ba|=1 \Rightarrow ba=b$.
(5.2) .- If $|H|=3$, $S_l(H)=\{a\}$, $S_r(H)=\emptyset$, and x is an element such that $\exists s \in H$: $|xs|=1$, then we have:

(i) $|xa| \geq 2 \Rightarrow xa=\{a,x\}$.

(ii) $|xa| \geq 2$, $|xy|=1 \Rightarrow xy=y$

(iii) $|xa|=1 \Rightarrow xa=x$.

CASE (5_1).-

For (5.2), we have the following starting table:

	a	b	c
a	a	b	c
b	a,b	M	c
c	a,c	b	N

From (cbc) we obtain $N=\{c\}$, absurd. So in this case, **there don't exist hypergroups**.

CASE (5_2).-

Always for (5.2), we can write:

	a	b	c
a	a	b	c
b	a,b	N	c
c	c	P	Q

We have the following equalities:

1. $Q=H$,
2. $N=\{a,b\}$,
3. $P=H$.

Proof.-

1. (1.2) implies that $\{a,b\} \subseteq Q$. If $Q=\{a,b\}$, then (ccc) gives $\{c\}=\{c\} \cup P$, a manifest contradiction. Therefore $Q=H$.
2. This fact is a consequence of (2.11), since (bbc) gives $|Nc|=1$.
3. For (1.2), $c \in P$. (bcb) gives $P=bP$. Since $\{a,b\} \cap P \neq \emptyset$, it follows that $P=H$.

□

Therefore this case leads to **one hypergroup**:

	a	b	c
a	a	b	c
b	a,b	a,b	c
c	c	H	H

CASE (5_3).-

By using (5.2), we obtain immediately the following table:

	a	b	c
a	a	b	c
b	a,b	b	M
c	a,c	N	c

For (1.2), we have **M=N=H**, and thus we obtain **one hypergroup**:

	a	b	c
a	a	b	c
b	a,b	b	H
c	a,c	H	c

CASE (5_4).

(5.2) allows us to start from the following table:

	a	b	c
a	a	b	c
b	a,b	b	M
c	c	N	P

We have:

1. M=H ,
2. N=H ,
3. P $\in$ {{a,c},H}.

Proof.-

1. Obviously $c \in M$. Moreover (bbc) gives M=bM and so: $a \in M \Rightarrow b \in M$. Analogously, starting from (bca) we obtain that $b \in M \Rightarrow a \in M$. Since $M \cap \{a,b\} \neq \emptyset$, the assertion follows.
2. For (1.2), $\{a,c\} \subseteq N$. Moreover (cbb) gives Nb=N, whence, since $a \in N$, the thesis.
3. Since (bcc) implies H=bP, one has that $c \in P$. From (cca) we obtain Pa=P and so $b \in P \Rightarrow a \in P$.

□

In conclusion, we have **two hypergroups**:

	a	b	c
a	a	b	c
b	a,b	b	H
c	c	H	P

with P $\in$ {{a,c},H}.

CASE (5_5).

For (5.2), we can start from the following table:

	a	b	c
a	a	b	c
b	a,b	b	c
c	M	N	P

For (1.2), $c \in M$.

(bca) $bM=M$ and so $a \in M \Rightarrow b \in M$.

(caa) $M=Ma$ and so $b \in M \Rightarrow a \in M$. Therefore **M=H.**

$\forall x \in \{b,c\}$, (cax) gives $cx = H$ and then **N=P=H.**

We obtain **one hypergroup**:

	a	**b**	**c**
a	**a**	**b**	**c**
b	**a,b**	**b**	**c**
c	**H**	**H**	**H**

CASE (5_6).

For (5.1), ba=b and so we have the following table:

	a	b	c
a	a	b	c
b	b	M	r
c	N	P	Q

(bca) $ra=bN$ and therefore $r \in \{b,c\}$.

If **r=b**, then $b=Nb$ and so **N={a,c}**. Moreover (bcc) implies that $b=bQ$, and thus **Q={a,c}**. For (1.2), $\{a,c\} \subseteq M$. But, if $M=\{a,c\}$, then from (bbb) we obtain $\{b\} \cup P=\{b\}$, which is impossible. Thus **M=H.** For (1.2), $b \in P$. (cbc) implies that

$Pc=P$ and finally, being $a \in P \Rightarrow c \in P$, $P=H$. So in case of $r=b$, one obtains the following hypergroup:

	a	b	c
a	a	b	c
b	b	H	b
c	a,c	H	a,c

If $r=c$, one finds: $\{a,b\} \subseteq Q$. From (cbc) we obtain $Pc=Q$ and $Q=H$. Moreover since $|Mc|=1$, $|bc|=1$, $|cc|>1$, one deduces, for (2.11), $\mathbf{M=\{a,b\}}$. Finally we have, for (1.2), $c \in P \cap N$; moreover the relations (bcb) and (bca) imply $P=bP$, $N=bN$. Necessarily it must be $P=N=H$. Therefore we have another hypergroup:

	a	b	c
a	a	b	c
b	b	a,b	c
c	H	H	H

In all, in this case there exist **two hypergroups**.

CASE (5_7).

For (5.1), $ba=b$. Thus we have this initial table:

	a	b	c
a	a	b	c
b	b	r	M
c	N	P	Q

(bba) gives $ra=r$, hence $r \neq c$. If $r=a$, then (bbc) implies $c=bM$, absurd. So we have $\mathbf{r=b}$.

For (1.2), $\{a,c\} \subseteq M$ and $\{a,c\} \subseteq P$. Moreover $b \in M \cap P$, because otherwise (bbc) and

(cab) lead to contradictions. Therefore M=P=H.
At once, (1.2) implies $c \in N$. Moreover (ccb) implies Qb=H and so $c \in Q$. Finally (cac) implies Nc=Q and so if $b \in N$ then Q=H.

Consequently, we obtain **five hypergroups**:

	a	b	c
a	a	b	c
b	b	b	H
c	a,c	H	Q

	a	b	c
a	a	b	c
b	b	b	H
c	N	H	H

with $Q \in \{\{a,c\},\{b,c\},H\}$ and $N \in \{\{b,c\},H\}$.

In all, there exist 12 **hypergroups** such that $|H|=3$, $S \neq \emptyset$.

6. THE CASE $S \neq \emptyset$; $|H|=4$.-

Put H={a,b,c,d}. Consider that there are three possibilities:

(i) $|S|=3$; S={a,b,c};
(ii) $|S|=2$; S={a,b};
(iii) $|S|=1$; S={a}.

Let us begin with the case (i).
We have only the following possibility:

(6_0) P(H)={(d,a),(d,b),(d,c),(d,d)}.

For (2.6), (2.8), we can consider two subcases: .

(i_1) **a,b,c are left scalar identities;**

(i_2) **a is the unique left scalar identity.**

In the first case, we have a table of the following type:

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	a	b	c	d
d	M	N	P	Q

(daa) $Ma=M$ gives $a \in M$ and by (1.2), $M=H$.
Analogously (dbb), (dcc) and (ddd) give $N=P=Q=H$.

Therefore in the case (i_1), one obtains **one hypergroup**:

	a	**b**	**c**	**d**
a	**a**	**b**	**c**	**d**
b	**a**	**b**	**c**	**d**
c	**a**	**b**	**c**	**d**
d	H	H	H	H

In the case (i_2), for (j) of (2.12) and (2.3), we have the following initial table:

	a	b	c	d
a	a	b	c	d
b	r	s	t	d
c	u	v	w	d
d	M	N	P	Q

with $\forall x \in \{b,c\}$, $\forall y \in \{a,b,c\}$, $xy \neq y$.

If $r=c$, then $s=a$ and $t=b$. This is impossible because (bba) gives $a=b$.
Therefore it follows $r=b$, $s=c$ and $t=a$. We can suppose $u=c$ (because otherwise, by exchanging b and c, we should obtain $ba=c$, that is the preceding absurdity) and so $v=a$ and $w=b$.
By (1.2), $\{a,b,c\} \subseteq Q$ and from (ddd) we obtain $Qd=dQ$. So $Q=H$.
(bda) $M=bM$ and so $a \in M \Rightarrow b \in M \Rightarrow c \in M \Rightarrow a \in M$. By (1.2) it follows $M=H$.
In an analogous way, we can find $N=P=H$.

Finally we obtain again **one hypergroup**:

	a	b	c	d
a	a	b	c	d
b	b	c	a	d
c	c	a	b	d
d	H	H	H	H

In conclusion, in the case $|S|=3$, one obtains **two hypergroups**.

Now consider the case : **(ii)** $|S|=2$; $S=\{a,b\}$.

For (2.6), we can suppose that **a is a left scalar identity**.

Up to isomorphisms, we have the following 6 cases selected so that there are at least two proper pairs (d,x) and (d,y) and listed in lexicographical ordering.

(6_1) $P(H) = \{(c,a),(d,b),(d,c),(d,d)\}$;
(6_2) $P(H) = \{(c,b),(c,c),(d,a),(d,d)\}$;
(6_3) $P(H) = \{(c,b),(c,d),(d,a),(d,c)\}$;
(6_4) $P(H) = \{(c,c),(c,d),(d,a),(d,b)\}$;
(6_5) $P(H) = \{(c,c),(d,a),(d,b),(d,d)\}$;
(6_6) $P(H) = \{(c,d),(d,a),(d,b),(d,c)\}$.

CASE (6.1).-

Always for (2.2) and (2.12), we obtain the following table:

	a	b	c	d
a	a	b	c	d
b	r	s	c	d
c	M	t	u	v
d	d	N	P	Q

with $\{r,s\}=\{a,b\}$.

If $r=a$, $s=b$, (jj) of (2.12) entails $t=c$. Furthermore, from (cab) it follows that

Mb=c. Absurd, since $|M| \geq 2$.

On the other hand, if r=b, s=a, from (cbb) one can deduce t=d and so (cab) gives Mb = d, another absurdity, since $|M| \geq 2$.

Therefore, in this case **there aren't hypergroups.**

CASE (6_2).

For (2.12), we have ca=bc=c and bd=d. Put ba=x, bb=y and cb=M. By (1.2) {x,y}={a,b}.
If x=a, then (cba) gives Ma=c. This is impossible since $|M| \geq 2$.
If x=b, then y=a and (cbb) gives c = Mb, which is again absurd.
Thus in this case, **there don't exist hypergroups.**

CASE (6_3).-

For (jj) of (2.12), ca=c. By putting the same positions of the preceding case and reasoning in the same manner, one concludes that **there are not hypergroups.**

CASE (6_4).-

By using (2.2) and (2.12), we can start from the following table:

	a	b	c	d
a	a	b	c	d
b	r	s	c	d
c	c	t	M	N
d	P	Q	u	v

with {r,s}={a,b}.

(dac) Pc = u and so $c \notin P$. From $\{a,b\} \cap P = \emptyset$ we obtain **u** = c .

(dad) $Pd = v$ gives $v = d$.

(cdd) and (ccc) give $Nd = N$ and $Mc = cM$. By (1.2) $\{a,b,c\} \subseteq N$ and $\{a,b,d\} \subseteq M$ and so N=H=M.

(cbd) td=H and so t=c.

Now we prove that :

$$P = Q = \{a,b,d\}.$$

Proof.-

If $r = a$ and $s = b$, by (1.2) it follows that $\{b,d\} \subseteq P$ and $\{a,d\} \subseteq Q$.

(daa) gives $P = Pa$ whence $P = \{a,b,d\}$. Being $c \notin P$, (dba) gives $c \notin Q$ and (dab) gives $Q = \{a,b,d\}$.

□

This leads to **one hypergroup.**

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	H	H
d	H\c	H\c	c	d

If $r = b$ and $s = a$, by (1.2) $d \in P \cap Q$. From (dbb) we obtain $Qb = P$ and since $c \notin P$ then $c \notin Q$. Finally (bda) and (dab) give $P = Q = \{a,b,d\}$.

We have so **another hypergroup.**

	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	c	H	H
d	H\c	H\c	c	d

CASE (6.5).

We can start from this table:

	a	b	c	d
a	a	b	c	d
b	r	s	c	d
c	c	t	M	u
d	N	P	v	Q

with $\{r,s\}=\{a,b\}$.

(dac) $v = Nc$ and so $c \notin N$ and $v = c$.

(dcd) $u = du$ and so $u = c$.

(dcb) $t = dt$ and so $t = c$.

(cdd) and (cdb) give $c \notin Q \cap P$.

By (1.2), $\{a,b,d\} \subseteq M$, $\{a,b\} \subseteq Q$ and $d \in N \cap P$.

Now we prove that :

$$N = P = Q = \{a,b,d\}.$$

Proof.-

If $r = a$ and $s = b$, by (1.2), $b \in N$ and $a \in P$.

(daa) gives $N = Na$ whence $N = \{a,b,d\}$.

(dbb) gives $Pb = P$ whence $P = \{a,b,d\}$.

(dad) gives $d \in Q$ and so $Q = \{a,b,d\}$.

If $r = b$ and $s = a$,

(bda) gives $N = bN$ whence , being $a \in N \Leftrightarrow b \in N$, $N = \{a,b,d\}$.

Analogously (bdb) gives $P = bP$ whence $P = \{a,b,d\}$.

Finally (dad) gives $\{d\} \cup Q = Q$ and so $Q = \{a,b,d\}$.

□

This leads to the following **four hypergroups**.

	a	b	c	d
a	a	b	c	d
b	r	s	c	d
c	c	c	M	c
d	H\c	H\c	c	H\c

with $M \in \{H, H\setminus\{c\}\}$ and $\{r,s\} = \{a,b\}$.

CASE (6.6).

This is the initial configuration of the table:

	a	b	c	d
a	a	b	c	d
b	r	s	c	d
c	c	t	u	M
d	N	P	Q	v

with $\{r,s\}=\{a,b\}$.

(dad) $Nd = v$ and so $v = d$ and $c \notin N$.

By (1.2), $\{a,b,c\} \subseteq M$ and by (cdd) we obtain $d \in M$ and so $M = H$.

(cbd) $td = H$ and so $t = c$.

(ccd) $ud = H$ and so $u = c$.

By (1.2), $\{a,b,d\} \subseteq Q$ and by (dac) we obtain $c \in Q$ and so $Q = H$.

(dbd) $Pd = d$ and so $c \notin P$.

If $r = a$ and $s = b$, by (1.2), $\{b,d\} \subseteq N$ and $\{a,d\} \subseteq P$.

(dab) $Nb = P$ whence, being $b \in N$, we obtain $b \in P$ and so $P = \{a,b,d\}$.

(dba) $Pa = N$ and so $N \cup \{a\} = N$, whence $N = \{a,b,d\}$.

The following one is the resulting hypergroup.

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	H
d	H\c	H\c	H	d

If $r = b$ and $s = a$, by (1.2), $d \in N \cap P$.

(dba) and (dbb) give $Pa = P$ and $Pb = N$, whence, being $d \in P$, we obtain $N = P$.

(bda) gives $bN = N$ and so, being $a \in N \Rightarrow b \in N$, we obtain $N = P = \{a,b,d\}$.

We have so another hypergroup:

	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	c	c	H
d	H\c	H\c	H	d

Ultimately, the case $|S|=2$, has given rise to **eight hypergroups**.

At last, we study the case **(iii)** $|S|=1$; $S=\{a\}$. By (2.6), **a is a left scalar identity**.

There are, unless isomorphisms, the following seven cases selected so that every type contains the proper pair (b,a) and listed in lexicographical ordering:

(6_7) P(H)={(b,a), (b,b), (c,c), (d,d)};
(6_8) P(H)={(b,a), (b,b), (c,d), (d,c)};
(6_9) P(H)={(b,a), (b,c), (c,b), (d,d)};
(6_{10}) P(H)={(b,a), (b,d), (c,b), (d,c)};
(6_{11}) P(H)={(b,a), (c,b), (c,c), (d,d)};
(6_{12}) P(H)={(b,a), (c,b), (d,c), (d,d)};
(6_{13}) P(H)={(b,a), (c,d), (d,b), (d,c)}.

(6.1) REMARK.- If $K=(K,\circ)$ is a hypergroup of size n, then it is possible to construct two hypergroups on the underlying set $H=K\cup\{x\}$, with $x\notin K$, by defining the following hyperoperation $*$:

$\forall(a,b)\in K^2$, $a*b=a\circ b$;

$\forall a\in K$, $a*x=x*a=\{x\}$;

$x*x\in\{K,H\}$.

In the rest, these hypergroups will be called **x-enlargements of** $(K,\circ)$.

Now we are ready to prove the following result:

(6.2) PROPOSITION.- In the cases (6_7), (6_9), (6_{11}), by putting x=d, one obtains all and alone the hypergroups, which are x-enlargements of the hypergroups K of size 3, with $|P(K)|=3$, $|S_1(K)|=1$, $|S(K)|=\emptyset$, whose tables are known (see [3]).

Proof.- In every case, it suffices to show that the following condition is valid:

(*) $\forall\alpha\in H\setminus\{d\}$, $\alpha d=d\alpha=d$.

In fact, if (*) is satisfied, then $\forall(x,y)\in\{a,b,c\}^2$, (xyd) gives xd=d, whence $d\notin xy$ and {a,b,c} results a hypergroup K with $|P(K)|=3$, $|S_1(K)|=1$, $|S(K)|=\emptyset$.

We begin with the case (6_7). For (2.14), we have only to show that cd=dc=d. (bdc) gives $dc\in\{c,d\}$, moreover, taking in account that bc=c (see (2.14)), (bcd) gives $cd\in\{c,d\}$. By (cdc) one deduces that **cd = dc.**

(cac) and (cbc) give **ca=cb=c.**

Without loss of generality we can set **cd=dc=d**, (since otherwise, by putting cd=dc=c, one obtains hypergroups which are isomorphic to those ones in which cd=dc=d).

□

Ultimately, this case gives rise to the following **four hypergroups**:

	a	b	c	d
a	a	b	c	d
b	a,b	a,b	c	d
c	c	c	a,b	d
d	d	d	d	M

	a	b	c	d
a	a	b	c	d
b	a,b	a,b	c	d
c	c	c	H\d	d
d	d	d	d	M

with $M\in\{H,H\setminus\{d\}\}$.

Now we come to the case (6_9). For (2.14), we can limit ourselves to determine the products cd, dc. (1.2) implies $a\in cb$ and so (cbd) gives **cd=d.** Finally (dcd) gives **dc=d.**

Therefore, this case leads to the ensuing **two hypergroups** :

	a	b	c	d
a	a	b	c	d
b	a,b	b	H\d	d
c	c	H\d	c	d
d	d	d	d	M

with $M \in \{H, H\setminus\{d\}\}$.

To sum up, consider the case (6_{11}). For (2.14), **ad=bd=cd=da=db=d.** So it remains to be proved that dc=d. Since dd=d(cd)=(dc)d and $(d,d) \in P(H)$, it descends that dc=d.

Therefore more **two hypergroups** complete the cases of the preposition :

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	d
c	c	H\d	H\d	d
d	d	d	d	M

with $M \in \{H, H\setminus\{d\}\}$.

□

Now we can go on with the remaining four cases:

CASE (6_8).

We have the following initial configuration:

	a	b	c	d
a	a	b	c	d
b	a,b	M	c	d
c	c	c	r	N
d	d	d	P	s

(bbd) and (bbc) give $\{c,d\} \cap M = \emptyset$ and so $M=\{a,b\}$.

(bcc) gives $r \notin \{a,b\}$ and so (for (1.2)) $\{a,b\} \subseteq P$; $\{a,b\} \subseteq N$.
We prove that :

1. **If $r=c$ then $N=P=H$ and $s=d$.**
2. **If $r=d$ then $N=P=\{a,b\}$ and $s=c$.**

Proof.-

1. By (1.2), $\{a,b,d\} \subseteq N \cap P$.
   (ccd) $c \in N$ and so $N=H$.
   (dcc) $c \in P$ and so $P=H$.
   (ddc) $sc=H$ and so $s=d$.

Then the following **hypergroup** remains determined.

	a	b	c	d
a	a	b	c	d
b	a,b	a,b	c	d
c	c	c	c	H
d	d	d	H	d

2. (dcc) $s=Pc$ and so, being $a \in P$, we obtain $s=c$.
   (ccd) $\{c,d\} \cap N = \emptyset$ and so $N=\{a,b\}$.
   (ccc) $P=N$ and so $P=\{a,b\}$.

□

Then the following **hypergroup** remains determined:

	a	b	c	d
a	a	b	c	d
b	a,b	a,b	c	d
c	c	c	d	a,b
d	d	d	a,b	c

Then the case (6_3) has given rise to **two hypergroups**. □

CASE (6_{10}).

For (2.14), we can write the following table:

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	M
c	c	N	r	s
d	d	d	P	t

with $d \in M$ and $\{a,c\} \subseteq N$.

(cbc) $d \notin N$, $r = c$ and by(1.2), $s = d$.

(bcd) $M = \{d\}$, which is impossible

Therefore in the case (6_{10}) **there don't exist hypergroups.**

CASE (6_{12}).

For (2.14), we can start from the table below:

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	d
c	c	M	c	d
d	d	d	N	P

By (1.2), $\{a,b,d\} \subseteq N$, $\{a,b,c\} \subseteq P$, $\{a,b,c\} \subseteq M$.

(cbc) $d \notin M$ and so $M = \{a,b,c\}$.

(cdc) being $a \in N$, we obtain $c \in N$ and so $N = H$.

(dcd) $P = H$.

Ultimately we have **one hypergroup**, whose table is represented below:

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	d
c	c	H\d	c	d
d	d	d	H	H

CASE (6_{13}).

By turning to account again (2.14), the following one can be taken as initial table:

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	d
c	c	c	r	M
d	d	N	P	d

We have:

(cca), (ccb) and (bcc) give $r = c$ and so by (1.2), we obtain $\mathbf{M} = H$, $\{a,b,d\} \subseteq P$ and $\{a,d\} \subseteq N$.

(bdb) $N = bN$ and so, being $a \in N$, we obtain $b \in N$.

(dbd) $c \notin N$ and so $\mathbf{N} = \{a,b,d\}$.

(dbc) $\{c\} \cup P = P$ and so $P = H$.

We have again one **hypergroup** :

	a	b	c	d
a	a	b	c	d
b	a,b	b	c	d
c	c	c	c	H
d	d	H\c	H	d

In conclusion, **in the case** $|S| = 1$, one obtains 12 hypergroups.

That is a recapitulatory list of the results obtained :

CASE	HYPERGROUPS	$\|S\|$	$\|H\|$
	1	0	2
(3_1)	4	0	3
(3_2)	0	0	3
(3_3)	0	0	3
(3_4)	1	0	3
(3_5)	0	0	3
(3_6)	12	0	3
(3_7)	1	0	3
(3_8)	1	0	3
(4_1)	0	0	4
(4_2)	4	0	4
(4_3)	0	0	4
(4_4)	2	0	4
(4_5)	0	0	4
(5_1)	0	1	3
(5_2)	1	1	3
(5_3)	1	1	3
(5_4)	2	1	3
(5_5)	1	1	3
(5_6)	2	1	3
(5_7)	5	1	3
(6_0)	2	3	4
(6_1)	0	2	4
(6_2)	0	2	4
(6_3)	0	2	4
(6_4)	2	2	4
(6_5)	4	2	4
(6_6)	2	2	4
(6_7)	4	1	4
(6_8)	2	1	4

(6_9)	2	1	4
(6_{10})	0	1	4
(6_{11})	2	1	4
(6_{12})	1	1	4
(6_{13})	1	1	4

CONCLUSIONS.-

Let $\mathcal{I}_l$ (respect. $\mathcal{I}_r$) be the family of hypergroups with left (right) scalars and without right (left) scalars.

Obviously $\forall (H,H') \in \mathcal{I}_l \times \mathcal{I}_r$, H cannot be isomorphic to H'.

Therefore we can resume all the results, by means of the following table where S_l is the set of left scalars of H and S_r is the set of right scalars.

	$\lvert H \rvert=2$	$\lvert H \rvert=3$	$\lvert H \rvert=4$
$S_l=S_r=\emptyset$	1	19	6
$S_l \neq \emptyset$; $S_r=\emptyset$		12	22
$S_l=\emptyset$; $S_r \neq \emptyset$		12	22
total	1	43	50

REMARK.-

We observe that only in one of the eight cases of section3 (the case 3_6) the number of hypergroups is not equal to the number of semi-hypergroups. In fact in the remaining seven cases, the hypothesis of reproducibility (i.e. the (1.2)) has not been used. The complexity of the cases studied in the subsequent sections and the final aim of the authors (to find the hypergroups which have exactly four proper pairs) in accordance with the papers [3], [4], [5], have prevented them from verifying if in some cases the hypergroups could be found without the use of (1.2). So, **to find all semi-hypergroups with at most four proper pairs** remains an open question.

ACKNOWLEDGEMENT.-

The authors thank the referee sincerely for the suggestions to improve this paper.

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7. SUBJECT INDEX.-

In this paper the authors study the hypergroups, such that there are exactly four pairs of elements, which define proper hyperproducts, and such that there are no scalars.

They solve the combinatorial problem of finding, up to isomorphism, all the tables of the aforesaid hypergroups.

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