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ON THE GENERATING FUNCTIONAL OF A CONVOLUTION SEMIGROUP ON A HILBERT-LIE GROUP

by Erdal Coşkun and Herbert Heyer

Abstract (English)

The authors establish a Lévy-Khintchine type representation for the generating functional of a continuous convolution semigroup of probability measures on a Hilbert-Lie group. The proof is inspired by the one given in the case of a locally compact group the additional technical problem to be handled being the construction of modified canonical coordinates within an appropriate space of twice differentiable functions on the group.

Abstract (French)

On établit une formule de représentation de type Lévy-Khinchine pour la fonctionnelle génératrice d'un semi-groupe continu de convolution des mesures de probabilité sur un groupe de Lie-Hilbert. La démonstration est stimulée par cette du cas d'un groupe localement compact le problème à résoudre étant la construction des coordonnées canoniques modifiées audedans un propre espace des fonctions deux fois différentiables sur le groupe.

1. Preliminaries

For any topological group G whose topology admits a complete left invariant metric d we denote the Banach space of bounded left d-uniformly continuous real-valued functions on G by $C_u(G)$. Given any real-valued function f on G and $a \in G$ the functions $f^*, f_a := R_a f$ and ${}_a f := L_a f$ are defined for all $b \in G$ by $f^*(b) := f(b^{-1}), f_a(b) := f(ba)$ and ${}_a f(b) := f(ab)$ respectively. In order to do measure theory on G we consider the Banach algebra M(G) of real-valued measures on the

Borel σ-field $\mathcal{B}(G)$ of G, M(G) being furnished with total variation and convolution. The symbols $M_+(G)$ and $M^1(G)$ stand for the semi-groups of positive measures and of probability measures on G respectively.

In what follows G will always be a Hilbert-Lie group modelled over a separable Hilbert space H. Interesting examples of Hilbert-Lie groups are

1.1 the *Sobolev groups* $H^k(M,G)$ introduced for a connected Riemannian manifold M and a finite dimensional compact Lie group G such that $k > \frac{1}{2} \dim M$, and

1.2 the *Kosyak groups* $GL_2(\alpha)$ for $\alpha := (a_{kn}) \in (\mathbb{R}_+^\times)^{\mathbb{Z}^2}$ such that there exists a constant $c > o$ satisfying $\alpha_{kn} \leq c^2 \alpha_{km}\, \alpha_{mn}$ whenever $k,n,m \in \mathbb{Z}$.

(See [1] and [4] respectively)

The tangent space T_e of G which is isomorphic to H serves as the domain of the exponential mapping Exp into G. Exp is an analytic homeomorphism from a neighborhood N_o of $o \in T_e$ onto a neighborhood U_e of $e \in G$. The inverse of Exp considered as a mapping from U_e onto N_o will be denoted by Log. Given an orthonormal basis $\{X_i : i \in \mathbb{N}\}$ of H one definies a system $\{a_i : i \in \mathbb{N}\}$ of canonical coordinates $a_i : U_e \to \mathbb{R}$ such that

$$a = \operatorname{Exp}\Big(\sum_{i \geq 1} a_i(a) X_i\Big)$$

for all $a \in U_e$. In fact, for each $i \in \mathbb{N}$ we put $a_i(a) := \langle \operatorname{Log}(a), X_i \rangle$ whenever $a \in U_e$.

Given $X \in H$ a function $f \in C_u(G)$ is called *left differentiable at* $a \in G$ with respect to X if

$$Xf(a) := \overline{\lim_{t \to o}} \frac{1}{t}(f(\operatorname{Exp}(tX)a) - f(a))$$

exists. f is called *continuously left differentiable* if Xf(a) exists for all $X \in H$, $a \in G$ and if $a \longmapsto Xf(a)$ as well as $X \longmapsto Xf(a)$ are continuous mappings. Derivatives of higher order are defined inductively. Now, let $f \in C_u(G)$ be a twice continuously left differentiable function.

For each $a \in G$ the mappings $Df(a): X \longmapsto Xf(a)$ and $D^2f(a): (X,Y) \longmapsto XYf(a)$ are continuous linear and symmetric continuous bilinear functionals on H and $H \times H$ respectively. One has the equalities $\langle Df(a),X\rangle = Xf(a)$ as well as $\langle D^2f(a)(X),Y\rangle = XYf(a)$ whenever $a \in G$ and $X,Y \in H$. Now the set $C_2(G)$ of all twice continuously left differentiable functions $f \in C_u(G)$ such that the mapping $a \longmapsto D^2f(a)$ is d-uniformly continuous, $||Df|| := \sup_{a\in G} ||Df(a)|| < \infty$ and $||D^2f|| := \sup_{a\in G} ||D^2f(a)|| < \infty$ turns out to be a Banach space with respect to the norm

$$f \longmapsto ||f||_2 := ||f|| + ||Df|| + ||D^2f||.$$

We note that each $f \in C_2(G)$ has a Taylor expansion of second order at $e \in G$ given by

$$f(a) = f(e) + \Sigma_{i\geq 1} a_i(a) X_i f(e) + \frac{1}{2} \Sigma_{i,j\geq 1} a_i(a) a_j(a) X_i X_j f(\bar{a})$$

for all $a \in U_e$ and some $\bar{a} \in U_e$.

The next aim of our discussion is a two-stage modification of the given canonical coordinate system $\{a_i : i \in \mathbb{N}\}$. It is not difficult to achieve an extension of $\{a_i : i \in \mathbb{N}\}$ to a canonical coordinate system $\{b_i : i \in \mathbb{N}\}$ in $C_2(G)$. For the second modification which has been the main work in [2] we start with a motivation valid for commutative G over H. Given a complete orthonormal system $\{X_i : i \in \mathbb{N}\}$ of H and $n \in \mathbb{N}$ we introduce $H_n := \langle\{X_1,\ldots,X_n\}\rangle$. Then $H/H_n^{\perp}$ and H_n are isomorphic spaces, $G_n := \mathrm{Exp}\, H_n^{\perp}$ is a closed subgroup of G and G/G_n is a finite dimensional Hilbert-Lie group. If p_n denotes the canonical projection from G onto G/G_n and $\{b_i^n : i=1,\ldots,n\}$ a canonical coordinate system with respect to $\{X_1,\ldots,X_n\}$ (in $C_2(G/G_n)$) then the functions $d_i^n := b_i^n \circ p_n \in C_2(G)$ have the properties that $X_j d_i^n$ exists and $= o$ for all $j>n$, $i=1,\ldots,n$. It is therefore reasonable to introduce for any Hilbert-Lie group G over H, any orthonormal basis $\{X_i : i \in \mathbb{N}\}$ of H and every $n \in \mathbb{N}$ the space $C_{(2),n}(G)$ of functions $f \in C_2(G)$ satisfying the equalities $X_i f = o$ for all $i>n$ and $X_i X_j f = o$

for all $i>n$ or $j>n$. The desired function space appears to be

$$C_{(2)}(G) := \bigcup_{n\in\mathbb{N}} C_{(2),n}(G).$$

Clearly, if G ist commutative,

$$C_{(2),n}(G) = \{f \circ p_n \in C_2(G) : f \in C_2(G/G_n)\}$$

for each $n\in\mathbb{N}$, and $C_{(2)}(G)$ coincides with its right counterpart $\tilde{C}_{(2)}(G)$ where differentiability is considered from the right rather than from the left.

In order to obtain a modification of the given canonical coordinate system $\{b_i : i\in\mathbb{N}\}$ in $C_2(G)$ to a modified one $\{d_i : i\in\mathbb{N}\}$ in $C_{(2)}(G)$ we proceed as follows: For each $n\in\mathbb{N}$ let $\{b_i^n : i=1,\dots,n\}$ be a canonical coordinate system in $C_2(G)$ (with respect to $\{X_1,\dots,X_n\}$). Then, if $b_i^n \in C_{(2),n}(G)$ for all $i=1,\dots,n$ and $n\geq n_o$ for some $n_o\in\mathbb{N}$ then the system $\{d_i : i\in\mathbb{N}\}$ given by

$$d_i := \begin{cases} b_i^{n_o} & \text{for all } i=1,2,\dots,n_o \\ b_n^n & \text{for all } n>n_o \end{cases}$$

lies in $C_{(2)}(G)$. $\{d_i : i\in\mathbb{N}\}$ is called a *modified canonical coordinate system* with respect to the basis $\{X_i : i\in\mathbb{N}\}$ of H.

Obviously every commutative Hilbert-Lie group and every finite dimensional Lie group admit modified canonical coordinate systems. In the finite dimensional case n_o equals the dimension of the group.

For Hilbert-Lie groups G admitting a modified canonical coordinate system one defines *Hunt functions* Φ_n by

$$\Phi_n(a) := \Sigma_{i=1}^n d_i(a)^2$$

for all $a\in G$. Clearly, $\Phi_n \in C_{(2),n}(G)$, $\Phi_n(a)>o$ for all $a\in G\setminus[\Phi_n=o]$, hence $X_i\Phi_n(e)=o$ and $X_iX_j\Phi_n(e)=2\delta_{ij}$ whenever $i,j=1,\dots,n$ ($n\in\mathbb{N}$). (cf. [3], Lemma 4.1.9 and 4.1.10).

2. The domain of the generating functional

For any measure $\mu \in M^1(G)$ one introduces the *translation operator* T_μ of μ on $C_u(G)$ by

$$T_\mu f := \int f_a \mu(da)$$

for all $f \in C_u(G)$.

2.1 Properties of the translation operator

2.1.1 $T_\mu C_u(G) \subset C_u(G)$

2.1.2 $T_{\mu * \nu} = T_\mu T_\nu$ if also $\nu \in M^1(G)$

2.1.3 $T_\mu C_2(G) \subset C_2(G)$, hence

2.1.4 $T_\mu C_{(2)}(G) \subset C_{(2)}(G)$.

A (continuous) *convolution semigroup on* G is a family $\{\mu_t \in \mathbb{R}_+\}$ in $M^1(G)$ such that $\mu_s * \mu_t = \mu_{s+t}$ for all $s,t \in \mathbb{R}_+^\times$ and $\lim_{t\to o}\mu_t =: \mu_o = \varepsilon_e$ the limit being taken in the weak topology in $M^1(G)$.

2.2 Proposition. Any convolution semigroup $\{\mu_t : t \in \mathbb{R}_+\}$ in $M^1(G)$ admits a *Lévy measure* η on G defined as a σ-finite measure in $M_+(G)$ satisfying the properties $\eta(\{e\}) = o$ and

$$\lim_{t\downarrow o} \frac{1}{t} \int f d\mu_t = \int f d\eta$$

valid for all $f \in C_u(G)$ with $e \notin \operatorname{supp}(f)$.
For a proof see [6].

2.3 Corollary. For every neighborhood U of e

$$\sup_{t>o} \frac{1}{t} \mu_t(\complement U) < \infty$$

Let $\{\mu_t : t \in \mathbb{R}_+\}$ be a convolution semigroup on G and $\{T_{\mu_t} : t \in \mathbb{R}_+\}$ the corresponding contraction semigroup on $C_u(G)$ with (infinitesimal) generator $(N, D(N))$. The *generating functional* $(A, D(A))$ of $\{\mu_t : t \in \mathbb{R}_+\}$ is given by

$$Af := \lim_{t\downarrow o} \frac{1}{t} (T_{\mu_t} f(e) - f(e))$$

for all f in the domain $D(A)$ of A. Plainly $Af=Nf(e)$ whenever $f\in D(N)$.

From now on we assume G to be a Hilbert-Lie group (over a separable Hilbert space H) admitting a system $\{d_i : i\in\mathbb{N}\}$ of modified canonical coordinates (with respect to an orthonormal system $\{X_i : i\in\mathbb{N}\}$ of H). Moreover, let $\{\mu_t : t\in\mathbb{R}_+\}$ be a convolution semigroup on G.

2.4 <u>Proposition</u>. For every $n\in\mathbb{N}$

$$\sup_{t\in\mathbb{R}_+^\times} \frac{1}{t}\int \Phi_n d\mu_t < \infty.$$

<u>Proof</u>. As a consequence of the Banach-Steinhaus theorem together with the Hille-Yoshida theory (cf [3] , Lemma 4.1.11) we obtain that for every $f\in C_{(2),n}(G)$ and every $\varepsilon>o$ there exists a $g:=g_\varepsilon \in C_{(2),n}(G)\cap D(N)$ such that $||f-g||_2<\varepsilon$, $f(e)=g(e)$, $X_if(e)=X_ig(e)$, and $X_iX_jf(e)=X_iX_jg(e)$ for all $i,j=1,\dots,n, n\geq 1$. Applying this statement to $\Phi_n\in C_{(2),n}(G)$ we obtain the existence of $\Psi_n\in C_{(2),n}(G)\cap D(N)$ satisfying $||\Phi_n-\Psi_n||_2<\infty$, $\Psi_n(e)=\Phi_n(e)=o$, $X_i\Psi_n(e)=X_i\Phi_n(e)=o$, and $X_iX_j\Psi_n(e)=X_iX_j\Phi_n(e)=2\delta_{ij}$ for all $i,j=1,\dots,n$. But the Taylor expansion of Ψ_n implies the existence of a constant $\delta_n>o$ and a neighborhod W of e such that

$$\Psi_n(a)\geq\delta_n\Sigma_{i=1}^n d_i^2(a)$$

valid for all $a\in W$. Then

$$\sup_{t\in\mathbb{R}_+^\times} \frac{1}{t}\int_W \Phi_n d\mu_t<\infty,$$

and, since Φ_n is bounded, Corollary 2.3 yields the assertion. __|

2.5 <u>Theorem</u>. $C_{(2)}(G)\subset D(A)$

<u>Proof</u>. Let $f\in C_{(2),n}(G)$ $(n\in\mathbb{N})$ and put

$$g(a) := f(a)-f(e) - \Sigma_{i=1}^n z_i(a)X_if(e)$$

for all $a\in G$, where the functions z_i are chosen in $C_{(2),n}(G)\cap D(N)$

such that $z_i(e)=d_i(e)=o$ and $X_j z_i(e)=X_j d_i(e)=\delta_{ij}$ for all $i,j=1,\dots,n$ (See the proof of Proposition 2.4). Then $g\in C_{(2),n}(G)$ with $g(e)=o$ and $X_i g(e)=o$. From an application of the Taylor expansion of G in a neighborhood W of e we obtain a constant $k_1\in \mathbb{R}_+^\times$ such that $|g(a)| \leq k_1||g||_2\Phi_n(a)$ for all $a\in W$. Now Proposition 2.4 implies that

$$\sup_{t\in \mathbb{R}_+^\times}\left|\frac{1}{t}\int_W g d\mu_t\right|<\infty.$$

Since g is bounded, Corollary 2.3 provides a constant $k_2\in \mathbb{R}_+^\times$ independent of t such that

$$\left|\frac{1}{t}\int_{\complement W} g d\mu_t\right|\leq k_2||g||_2$$

for all $t\in \mathbb{R}_+^\times$. Adding these two inequalities yields a constant $k_3\in \mathbb{R}_+^\times$ independent of t such that

$$\left|\frac{1}{t}(T_{\mu_t}f(e)-f(e))-\frac{1}{t}\Sigma_{i=1}^n X_i f(e)T_{\mu_t}z_i(e)\right|\leq k_3||f||_2$$

for all $t\in \mathbb{R}_+^\times$. Since $z_i\in D(N)$ and $z_i(e)=o$ there is a constant $k(n)\in \mathbb{R}_+^\times$ depending only on n such that

$$\left|\frac{1}{t}(T_{\mu_t}f(e)-f(e))\right|\leq k(n)||f||_2$$

for all $t\in \mathbb{R}_+^\times$. This inequality holds for all $f\in C_{(2),n}(G)$. From the Banach-Steinhaus theorem we finally conclude that Af exists for all $f\in C_{(2)}(G)$. $\_|$

2.6 Corollary. For every $n\in \mathbb{N}$ the measures $\Phi_n.\eta$ are bounded.

Proof. Let $(f_k)_{k\geq 1}$ be a sequence of functions in $C_u(G)$ satisfying $o\leq f_k\leq 1$, $e\notin supp(f_k)$ $(k\geq 1)$ and $f_k\uparrow 1_{G^\times}$ for $k\to\infty$ (where $G^\times:=G\setminus\{e\}$). Then, since $e\notin supp(f_k\Phi_n)$,

$$A(f_k\Phi_n) = \int f_k\Phi_n d\eta,$$

and by the theorem $A(f_k\Phi_n)\leq A(f_{k+1}\Phi_n)\leq\dots\leq A(1_{G^\times}\Phi_n)<\infty$ $(k\geq 1)$. The monotone convergence theorem yields the assertion. $\_|$

3. The representation of the generating functional

G remains to be a given Hilbert-Lie group over a separable Hilbert space H. We assume the existence of a system $\{d_i : i \in \mathbb{N}\}$ of modified canonical coordinates in $C_{(2),n}(G)$. For every $f \in C_{(2),n}(G)$ we define functions D_f^n on G by

$$D_f^n(a) := \begin{cases} \left(f(a)-f(e)-\Sigma_{i=1}^n d_i(a)X_i f(e) - \frac{1}{2}\Sigma_{i,j=1}^n d_i(a)d_j(a)X_iX_j f(e)\right)\Phi_n(e)^{-1} & \text{if} \quad \Phi_n(a)>o \\ o & \text{otherwise.} \end{cases}$$

They are measurable, continuous at e, and bounded in a neighborhood of e. In fact, the Taylor expansion of f at e yields

$$f(a)=f(e)+\Sigma_{i=1}^n d_i(a)X_i f(e) + \frac{1}{2}\Sigma_{i,j=1}^n d_i(a)d_j(a)X_iX_j f(e)+\Phi_n(a)\theta_n(f,a)$$

for all a in a neighborhood W of e, where $\theta_n(f,.)$ satisfies $\lim_{a\to e}\theta_n(f,a)=:\theta_n(f,e)=o$. Thus

$$D_f^n(a) = \begin{cases} \theta_n(f,a) & \text{if } a\in W\setminus[\Phi_n=o] \\ o & \text{if } a\in[\Phi_n=o] \end{cases}$$

for all $a\in W$ and $\lim_{a\to e} D_f^n(a)=o$, hence $\sup_{a\in W}|D_f^n(a)|<\infty$. The measurability of D_f^n is clear.

Also note that there exist a neighborhood V of e with $\bar{V}\subset W$ and a function $\zeta\in C_u(G)$ with $o\leq\zeta\leq 1$, $\zeta(V)=\{1\}$ and $\zeta(\complement W)=\{o\}$. The functions $B_f^n:=D_f^n$ are also measurable, continuous at e and bounded with $B_f^n(e)=o$, and they satisfy

$$B_f^n = \left(f-f(e) - \Sigma_{i=1}^n d_iX_i f(e) - \frac{1}{2}\Sigma_{i,j=1}^n d_id_jX_iX_j f(e)\right)\Phi_n^{-1}$$

on $V\setminus[\Phi_n=o]$.

We are returning to the discussion of a convolution semigroup $\{\mu_t : t\in \mathbb{R}_+\}$ on G with associated Lévy measure $\eta\in M_+(G)$.

3.1 Proposition. For every $f\in C_{(2)}(G)$ the integral

$$\int_{G^\times}(f-f(e)-\Sigma_{i\geq 1}d_iX_if(e)-\tfrac{1}{2}\,\Sigma_{i,j\geq 1}d_id_jX_iX_jf(e))d\eta$$

exists.

Proof. Let $f\in C_{(2),n}(G)$ for some $n\in\mathbb{N}$. By Corollary 2.6 together with the properties of B_n^f we obtain that

$$\int_{G\setminus[\Phi_n=o]} B_f^n\, d(\Phi_n.\eta)<\infty.$$

Now, let V be a neighborhood of e chosen as in the definition of B_n^f. Without loss of generality we assume that $\eta(\partial V)=o$. Then

$$\int_{V^\times}(f-f(e)-\Sigma_{i=1}^n d_iX_if(e)-\tfrac{1}{2}\,\Sigma_{i,j=1}^n d_id_jX_iX_jf(e))d\eta$$

$$=\int_{V\setminus[\Phi_n=o]} B_f^n\, d(\Phi_n.\eta)<\infty$$

Applying Corollary 2.3 we also obtain that

$$\int_{\complement V}(f-f(e)-\Sigma_{i=1}^n d_iX_if(e)-\tfrac{1}{2}\,\Sigma_{i,j=1}^n d_id_jX_iX_jf(e))d\eta<\infty,$$

hence that the integral in question exists. $\rule{1em}{0.4pt}|$

3.2 Proposition. Let $n\in\mathbb{N}$. Then

(i) for $f\in C_{(2),n}(G)$ the integral $\int_{V\setminus[\Phi_n=o]} f d\eta$ is bounded provided $f\Phi_n^{-1}$ is bounded on $V\setminus[\Phi_n=o]$.

(ii) For every bounded measurable function f on G which is continuous at e, $(\Phi_n.\eta)$-a.e. continuous and satisfies $f(e)=o$,

$$\lim_{t\downarrow o}\frac{1}{t}\int_G f\Phi_n d\mu_t=\int_{G^\times} f\Phi_n d\eta.$$

Proof. (i) follows from

$$\int_{V\setminus[\Phi_n=o]}|f|d\eta = \int_{V\setminus[\Phi_n=o]}|f|\Phi_n^{-1}d(\Phi_n.\eta)$$

$$= \int h_f^n \, d(\Phi_n.\eta)<\infty,$$

since $h_f^n:=1_{V\setminus[\Phi_n=o]}f\Phi_n^{-1}$ is lower semicontinuous and bounded on G.

(ii) By Corollary 2.6 the measure $\nu^n:=\Phi_n.\eta$ is bounded on $G^\times$. On the other hand we infer from Theorem 2.5 that for the measures $\nu_t^n:=(\frac{1}{t}\Phi_n).\mu_t\,(t>o)$ the inequalities

$$\lim_{t\downarrow o}\nu_t^n(G^\times) = A(\Phi_n)\geq\nu^n(G^\times)$$

hold. It follows that $\nu^n(G^\times)\leq c:=\sup_{t\in\mathbb{R}_+^\times}\nu_t^n(G^\times)<\infty$. If f is a bounded measurable function on G which is ν^n-a.e. continuous and satisfies $e\notin\operatorname{supp}(f)$ then clearly

$$\lim_{t\downarrow o}\int f d\nu_t^n = \int f d\nu^n.$$

A slightly more sophisticated argument yields the validity of this limit relationship also for bounded measurable functions f that are ν^n-a.e. continuous, continuous at e and satisfy $f(e)=o$. __|

3.3 Corollary. For every $n\in\mathbb{N}$

$$\lim_{t\downarrow o}\frac{1}{t}\int B_f^n\,\Phi_n d\mu_t = \int_{G^\times}B_f^n\,\Phi_n d\eta.$$

The proof follows from the discussion preceding Proposition 3.1 together with (ii) of the Proposition. __|

3.4 Theorem. Let G be a Hilbert-Lie group over a separable Hilbert space H. We assume that there exists a modified canonical coordinate system $\{d_i:i\in\mathbb{N}\}$ with respect to an orthonormal basis $\{X_i:i\in\mathbb{N}\}$ of H. On G we are given a convolution semigroup $\{\mu_t:t\in\mathbb{R}_+\}$ with Lévy measure η and generating functional A.

Then there exist a vector $r=(r_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$ and a symmetric positive-semidefinite matrix $\alpha=(\alpha_{ij})_{i,j\in\mathbb{N}}\in\mathbb{M}(\mathbb{N},\mathbb{R})$ such that for all $f\in C_{(2)}(G)$ one has

$$Af = \Sigma_{i\geq 1} r_i X_i f(e) + \Sigma_{i,j\geq 1}\alpha_{ij}X_iX_jf(e)$$

$$+\int_{G^\times}(f-f(e) - \Sigma_{i\geq 1}d_iX_if(e))d\eta.$$

<u>Proof.</u> Let $f\in C_{(2)}(G)$, hence $\in C_{(2),n}(G)$ for some $n\in\mathbb{N}$. Then by Corollary 2.6 together with the discussion preceding Proposition 3.1 we obtain that for the function $g:=B_f^n\,\Phi_n$ the integral

$$\int_{G^\times} g\,d\eta = \int_{G^\times} B_f^n\,\Phi_n d\eta$$

exists. From Corollary 3.3 we infer that

$$\int_{G^\times} g\,d\eta = \lim_{t\downarrow o}\frac{1}{t}\int_G g\,d\mu_t.$$

Now let V be a neighborhood of e chosen as in the definition of B_f^n. Since for all $i,j=1,\ldots,n$ the functions $d_id_j\Phi_n^{-1}$ are bounded and continuous on $V^\times:=V\setminus\{e\}$ the integrals

$$\int_{V^\times} d_id_j d\eta = \int_{V\setminus[\Phi_n=o]} d_id_j\Phi_n^{-1}d(\Phi_n\cdot\eta)$$

exist, as follows from (i) of Proposition 3.2. Moreover we have

$$g = f-f(e) - \Sigma_{i=1}^n d_iX_if(e) - \frac{1}{2}\Sigma_{i,j=1}^n d_id_jX_iX_jf(e)$$

on V, hence

$$\lim_{t\downarrow o}\frac{1}{t}\int_V g\,dy_t$$

$$= \int_{V^\times}(f-f(e)-\Sigma_{i=1}^n d_iX_if(e))d\eta - \frac{1}{2}\Sigma_{i,j=1}^n\int_{V^\times} d_id_jX_iX_jf(e)d\eta.$$

Consequently,

$$\lim_{t\downarrow o}\frac{1}{t}\int_V(f-f(e))d\mu_t$$

$$= \int_{V^\times} g\,d\eta + \Sigma_{i=1}^{n} (\lim_{t\downarrow o} \frac{1}{t} \int_V d_i d\mu_t) X_i f(e)$$

$$+ \frac{1}{2} \Sigma_{i,j=1}^{n} (\lim_{t\downarrow o} \frac{1}{t} \int_V d_i d_j d\mu_t) X_i X_j f(e).$$

On the other hand, since $\eta(\partial V)=o$, we obtain that

$$\int_{\complement V} (f - f(e) - \Sigma_{i=1}^{n} d_i X_i f(e) - \frac{1}{2} \Sigma_{i,j=1}^{n} d_i d_j X_i X_j f(e)) d\eta$$

$$= \lim_{t\downarrow o} \frac{1}{t} \int_{\complement V} (f-f(e)) d\mu_t$$

$$- \lim_{t\downarrow o} \frac{1}{t} \int_{\complement V} (\Sigma_{i=1}^{n} d_i X_i f(e) - \Sigma_{i,j=1}^{n} d_i d_j X_i X_j f(e)) d\mu_t$$

which altogether implies that

$$Af = \Sigma_{i=1}^{n} A(d_i) X_i f(e) + \frac{1}{2} \Sigma_{i,j=1}^{n} A(d_i d_j) X_i X_j f(e)$$

$$+ \int_{G^\times} (f-f(e) - \Sigma_{i=1}^{n} d_i X_i f(e)) d\eta$$

$$- \frac{1}{2} \int_{G^\times} (\Sigma_{i,j=1}^{n} d_i d_j X_i X_j f(e)) d\eta.$$

Defining $r_i^n := A(d_i)$ and

$$\alpha_{ij}^n := \frac{1}{2}(A(d_i d_j) - \int_{G^\times} d_i d_j d\eta)$$

for $i,j=1,\dots,n$ we then arrive at the representation

$$Af = \Sigma_{i=1}^{n} r_i^n X_i f(e) + \Sigma_{i,j=1}^{n} \alpha_{ij}^n X_i X_j f(e)$$

$$+ \int_{G^\times} (f-f(e) - \Sigma_{i=1}^{n} X_i f(e)) d\eta.$$

As in the proof of Theorem 4.2.4 in [3] one shows that the matrix $\alpha^n := (\alpha_{ij}^n)_{i,j=1,\dots,n} \in \mathbb{M}(n, \mathbb{R})$ is symmetric and positive-semi-definite. Moreover, from the definition of the system $\{d_i : i \in \mathbb{N}\}$ of modified canonical coordinates we conclude that $r_i^n = r_i^{n+1}$ and $\alpha_{ij}^n = \alpha_{ij}^{n+1}$ for all $i,j=1,\dots,n$ and $n \in \mathbb{N}$. Since $f \in C_{(2)}(G)$ was chosen arbitrarily, there exist a vector $(r_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and a symmetric positive-semidefinite matrix $\alpha:=(\alpha_{ij})_{i,j\in\mathbb{N}}\in\mathbb{M}(\mathbb{N},\mathbb{R})$ such that $r_i=r_i^n$ and $\alpha_{ij}=\alpha_{ij}^n$ for all $i,j=1,\dots,n, n\in\mathbb{N}$. The proof is complete. $\_|$

3.5 Remark. If G is commutative one can show that the space $C_{(2)}(G)$ which in this case coincides with its right counterpart $\tilde{C}_{(2)}(G)$ is contained in the domain D(N) of the generator N of the given convolution semigroup $\{\mu_t:t\in\mathbb{R}_+\}$ on G, and a representation of N analoguous to that of A is available. As for finite dimensional Lie groups also for Hilbert-Lie groups G Gaussian semigroups can be defined and characterized by the locality of their generators. (cf.[3],§ 6.2).

3.6 Remark. In the special case that G itself is a separable Hilbert space H the representation of the generator N of a convolution semigroup $\{\mu_t:t\in\mathbb{R}_+\}$ on H has been established previously in [5] and [7]. In fact, in [5] the space $C_u^{(2)}(H)$ of all twice Fréchet differentiable functions $f\in C_u(H)$ such that $||f'||:=\sup_{x\in H}||f'(x)||<\infty$, $||f''||:=\sup_{x\in H}||f''(x)||<\infty$ and f'' is uniformly continuous has been introduced, and it has been shown that $C_u^{(2)}(H)\subset D(N)$. Note that $C_{(2)}(H)\subset C_2(G)=C_u^{(2)}(H)$, and that our result yields the representation of [5] at least for functions in $C_{(2)}(H)$.

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