KAMAL BOUSSAF
ALAIN ESCASSUT

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ABSOLUTE VALUES ON ALGEBRAS $H(D)$

by Kamal Boussaf and Alain Escassut

Abstract. Let $K$ be an algebraically closed complete ultrametric field, and let $D$ be an infraconnected set in $K$ such that the set $H(D)$ of the analytic elements on $D$ is a ring. Among the continuous multiplicative semi-norms on $H(D)$, we look for the ones that are absolute values. They are characterized by the location of the $T$-filters on $D$. Besides, we characterize the sets $D$ such that $H(D)$ admits at least one continuous absolute value $|\cdot|$.

Notations: Let $K$ be an algebraically closed field complete for an ultrametric absolute value.

Given $a \in K$ and $r > 0$, $d(a, r)$ (resp. $d(a, r^-)$, resp. $C(a, r)$) denotes the disk \( \{x \in K \mid |x - a| \leq r\} \) (resp. $\{x \in K \mid |x - a| < r\}$, resp. the circle $\{x \in K \mid |x - a| = r\}$).

Given $a \in K$, $r' > 0$ and $r'' > r'$, $\Gamma(a, r', r'')$ denotes the annulus $\{x \in K \mid r' < |x - a| < r''\}$.

Given a set $A$ in $K$ and a point $a \in K$, we denote by $\delta(a, A)$ the distance from $a$ to $A$. Let $E$ be an infinite set in $K$, and let $a \in E$. If $E$ is bounded of diameter $r$, we denote by $\bar{E}$ the disk $d(a, r)$, and if $E$ is not bounded, we put $\bar{E} = K$. Then, $\bar{E} \setminus \hat{E}$ is known to admit a partition of the form $(d(a_i, r^-_i)_{i \in J})$, with $r_i = \delta(a_i, D)$ for each $i \in J$. The disks $d(a_i, r^-_i)_{i \in J}$, are named the holes of $E$.

$R(E)$ denotes the set of rational functions $h \in K(x)$ with no poles in $E$. This is a $K$-subalgebra of the algebra $K^D$ of all functions from $E$ into $K$. Then $R(E)$ is provided with the topology $\mathcal{U}_E$ of uniform convergence on $E$, and is a topological group for this topology. $H(E)$ denotes the completion of $R(E)$ for this topology and its elements are named the analytic elements on $E$ [1], [2], [3], [9].

By [3], we remember that $H(E)$ is a $K$-subalgera of the algebra $K^D$ if and only if $E$ satisfies the following conditions:

A) $\bar{E} \setminus \hat{E}$ is bounded,

B) $\bar{E} \setminus \hat{E} \subset \hat{E}$.
Henceforth, D will denote an infraconnected set satisfying Conditions A) and B).

In [7], [4], the continuous multiplicative semi-norms of an algebra $H(D)$ were characterized by means of the circular filters on D. So we have to recall the definitions of monotonous and circular filters.

Definitions and notations: The set of those multiplicative semi-norms that are continuous with respect to the topology of uniform convergence on D is denoted by $\text{Mult}(H(D),\mathcal{U}_D)$. Given a continuous multiplicative semi-norm $\psi \in \text{Mult}(H(D),\mathcal{U}_D)$ we denote by $\text{Ker}\psi$ the closed prime ideal of the $f \in H(D)$ such that $\psi(f) = 0$.

$\psi$ will be said to be punctual if $\text{Ker}\psi$ is a maximal ideal of codimension 1 of $H(D)$. We know that there exists a bijection $M$ from D onto the set of maximal ideals of codimension 1 of $H(D)$, defined as $M(a) = \{ f \in H(D) | f(a) = 0 \}$ (indeed, this was shown in [3], Proposition II.6, when $D$ is closed and bounded, and it is easily extended to all sets $D$ satisfying Conditions A) and B)). As a consequence, there exists a bijection $S$ from D onto the set of punctual continuous multiplicative semi-norms of $H(D)$ defined as $S(a)(f) = |f(a)|$, whenever $f \in H(D)$.

In order to recall the characterization of the continuous multiplicative semi-norms of $H(D)$, we first have to recall the definition of monotonous and circular filters.

Given a filter $\mathcal{F}$ on D, we will denote by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{\mathcal{F}} f(x) = 0$.

Let $a \in \bar{D}$ and $S \in \mathbb{R}_+^*$ be such that $\Gamma(a,r,S) \cap D \neq \emptyset$ whenever $r \in ]0,S[$ (resp. $\Gamma(a,S,r) \cap D \neq \emptyset$ whenever $r > S$). We call an increasing (resp. a decreasing) filter of center a and diameter S, on D the filter $\mathcal{F}$ on D that admits for base the family of sets $\Gamma(a,r,S) \cap D$ (resp. $\Gamma(a,S,r) \cap D$). For every sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n < r_{n+1}$ (resp. $r_n > r_{n+1}$) and $\lim_{n \to \infty} r_n = S$, it is seen that the sequence $\Gamma(a,r_n,S) \cap D$ (resp. $\Gamma(a,S,r_n) \cap D$) is a base of $\mathcal{F}$ and such a base is called a canonical base.

We call a decreasing filter with no center of canonical base $(D_n)_{n \in \mathbb{N}}$ and diameter $S > 0$, on D a filter $\mathcal{F}$ on D that admits for base a sequence $(D_n)_{n \in \mathbb{N}}$ of the form $D_n = d(a_n,r_n) \cap D$ with $D_{n+1} \subset D_n$ , $r_{n+1} < r_n$, $\lim_{n \to \infty} r_n = S$, and $\bigcap_{n \in \mathbb{N}} d(a_n,r_n) = \emptyset$.

Given an increasing (resp. a decreasing) filter $\mathcal{F}$ on D of center a and diameter r, we will denote by $\mathcal{P}(\mathcal{F})$ the set $\{ x \in D | |x-a| \geq r \}$ (resp. the set $\{ x \in D | |x-a| \leq r \}$ and by $\mathcal{C}(\mathcal{F})$ the set $\{ x \in D | |x-a| < r \}$ (resp. the set $\{ x \in D | |x-a| > r \}$). Besides $\mathcal{C}(\mathcal{F})$ will be named the body of $\mathcal{F}$ and $\mathcal{P}(\mathcal{F})$ will be named the beach of $\mathcal{F}$.

We call a monotonous filter on D a filter which is either an increasing filter or a decreasing filter (with or without a center).

Given a monotonous filter $\mathcal{F}$ we will denote by $\text{diam}(\mathcal{F})$ its diameter.
The field $K$ is said to be spherically complete if every decreasing filter on $K$ has a center in $K$. (The field $\mathbb{F}_p$ for example is not spherically complete). However, every algebraically closed complete ultrametric field admits a spherically complete algebraically closed extension [10], [11].

Two monotonous filters $\mathcal{F}$ and $\mathcal{G}$ are said to be complementary if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$.

Let $\mathcal{F}$ be an increasing (resp. a decreasing) filter of center $a$ and diameter $S$ on $D$. $\mathcal{F}$ is said to be pierced if for every $r \in [0, S]$, (resp. $r < S$), $\Gamma(a, r, S)$ (resp. $\Gamma(a, S, r)$) contains some hole $T_m$ of $D$. A decreasing filter with no center $\mathcal{F}$, and canonical base $(D_n)_{n \in \mathbb{N}}$, on $D$ is said to be pierced if for every $m \in \mathbb{N}$, $D_m \setminus D_{m+1}$ contains some hole $T_m$ of $D$.

Let $a \in \partial D$, let $\rho = \delta(a, D)$ be such that $\rho \leq S \leq \text{diam}(D)$. We call circular filter of center $a$ and diameter $S$ on $D$ the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a, S)$, $r' < S < r''$, i.e. $\mathcal{F}$ is the filter which admits for base the family of sets of the form $D \cap \left( \bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i) \right)$ with $\alpha_i \in d(a, S), r'_i < S < r''_i$ ($1 \leq i \leq q$, $q \in \mathbb{N}$).

A decreasing filter with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$ is also called circular filter on $D$ with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$.

Finally the filter of the neighbourhoods of a point $a \in D$ will be called circular filter of the neighbourhoods of $a$ on $D$. It will be also named circular filter of center $a$ and diameter 0.

A circular filter on $D$ will be said to be large if it has diameter different from 0.

Given a circular filter $\mathcal{F}$, its diameter will be denoted by $\text{diam}(\mathcal{F})$.

The set of the circular filters on $D$ will be denoted by $\Phi(D)$.

Now let $\mathcal{F}$ be a circular filter on $D$. By [7], [4], we have the following characterization of continuous multiplicative semi-norms of $H(D)$.

**Theorem 0:** Let $\mathcal{F}$ be a circular filter on $D$. For every $f \in H(D)$, $|f(x)|$ admits a limit along $\mathcal{F}$, and this limit, denoted by $\varphi_\mathcal{F}(f)$, defines a continuous multiplicative semi-norm $\varphi_\mathcal{F}$ on $H(D)$. Further, the mapping $\Theta$ from $\Phi(D)$ into $\text{Mult}(H(D), \mathcal{U}_D)$ defined as $\Theta(\mathcal{F}) = \varphi_\mathcal{F}$ is a bijection.

**Notations:** For convenience, when $\mathcal{F}$ is the circular filter of center $a$ and diameter $r$, we also denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_\mathcal{F}$.

Here, assuming $H(D)$ to be a $K$-algebra, we study what continuous multiplicative semi-norms of $H(D)$ are norms, i.e. are absolute values on $H(D)$. Of course, this requires $H(D)$ to have no divisors of zero. But then, as a transcendental extension of the field $K$, the field of quotients $L$ of $H(D)$ does admit absolute values extending the
one of $K$. Hence so does $H(D)$. The problem, here, is whether such absolute values are continuous with respect to the topology of $H(D)$, i.e. are defined by circular filters on $D$. So, we will give the condition a circular filter has to satisfy in order that its continuous multiplicative semi-norm be an absolute value, and next, we will characterize the sets $D$ such that at least one of the continuous multiplicative semi-norms is an absolute value.

All this study involves $T$-filters, and now we have to introduce them.

**Definition:** Let $F$ be an increasing (resp. a decreasing) filter on $D$, of center $a$ and diameter $s$. An element $f \in I(F)$ is said to be strictly vanishing along $F$ if there exists $t < s$ (resp. $t > s$) such that $\varphi_{a,r}(f) > 0$ for all $r \in [t,s]$ (resp. $]s,t]$).

Let $F$ be a decreasing filter on $D$, with no center, of diameter $r$, of canonical base $(D_n)_{n \in \mathbb{N}}$, with $D_n = d(a_n,r_n) \cap D$. Then an element $f \in I(F)$ is said to be strictly vanishing along $F$ if there exists $t > s$ such that $\varphi_{a_n,r}(f) > 0$ for all $r \in [r_n,t]$, for every $n \in \mathbb{N}$.

Let $F$ be a filter on $D$, and let $A \subset D$. $F$ will be said to be secant with $A$ if for every $F \in F$, $A \cap F$ is not empty.

$T$-filters are certain pierced monotonous filters satisfying particular properties linked to the holes of $D$, and were defined in [1], [2], [5]. Here we will only use the following characterization:

A monotonous filter $F$ is a $T$-filter if and only if there exists $f \in H(D)$ strictly vanishing along $F$.

From [2], [5], [6], we can easily deduce the following technical propositions that will be indispensable.

**Proposition P:** Let $b \in D$, $l > 0$ and let $f \in H(D)$ satisfy $f(b) \neq 0$ and $\varphi_{b,l} = 0$. There exists an increasing $T$-filter $F$ of center $b$ and diameter $t \in ]0,l]$ such that $f$ is strictly vanishing along $F$ and satisfies $\varphi_{b,s}(f) > 0$ for every $s \in ]0,t]$.

Let $a \in \bar{D}$ and let $r,s \in \mathbb{R}$ satisfy $\delta(a,D) \leq r \leq \text{diam}(D)$. If $f$ satisfies $f(x) = 0$ whenever $x \in d(a,r) \cap D$, then $f$ is strictly vanishing along a $T$-filter $F$ such that $d(a,r) \cap D \subset \mathcal{P}(F)$ and $b \in C(F)$.

Now, we can characterize absolute values among continuous multiplicative semi-norms.

**Theorem 1:** Let $F$ be a large circular filter on $D$. Then $\varphi_F$ is not an absolute value if and only if it satisfies one of the following conditions:

a) There exists a $T$-filter $G$ on $D$ such that $F$ is secant with $\mathcal{P}(G)$,

b) $F$ is a $T$-filter.

**Proof:** First, suppose that there exists a $T$-filter $G$ satisfying a). By Lemma 1.6 A of [5], there exists $f \in H(D)$, strictly vanishing along $G$, equal to 0 in all of $\mathcal{P}(G)$. Hence we have $\varphi_F(f) = 0$ and then $\varphi_F$ is not a norm. Now if $F$ is a $T$-filter, then there exists
$f \in H(D)$ strictly vanishing along $\mathcal{F}$ and therefore we have $\lim_{x \to a} f(x) = 0$, hence $\varphi_x$ is not a norm.

Now we suppose that there exists no $T$-filter $\mathcal{G}$ satisfying a) and that $\mathcal{F}$ is not a $T$-filter, and we suppose that $\varphi_x$ is not a norm. Let $f \in H(D) \setminus \{0\}$ satisfying $\varphi_x(f) = 0$. Let $S = \text{diam}(\mathcal{F})$. Let $b \in D$ be such that $f(b) \neq 0$.

We first assume that $\mathcal{F}$ has a center $a$.

On the first hand, we suppose that $b \in d(a, S)$. Since $\varphi_{a,r}(f) \neq 0$, when $r$ approaches 0 there does exist $s \in ]0, S[$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in ]0, s[$. Hence $f$ is strictly vanishing along the increasing filter $\mathcal{G}$ of center $b$ and diameter $s$, and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.

On the second hand, we suppose that $|a - b| > S$. Let $t = |a - b|$. If $\varphi_{b,t}(f) = 0$, there exists $s \in ]0, t]$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in ]0, s[$, hence $f$ is strictly vanishing along an increasing $T$-filter $\mathcal{G}$ of center $b$ and diameter $s$ and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$. Now we may assume $\varphi_{b,t}(f) \neq 0$. But $\varphi_{b,t}(f) = \varphi_{a,t}(f)$ and therefore there exists $s \in [S, t]$ such that $\varphi_{a,s}(f) = 0$ and $\varphi_{a,r}(f) \neq 0$ whenever $r \in ]s, t]$. Hence $f$ is strictly vanishing along a decreasing $T$-filter $\mathcal{G}$ of center $a$ and diameter $s$, and $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.

Now, assume that $\mathcal{F}$ is a decreasing filter with no center. Let $(D_n)_{n \in \mathbb{N}}$ be a canonical base of $\mathcal{F}$, and for each $n \in \mathbb{N}$, let $D_n = d(a_n, r_n) \cap D$ and let $u_n = \varphi_{a_n,r_n}(f)$. If there exists $q \in \mathbb{N}$ such that $u_q = 0$, by Proposition P, $D$ admits a $T$-filter $\mathcal{G}$ such that $D_q$ is included in $\mathcal{P}(\mathcal{G})$, and therefore, $\mathcal{F}$ is obviously secant with $\mathcal{P}(\mathcal{G})$. Hence we can assume that $u_n > 0$ for every $n \in \mathbb{N}$. Now, suppose that there exist $q \in \mathbb{N}$ and $r \in [r_{q+1}, r_q]$ such that $\varphi_{a_{q+1},r}(f) = 0$. As we just saw, there exists a $T$-filter $\mathcal{G}$ on $D$ such that the circular filter $\mathcal{F}_q$ is secant with $\mathcal{P}(\mathcal{G})$, and therefore so is $\mathcal{F}$.

Thus, without loss of generality, we can assume that $\varphi_{a_{q+1},r}(f) > 0$ for every $r \in [r_{q+1}, r_q]$, for every $q \in \mathbb{N}$. Hence $f$ is just strictly vanishing along the decreasing filter $\mathcal{F}$ and therefore $\mathcal{F}$ is a $T$-filter. This ends the proof of Theorem 1.

**Corollary a:** All the not punctual continuous multiplicative semi-norms of $H(D)$ are absolute values if and only if $D$ has no $T$-filter.

**Definitions and notations:** Let $\text{inc}T(D)$ (resp. $\text{dec}T(D)$) be the set of increasing (resp. decreasing) $T$-filters on $D$. We will denote by $\preceq$ the relation defined on $\text{inc}T(D)$ (resp. $\text{dec}T(D)$) by $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{C}(\mathcal{F}_2) \subset \mathcal{C}(\mathcal{F}_1)$. This relation is obviously seen to be an order relation on $\text{inc}T(D)$ (resp. $\text{dec}T(D)$).

An increasing (resp. a decreasing) $T$-filter $\mathcal{F}$ will be said to be *maximal* if it is maximal in $\text{inc}T(D)$ (resp. in $\text{dec}T(D)$) with respect to this relation.

We will denote by $\prec$ the strict order associated to $\preceq$ by $\mathcal{F}_1 \prec \mathcal{F}_2$ if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$.

We will call an ascending chain of increasing (resp. decreasing) $T$-filters a sequence of increasing (resp. decreasing) $T$-filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \prec \mathcal{F}_{n+1}$ whenever $n \in \mathbb{N}$.
Let \((F_n)_{n \in \mathbb{N}}\) be an ascending chain of increasing T-filters. For each \(n \in \mathbb{N}\) let \(r_n = \text{diam}(F_n)\). Since the sequence \((r_n)_{n \in \mathbb{N}}\) is decreasing, we put \(r = \lim_{n \to \infty} r_n\), and then \(r\) will be named the diameter of the chain.

We put \(A := \bigcap_{n \in \mathbb{N}} C(F_n)\) and for each \(n \in \mathbb{N}\), \(D_n := C(F_n) \setminus A\). The sequence \((D_n)_{n \in \mathbb{N}}\) is then a base of a filter \(\mathcal{F}\) on \(D\) of diameter \(r\).

If \(r = 0\), since \(D\) (Condition Condition B), \(A\) is a point \(a\) of \(D\), hence \(\mathcal{F}\) is the filter of the neighbourhoods of \(a\) in \(D\).

If \(r > 0\) \(\mathcal{F}\) is a decreasing filter on \(D\) of diameter \(r\).

In both cases \(\mathcal{F}\) will be called the returning filter of the ascending chain \((F_n)_{n \in \mathbb{N}}\).

Now let \((F_n)_{n \in \mathbb{N}}\) be an ascending chain of decreasing T-filters and let \(a \in \mathcal{P}(F_n)\) for some \(n \in \mathbb{N}\). The sequence \((r_n)_{n \in \mathbb{N}}\) is an increasing sequence of limit \(r \in [0, +\infty]\), and \(r\) will be named the diameter of the chain. Since \(D\) belongs to \(A\), by Condition A) we notice that \(r < +\infty\), and then we will call the returning filter of the ascending chain \((F_n)_{n \in \mathbb{N}}\) the increasing filter \(\mathcal{F}\) of center \(a\) and diameter \(r\) (it is seen that \(\mathcal{F}\) does not depend on the point \(a \in \mathcal{P}(F_n)\), whenever \(n \in \mathbb{N}\)).

Lemma 1: Let \(H(D)\) have no divisors of zero. Then \(\text{inc}T(D)\) is totally ordered with respect to the order \(\preccurlyeq\).

Proof: Suppose that \(\mathcal{F}, \mathcal{G}\) are increasing T-filters on \(D\) that are not comparable. We put \(A = C(\mathcal{F})\) and \(B = C(\mathcal{G})\). Then \(A, B\) are two disks of \(D\) which satisfy neither \(A \subset B\), nor \(B \subset A\). Hence we have \(A \cap B = \emptyset\). As a consequence, \(\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G})\) is equal to \(D\) and then by [6] \(H(D)\) has divisors of zero. Hence this contradicts the hypothesis and ends the proof.

Corollary b: Let \(H(D)\) have no divisors of zero, and let \(\mathcal{F}\) (resp. \(\mathcal{G}\)) be an increasing T-filter on \(D\), of diameter \(r\) (resp. \(s\)). If \(r > s\) then \(\mathcal{F} \prec \mathcal{G}\). If \(r = s\), then \(\mathcal{F} = \mathcal{G}\).

We are now able to characterize the sets \(D\) such that \(H(D)\) admits continuous absolute values. Let us recall the following theorem of [6]:

The algebra \(H(D)\) has no divisors of zero if and only if \(D\) does not admit two complementary T-filters.

We will use comparison between filters. Here, a filter \(\mathcal{F}\) will be said thinner than a filter \(\mathcal{G}\) every element of \(\mathcal{G}\) belongs to \(\mathcal{F}\).

Theorem 2: Let \(H(D)\) have no divisors of zero. Then \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm if and only if \(D\) admits an ascending chain of T-filters \((F_n)_{n \in \mathbb{N}}\) whose returning filter is either a T-filter or a Cauchy filter.

Proof: On the first hand, we suppose that \(D\) admits an ascending chain of T-filters \((F_n)_{n \in \mathbb{N}}\) whose returning filter \(\mathcal{F}\) is either a T-filter or a Cauchy filter and we will prove that \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm. We denote by \(r\) the diameter
of this ascending chain \((F_n)_{n \in \mathbb{N}}\). Let \(G\) be a circular filter on \(D\), of diameter \(s > 0\). By Theorem 1, we only have to show that either \(G\) is a \(T\)-filter, or \(G\) is secant with the beach of a \(T\)-filter.

First we suppose \(F\) is a Cauchy filter. Then the \(F_n\) are increasing \(T\)-filters. Let \(q \in \mathbb{N}\) be such that \(r_q < s\). Then \(G\) is clearly secant with \(\mathcal{P}(F_q)\).

Now we suppose that \(F\) is a \(T\)-filter. Then we just have to consider the case when \(G\) is secant with \(C(F)\) and is not equal to \(F\).

First we suppose \(F\) increasing, of center \(b\) and diameter \(r\). Hence we have \(C(F) = d(b, r^-) \cap D\). Then \(G\) has a diameter \(s \in ]0, r[\), and then it admits elements \(E\) of diameter \(t \in ]s, r[\), included in \(d(b, r^-) \cap D\). Since \(E \cap C(F) \neq \emptyset\), given a point \(a \in E \cap C(F)\) we have \(E \subset d(a, t)\). Let \(q \in \mathbb{N}\) be such that \(r_q > \max(t, |a - b|)\). Then \(E\) is included in \(d(b, r_q)\) and therefore is included in \(\mathcal{P}(F_q)\). Hence \(G\) is secant with \(\mathcal{P}(F_q)\).

Finally we suppose \(F\) decreasing, of diameter \(r\). Since \(G\) is secant with \(C(F)\), and is not less thin than \(F\), we have \(r < s\), and there exists \(t \in ]r, s[\) and \(a \in D\) such that \(G\) is secant with \((K \setminus d(a, t^-)) \cap D\) while \(F\) is secant with \(d(a, t^-) \cap D\). Let \(q \in \mathbb{N}\) be such that \(r_q < t\) and let \(a_q\) be a center of \(F_q\). Then \(a_q\) belongs to \(d(a, t)\) and we have \(\mathcal{P}(F_q) = (K \setminus d(a, r_q^-)) \cap D\). Hence \(G\) is secant with \(\mathcal{P}(F_q)\) and this finishes showing that \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm.

On the second hand, reciprocally, we suppose that \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm and we will show that \(D\) admits an ascending chain of \(T\)-filters \((F_n)_{n \in \mathbb{N}}\) whose returning filter is either a \(T\)-filter or a Cauchy filter.

We denote by \(R'\) the set of the diameters of the \(F \in \text{inc} T(D)\), by \(R''\) the set of the diameters of the \(F \in \text{dec} T(D)\), and we put \(R = R' \cup R''\). Since \(H(D)\) has no norm, by Theorem 1 \(R\) is not empty. Since \(D\) belongs to \(A\), by Condition A) \(R\) is obviously bounded. We put \(t = \sup(R)\). Let \(a \in D\). We will show that \(\text{inc} T(D) \neq \emptyset\). Indeed, suppose \(\text{inc} T(D) = \emptyset\). First let \(D\) be bounded, of diameter \(S\). Any decreasing filter on \(D\) has a diameter \(r < S\), and therefore the circular filter \(G\) of center \(a\) and diameter \(S\) is secant with \(D\), but (of course) is not a \(T\)-filter on \(D\), and is not secant with the beach of any decreasing \(T\)-filter on \(D\).

As a consequence, by Theorem 1 \(\varphi_G\) is a norm. Thus we see that \(D\) is not bounded. Then, any circular filter \(G\) of center \(a\) and diameter \(r > t\) is a \(T\)-filter and is not secant with the beach of any \(T\)-filter. Finally this shows that \(\varphi_G\) is a norm again. Thus we see that \(\text{inc} T(D)\) is not empty, and neither is \(R'\).

Now, we put \(s = \inf(R')\). First we suppose \(s \in R'\). Let \(T \in \text{inc} T(D)\) satisfy \(\text{diam}(T) = s\), and let \(b \in C(T)\). Then for every \(r \in ]0, s[\), the circular filter of center \(b\) and diameter \(r\) is not a decreasing \(T\)-filter, and therefore is secant with the beach of a decreasing \(T\)-filter. Hence there exists a decreasing \(T\)-filter \(F\) of center \(b\) and diameter \(\ell \geq s\). But since \(H(D)\) has no divisors of zero, \(F\) is not complementary with \(T\), hence we have \(s \leq \ell\), i.e. \(s \leq \ell < r\). So, we clearly deduce the existence of a sequence of decreasing \(T\)-filters \((F_n)_{n \in \mathbb{N}}\), such that each one admits \(b\) as a center and has a diameter \(r_n\) satisfying \(r_n < r_{n+1} < s\), \(\lim_{n \to \infty} r_n = s\). Therefore the sequence \((F_n)_{n \in \mathbb{N}}\)
is an ascending chain of decreasing \(T\)-filters such that \(\bigcap_{n \in \mathbb{N}} \mathcal{P}(\mathcal{F}_n) = \mathcal{C}(T)\), hence the returning filter of the ascending chain \((\mathcal{F}_n)_{n \in \mathbb{N}}\) is a \(T\)-filter.

Now, we suppose \(s \notin \mathcal{R}'\). Let \((T_n)_{n \in \mathbb{N}}\) be a sequence in \(\text{inc}T(D)\) such that 
\[
\lim_{n \to \infty} \text{diam}(T_n) = s, \text{ with } \text{diam}(T_{n+1}) < \text{diam}(T_n) \text{ for all } n \in \mathbb{N}.
\]
For every \(n \in \mathbb{N}\), we put \(A_n = \mathcal{C}(T_n)\), and \(r_n = \text{diam}(A_n)\). By Lemma 1 the sequence \(\mathcal{C}(T_n)_{n \in \mathbb{N}}\) is strictly decreasing, and so is the sequence \((A_n)_{n \in \mathbb{N}}\). Besides, the sequence \((T_n)_{n \in \mathbb{N}}\) is an ascending chain of increasing \(T\)-filters. Obviously each \(A_n\) contains a hole \(T_n\) of \(D\).

If \(s = 0\), then we have \(\delta(b, T_n) \leq r_n\), and therefore \(b\) does not belong to \(\overline{D}\), but then, by Condition A) \(b\) must belong to \(\overline{D}\). Thus, the sequence \((T_n)_{n \in \mathbb{N}}\) is an ascending chain of increasing \(T\)-filters that converges to \(b\), and therefore the returning filter of the ascending chain \((T_n)_{n \in \mathbb{N}}\) is a Cauchy filter.

Finally, it only remains to consider the case when \(s > 0\), with \(s \notin \mathcal{R}'\). Let \(\mathcal{F}\) be the returning filter of the sequence \((T_n)_{n \in \mathbb{N}}\). If \(\mathcal{F}\) were a \(T\)-filter, or were secant with the beach of an increasing \(T\)-filter, this increasing \(T\)-filter would have a diameter inferior or equal to \(s\). Hence \(\mathcal{F}\) either is a decreasing \(T\)-filter or is secant with the beach of a decreasing \(T\)-filter. Of course, if \(\mathcal{F}\) is a \(T\)-filter, it is just the returning filter of the chain \((T_n)_{n \in \mathbb{N}}\). Finally if \(\mathcal{F}\) is secant with the beach of a decreasing \(T\)-filter \(\mathcal{G}\), then we have \(\text{diam}(\mathcal{G}) \leq s\) because if \(\text{diam}(\mathcal{G})\) were strictly superior to \(s\), then \(\mathcal{G}\) would be complementary to \(T_n\) when \(n\) is big enough, and therefore \(H(D)\) would have divisors of zero. Hence we have \(\text{diam}(\mathcal{G}) = s\), and therefore \(\mathcal{G}\) is just the returning filter of the sequence \((T_n)_{n \in \mathbb{N}}\). This finishes proving that \(D\) admits an ascending chain of \(T\)-filters \((\mathcal{F}_n)_{n \in \mathbb{N}}\) whose returning filter is either a \(T\)-filter or a Cauchy filter, and this ends the proof of Theorem 2.

REFERENCES


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Laboratoire de Mathématiques Pures
Université Blaise Pascal
(Clermont-Ferrand)
Complexe Scientifique des Cézeaux
F 63177 AUBIERE CEDEX
FRANCE