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$p$-adic analytic interpolation


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p-ADIC ANALYTIC INTERPOLATION

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Abstract. Let $K$ be a complete ultrametric algebraically closed field. We study the Kernel of infinite van der Monde Matrices and show close connections with the zeroes of analytic functions. We study when such a matrix is invertible. Finally we use these results to obtain interpolation processes for analytic functions. They are more accurate if $K$ is spherically complete.

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1. NOTATIONS, DEFINITIONS AND THEOREMS

$K$ denotes an algebraically closed field complete for an ultrametric absolute value. Given $a \in K$, $r > 0$, we denote by $d(a, r)$ (resp. $d(a, r^-)$) the disk $\{x \in K : |x - a| \leq r\}$ (resp. $\{x \in K : |x - a| < r\}$). Given $r > 0$ we denote by $C(0, r)$ the circle $d(0, r) \setminus d(0, r^-)$. Given $r_1, r_2 \in \mathbb{R}_+$ such that $0 < r_1 < r_2$, we denote by $\Gamma(0, r_1, r_2)$ the set $d(0, r_2^-) \setminus d(0, r_1^-)$.

Given $r > 0$, we denote by $A(d(0, r^-))$ the algebra of the power series $\sum_{n=0}^{\infty} b_n x^n$ converging for $|x| < r$. Given $K$-vector spaces $E$, $F$, $\mathcal{L}(E, F)$ will denote the space of the $K$-linear mappings from $E$ into $F$.

$\mathcal{E}$ will denote the $K$-vector space of the sequences in $K$, and $\mathcal{E}_0$ will denote the subspace of the bounded sequences. The identically zero sequence will be denoted by $(0)$.

$\mathcal{E}_1$ will denote the set of the sequences $(a_n)$ such that $\limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq 1$. So $\mathcal{E}_1$ is seen to be a subspace of $\mathcal{E}$ isomorphic to the space $A(d(0, 1^-))$, and obviously contains $\mathcal{E}_0$.

Let $M_\infty$ be the set of the infinite matrices $(\lambda_{i,j})$ with coefficients in $K$.

$\delta_{i,j}$ will denote the Kronecker symbol. $I_\infty$ will denote the infinite identical matrix defined as $\lambda_{i,j} = \delta_{i,j}$.

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In this paper, \((a_n)\) will denote an injective sequence in \(d(0,1^-)\) such that \(a_n \neq 0\) for every \(n > 0\). and we will denote by \(M(a_n)\) the infinite matrix \(M = (\lambda_{i,j})\) defined as 
\[
\lambda_{i,j} = (a_i)^j, (i,j) \in \mathbb{N} \times \mathbb{N}.
\]
A matrix \(M = (\lambda_{i,j}) \in M_\infty\) will be said to be bounded if there exists \(A \in \mathbb{R}_+\) such that \(|\lambda_{i,j}| \leq A\) whenever \((i,j) \in \mathbb{N} \times \mathbb{N}\).

\(M\) will be said to be line-vanishing if for each \(i \in \mathbb{N}\), we have \(\lim_{j \to \infty} \lambda_{i,j} = 0\).

A line-vanishing matrix \(M\) is seen to define a \(K\)-linear mapping \(\psi_M\) from \(\mathcal{E}_0\) into \(\mathcal{E}\).

So the matrix \(M = M(a_n)\) clearly defines a \(K\)-linear mapping \(\psi_M\) from \(\mathcal{E}_1\) into \(\mathcal{E}\), because given a sequence \((b_n) \in \mathcal{E}_1\), the series \(\sum_{n=0}^{\infty} b_n(a_j)^n\) is obviously convergent.

Lemmas 1 and 2 are immediate:

**Lemma 1:** Let \(M \in M_\infty\) be line vanishing.

The three following statements are equivalent:

- \(\psi_M\) is continuous
- \(\psi_M\) is an endomorphism of \(\mathcal{E}_0\)
- \(M\) is bounded.

In particular, Lemma 1 applies to matrices of the form \(M(a_n)\).

**Lemma 2:** Let \(M = M(a_n)\) and let \((b_n) \in \mathcal{E}_1\). Then \((b_n)\) belongs to \(\ker \phi_M\) if and only if the analytic function \(f(t) = \sum_{n=0}^{\infty} b_n t^n\) admits each point \(a_j\) for zero.

**Theorem 1:** Let \(M = M(a_n)\). Then \(\ker \phi_M \neq \{0\}\) if and only if \(\lim_{n \to \infty} |a_n| = 1\).

Besides \(\ker \psi_M \neq \{0\}\) if and only if \(\prod_{n=0}^{\infty} |a_n| > 0\).

**Theorem 2:** Let \(b = (b_n) \in \mathcal{E}_0\). There exists an injective sequence \((\alpha_n)\) in \(d(0,1^-)\) such that \(b \in \ker \psi_{M(\alpha_n)}\) if and only if \(b\) satisfies \(|b_j| < \sup_{n \in \mathbb{N}} |b_n|\) for all \(j \in \mathbb{N}\).

**Definitions and notations:** An injective sequence \((a_n)\) in \(d(0,1^-)\) will be called a regular sequence if \(\inf_{n \neq m} |a_n - a_m| > 0\) and \(\lim_{n \to \infty} |a_n| = 1\).

Let \((a_n)\) be a regular sequence and let \(\rho = \inf_{n \neq m} |a_n - a_m|\). For every \(r \in [0,1]\), we will denote by \(\Omega((a_n),r)\) the set \(d(0,1^-) \setminus (\bigcup_{n \in \mathbb{N}} d(a_n, r^-))\), and by \(\Omega(a_n)\) the set \(d(0,1^-) \setminus (\bigcup_{n \in \mathbb{N}} d(a_n, \rho^-))\).
Let \( a = (a_n) \) and \( b = (b_n) \) be two sequences in \( K \). We will denote by \( a \ast b \) the convolution product \( (c_n) \) defined as \( c_n = \sum_{j=0}^{n} a_j b_{n-j} \).

**Theorem 3:** Let \( (\alpha_n) \) be a regular sequence of \( d(0,1^-) \) such that there exists \( g \in A(d(0,1^-)) \) satisfying

(i) \( \alpha_n \) is a zero of order 1 of \( g \) for all \( n \in \mathbb{N} \).

(ii) \( g(x) \neq 0 \) whenever \( x \in d(0,1^-) \setminus \{\alpha_n : n \in \mathbb{N}\} \).

(iii) \( \lim_{|x| \to 1^-} |g(x)| = +\infty \).

Let \( M = M(\alpha_n) \). Then \( \psi_M \) is injective but its image does not contain \( E_0 \). Also there exists \( P = (\lambda_{i,j}) \in M_\infty \) (not unique) satisfying

1. \( P \) is line-vanishing.
2. \( \lim_{n \to \infty} \lambda_{n,j} \alpha_h^n = 0 \) for all \( (j, h) \in \mathbb{N} \times \mathbb{N} \).
3. \( \sum_{n=0}^{\infty} \lambda_{n,j} \alpha_h^n = \delta_{j,h} \) for all \( (j, h) \in \mathbb{N} \times \mathbb{N} \).
4. \( MP = PM = I_\infty \).
5. \( P(b) \in E_1 \) for all \( b \in E_0 \).
6. \( MP(b) = b \) for all \( b \in E_0 \).
7. \( \psi_P \) is injective.

Let \( (\nu_n) \) be a sequence in \( K \) such that \( |\nu_0| \geq |\nu_n| \) for every \( n > 0 \). For every \( j \in \mathbb{N} \), let \( (\mu_{n,j})_{n \in \mathbb{N}} \) denote the sequence \( \left( \frac{1}{\sum_{m=0}^{\infty} \nu_m \alpha_j^m} \right)((\lambda_{n,j}) \ast (\nu_n)) \). Then the matrix \( Q = (\mu_{i,j}) \) also satisfies properties (1) – (7) and is not equal to \( P \) for infinitely many sequences \( (\nu_n) \).

**Remarks.**

1. Mainly, the proof of Theorem 3 takes inspiration from that of Lemma 3 in [7]. However, in this lemma, the considered matrix, roughly, was \( P \). Here the matrix we consider is a van der Monde matrix \( M \) and we look for \( P \).

2. Given \( M \), the matrix \( P \) depends on \( g \) and therefore is not unique satisfying (1) – (7). Indeed \( M_\infty \) is not a ring because the multiplication of matrices is not always defined and even when it is defined, is not always associative. As a consequence, if \( P, P' \) satisfy \( MP = MP' = PM = P'M = I_\infty \), we cannot conclude \( P' = P \).

Actually we can consider \( \phi_M \circ \psi_P \in \mathcal{L}(E_0, E) \) and then this is the identity in \( E_0 \). Next we can consider \( \psi_P \circ \psi_M \in \mathcal{L}(E_0, E_1) \) and this is the identity in \( E_0 \). But we cannot consider \( \psi_P \circ \phi_M \circ \psi_P \) because \( \psi_P \) is not defined in \( E_1 \). In the same way, we cannot consider \( (\psi_P \circ \psi_M) \circ \psi_P \) because \( \psi_P \circ \psi_M \) is only defined in \( E_0 \).

We consider the matrix \( P \) and look for "inverses" \( M \) such that \( MP = PM = I_\infty \). Suppose that there exists a bounded matrix \( M' \neq M \) such that \( PM' = M'P = I_\infty \). Now we can consider \( \phi_M' \circ (\psi_P \circ \psi_M) \in \mathcal{L}(E_0, E) \). Since \( \psi_P \circ \psi_M \) is the identity in \( E_0 \), then \( \phi_M' \circ (\psi_P \circ \psi_M) \) is equal to \( \psi_M' \). Next we can consider \( (\phi_M' \circ \psi_P) \circ \psi_M \in \mathcal{L}(E_0, E) \). Since
\( \phi_{M'} \circ \psi_P \) is the identity on \( \mathcal{E}_0 \), we have \((\phi_{M'} \circ \psi_P) \circ \psi_M = \psi_M \) and therefore \( \psi_M = \psi_M' \), hence \( M = M' \).

3. Let \( P, Q \in \mathcal{M}_\infty \) satisfy (1) – (7). Let \( \mathcal{E}' = \psi_P(\mathcal{E}_0) \), let \( \mathcal{E}'' = \psi_Q(\mathcal{E}_0) \). Then the restriction of \( \phi_M \) to \( \mathcal{E}' \) (resp. \( \mathcal{E}'' \)) is just the reciprocal of \( \psi_P \) (resp. \( \psi_Q \)).

**Conjecture.** Under the hypothesis of Theorem 1, every matrix satisfying properties (1) – (7) is of the form

\[
\mu_{n,j} = \left( \frac{1}{\sum_{m=0}^{\infty} \nu_m \lambda_{m,j}^{n,m}} \right) \left( (\lambda_{n,j}) \ast (\nu_n) \right).
\]

**Theorem 4 :** Let \( K \) be spherically complete, and let \((\alpha_n)\) be a sequence in \( d(0, 1^-) \) satisfying \(|\alpha_n - \alpha_m| \geq \min(|\alpha_n|, |\alpha_m|) \) whenever \( n \neq m \), \( \lim_{n \to \infty} |\alpha_n| = 1 \), and \( \prod_{n=0}^{\infty} |\alpha_n| = 0 \).

Then \( \mathcal{M}(\alpha_n) \) admits inverses \( P \) such that, for every bounded sequence \( b := (b_n) \) in \( K \), the sequence \( a := (a_n) = P(b) \) defines a function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in A(d(0, 1^-)) \) satisfying \( f(\alpha_n) = b_n \).

**Theorem 5 :** Let \((\alpha_n)\) be a regular sequence in \( d(0, 1^-) \). There exists a regular sequence \((\gamma_n)\) in \( d(0, 1^-) \) such that \((\alpha_n)\) is a subsequence of \((\gamma_n)\) satisfying: for every inverse matrix \( P \) of \( \mathcal{M}(\gamma_n) \) and for every bounded sequence \( b = (b_n) \) of \( K \), the sequence \( a = P(b) := (a_n) \) defines an analytic function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) such that \( f(\gamma_j) = b_j \) whenever \( j \in \mathbb{N} \).

2. PROVING THEOREMS 1 AND 2.

For each set \( D \) in \( K \), we denote by \( H(D) \) the set of the analytic elements in \( D \) (i.e., the completion of the set of the rational functions with no pole in \( D \)).

Given \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0, 1^-)) \), one defines the valuation function \( v(f, \mu) \) in the interval \([0, +\infty[\) as \( v(f, \mu) = \inf_{n \in \mathbb{N}} (v(b_n) + n\mu) \).

**Lemma 3** Let \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0, 1^-)) \). For every \( \mu > 0 \), \( f \) satisfies \( v(f, \mu) = \lim_{v(t) \to \mu, v(t) \neq \mu} v(f(t)) \). For every \( x \in d(0, 1^-) \), \( f \) satisfies \( v(f(x)) \geq v(f, v(x)) \).

For every \( r \in [0, 1[\), \( f \) satisfies \( -\log \|f\|_{d(0,r)} = v(f, -\log r) \).

Besides \( f \) is bounded in \( d(0, 1^-) \) if and only if the sequence \((b_n)\) belongs to \( \mathcal{E}_0 \). If \( f \) is bounded in \( d(0, 1^-) \), then \( \|f\|_{d(0,1^-)} = \sup_{n \in \mathbb{N}} |b_n| \) and \( -\log \|f\|_{d(0,1^-)} = \lim_{\mu \to 0} v(f, \mu) \).
Lemma 4: Let \( f(t) \in A(d(0,1^-)) \) and let \( r_1, r_2 \in (0,1) \) satisfy \( r_1 < r_2 \). If \( f \) admits \( q \) zeros in \( d(0,r_1) \) (taking multiplicities into account) and \( t \) distinct zeros \( \alpha_1, \ldots, \alpha_t \), of multiplicity order \( \zeta_j \) \((1 \leq j \leq t)\) respectively in \( \Gamma(0,r_1,r_2) \), then \( f \) satisfies

\[
v(f, -\log r_2) - v(f, -\log r_1) = - \sum_{j=1}^{t} \zeta_j (v(\alpha_j) + \log r_2) - q(\log r_2 - \log r_1).
\]

Proof of Theorem 1. Let \( b = (b_n) \in E_1 \setminus \{(0)\} \) and let \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \).

First we suppose \( \text{Ker}\phi_M \neq \{(0)\} \) and therefore we can assume \( b \in \text{Ker}\phi_M \). Then, by Lemma 2, \( f \) satisfies \( f(\alpha_j) = 0 \) for every \( j \in \mathbb{N} \). But for every \( r \in ]0,1[ \), we know that \( f \) belongs to \( H(d(0,r)) \) and has finitely many zeros in \( d(0,r) \). Hence we have \( \lim_{n \to \infty} |a_n| = 1 \).

Reciprocally, let the sequence \( (a_n) \) satisfy \( \lim_{n \to \infty} |a_n| = 1 \). By Proposition 5 in [4], we know that there exists a not identically zero analytic function \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \) which admits each \( \alpha_j \) as a zero. Hence we have \( \sum_{n=0}^{\infty} b_n a_j^n = 0 \), and of course the sequence \( (b_n) \) belongs to \( E_1 \), hence to \( \text{Ker}\phi_M \).

Now we suppose that \( \text{Ker}\psi_M \neq (0) \) and we assume that the sequence \( (b_n) \) belongs to \( \text{Ker}\psi_M \). In particular \( \text{Ker}\phi_M \neq (0) \) and therefore \( \lim_{n \to \infty} |a_n| = 1 \). Without loss of generality we may clearly assume \( |a_n| \leq |a_{n+1}| \) for all \( n \in \mathbb{N} \). Besides, by definition we have \( |a_1| > 0 \).

By Lemma 3 we know that \( \inf_{n \in \mathbb{N}} v(b_n) = \lim_{\mu \to 0^+} v(f,\mu) = \lim_{|z| \to 1, z \in D} v(f(x)) = -\log \|f\|_{d(0,1^-)} \).

Now for each \( \mu > 0 \), let \( q(\mu) \) be the unique integer such that \( v(a_n) \geq \mu \) for every \( n \leq q(\mu) \) and \( v(a_n) < \mu \) for every \( n > q(\mu) \). By Lemma 4, we check

\[
v(f,\mu) - v(f, v(a_1)) \leq \sum_{j=2}^{q(\mu)} \mu - v(a_j) + 2(\mu - v(a_1)).
\]

Since \( v(f,\mu) \) is bounded when \( \mu \) approaches 0, by (1) it is seen that \( \sum_{j=1}^{\infty} v(a_j) \) must be bounded and therefore we have \( \prod_{n=1}^{\infty} |a_n| > 0 \).

Reciprocally we suppose \( \prod_{n=1}^{\infty} |a_n| > 0 \). We can easily check that \( \lim_{n \to \infty} |a_n| = 1 \), and then we can assume \( |a_n| \leq |a_{n+1}| \) for all \( n \in \mathbb{N} \) without loss of generality. For each
j \in \mathbb{N} \text{ we put } P_j(x) = \prod_{m=1}^{j} (1 - x/a_m). \text{ By Theorem 1 in [2], we can check that there exists } f \in A(d(0,1^-)) \text{ (f not identically zero) satisfying }

(3) f(a_m) = 0 \text{ for all } m \in \mathbb{N}, \text{ and }
(4) v(f, \mu) \geq v(P_{q(\mu)}, \mu) - 1 \text{ for all } \mu > 0.

Now we notice that if } \mu_1 > \mu_2 > 0 \text{ then we have } v(P_{q(\mu_1)}, \mu_1) = v(P_{q(\mu_2)}, \mu_1) \text{ and then we see that } \lim_{\mu \to 0^+} v(P_{q(\mu)}, \mu) = \sum_{j=1}^{\infty} v(a_j). \text{ But by (2) we have } \sum_{j=1}^{\infty} v(a_j) < +\infty \text{ and therefore by (4), } v(f, \mu) \text{ is bounded in } [0, +\infty[. \text{ Let } f(t) = \sum_{n=0}^{\infty} b_n t^n. \text{ By Lemma 3 the sequence } (b_n) \text{ is bounded and by (3) it clearly belongs to Ker.}\psi_M. \text{ This finishes the proof of Theorem 1.}

Lemma 5 : Let } f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \text{ and let } r \in (0,1). \text{ Then } f \text{ admits at least one zero in } C(0,r) \text{ if and only if there exist } k, l \in \mathbb{N} (k < l) \text{ such that } |b_k| r^k = |b_l| r^l.

Proof of Theorem 2. As a consequence of Lemma 5, a function } f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \text{ admits infinitely many zeros in } d(0,1^-) \text{ if and only if } |b_j| < \sup_{n \in \mathbb{N}} |b_n| \text{ for every } j \in \mathbb{N}. \text{ Then the conclusion comes from Lemma 2.}

3. PROVING THEOREM 3.

As an application of Corollary (of Theorem 5) in [8], we have this lemma.

Lemma 6 : Let } f \in A(d(0,1^-)) \text{ have a regular sequence of zeros } (b_n) \text{ and satisfy } \lim_{|x| \to 1^- \atop x \in \Omega(b_n)} |f(x)| = +\infty. \text{ Then } 1/f \text{ belongs to } H(\Omega(b_n)).

Proof of Theorem 3. We may obviously assume } |\alpha_n| \leq |\alpha_{n+1}| \text{ and therefore } \alpha_n \neq 0 \text{ whenever } n > 0. \text{ Since } g \text{ is not bounded in } d(0,1^-), \text{ by Lemma 3 we have } \lim_{\mu \to 0^+} v(g, \mu) = -\infty, \text{ and by Lemma 4 the sequence of the zeros } (\alpha_n) \text{ satisfies } \prod_{n=1}^{\infty} |\alpha_n| = 0, \text{ hence } \psi_M \text{ is injective.}

Now we look for } P. \text{ Since } g \text{ admits each } \alpha_j \text{ as a simple zero, it factorizes in } A(d(0,1^-)) \text{ in the form } \psi_j(x)(1 - x/\alpha_j) \text{ and we have } \psi_j(\alpha_j) \neq 0. \text{ We put } g_j(x) = \frac{\psi_j(x)}{\psi_j(\alpha_j)}. \text{ Then } g_j \text{ belongs to } A(d(0,1^-)) \text{ and may be written as } \sum_{n=0}^{\infty} \lambda_{n,j} x^n. \text{ We denote by } P \text{ the matrix}
and we will show this satisfies Properties (1) - (7).

For convenience, we put $D = \Omega(\alpha_n)$. Since $\lim_{|x| \to 1^} |g(x)| = +\infty$, by Lemma 6, we know that $1/g$ belongs to $H(D)$. For each $n \in \mathbb{N}$, we put $u_n = x^n/g$. Then in $H(D)$, $u_n$ has a Mittag-Leffler series ([3], [5]) of the form $\sum_{j=0}^{\infty} \beta_{j,n}$. Now we put $\theta_j = \psi_j(\alpha_j)$ and we have $g(x) = \theta_jg_j(x)(1-x/\alpha_j)$. We will compute the $\beta_{j,n}$. Let $v_{j,n} = (1-x/\alpha_j)u_n$. Then we have $v_{j,n}(\alpha_i) = \frac{\alpha^n_j}{g_j(\alpha_j)\theta_j}$. But since $g_j(\alpha_j) = 1$ whenever $j \in \mathbb{N}$, we see that $\beta_{j,n} = \alpha^n_j/\theta_j$, hence $x^n g(x) = \sum_{i=0}^{\infty} \alpha^n_j \theta_j(1-x/\alpha_j)$. We notice that $\|g_j/1-x/\alpha_j\|_D = \|\alpha_j\|^{n+1}_D$ and then we have $\lim_{j \to \infty} |\theta_j| = +\infty$, because the sequence of the terms $x^n/g(x)$ must tend to 0. Now we have $x^n = \sum_{j=0}^{\infty} \frac{\alpha^n_j g(x)}{\theta_j(1-x/\alpha_j)}$, while $g_j(x) = \frac{g(x)}{\theta_j(1-x/\alpha_j)}$. Since $g_j(x) = \sum_{n=0}^{\infty} \lambda_{n,j} x^n$, we obtain

\[(8) \quad x^n = \sum_{j=0}^{\infty} \alpha^n_j (\sum_{h=0}^{\infty} \lambda_{h,j} x^h).\]

In particular, (8) holds in every disk $d(0,r)$ with $r \in ]0,1[$. But then we know that $\|g_j\|_{d(0,r)} = \sup_{h \in \mathbb{N}} |\lambda_{j,h}| r^h \leq \frac{\|g\|_{d(0,r)}}{\|\theta_j\|_D}$. Now, we have $\|\phi_j\|_{d(0,r)} \leq \|g\|_{d(0,r)}$ as soon as $|\alpha_i| > r$ because then $1/(1-x/\alpha_j)\|_{d(0,r)} = 1$ and therefore the sequence $\|\phi_j\|_{d(0,r)}$ is bounded. Then the family $\{\lambda_{j,h} x^h\}_{j,h \in \mathbb{N}}$ tends to zero when $j$ tends to $+\infty$, uniformly with respect to $h$. In particular, $P$ is line-vanishing. For each $h \in \mathbb{N}$, we put $s_h = \sup_{j \in \mathbb{N}} |\lambda_{h,j}|$. We will show

\[(9) \quad \lim_{h \to +\infty} \sup_{h \in \mathbb{N}} s_h^{1/h} \leq 1.\]

Indeed this is equivalent to show that for every $r \in ]0,1[$, we have

\[(10) \quad \lim_{h \to +\infty} s_h r^h = 0.\]
Let $r \in ]0, 1[$ and let $\varepsilon > 0$. Since the family $\left( r^j \right)_{j,h \in \mathbb{N}}$ tends to zero uniformly with respect to $h$ when $j$ tends to $+\infty$, there clearly exists $N$ such that $|\lambda_{h,j}| r^h < \varepsilon$ whenever $j > N$, whenever $h \in \mathbb{N}$, hence for every $h \in \mathbb{N}$, we have $s_h r^h \leq \max_{1 \leq j \leq N} |\lambda_{h,j}| r^h$. But for each fixed $i \in \mathbb{N}$, we know that $\lim_{h \to \infty} |\lambda_{h,j}| r^h = 0$, hence $\lim_{h \to \infty} \left( \max_{1 \leq j \leq N} |\lambda_{h,j}| r^h \right) = 0$. This finishes showing (10). Therefore (9) is proven and so is (2).

Now, we can apply the limits inversion theorem and, then, by (8), we have

$$x^n = \sum_{h=0}^{\infty} \left( \sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j} \right) x^h,$$

whenever $x \in d(0, r)$. Actually this is true for all $r \in ]0, 1[$ and therefore (11) holds for all $x \in d(0, 1^-)$. Hence we have $\sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j} = 0$ whenever $n \neq h$ and $\sum_{j=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1$. So (3) is satisfied.

Thus we have proven that $PM = I_{\infty}$. Now we check that $MP = I_{\infty}$. For every $h \neq j$, we have $g_j(\alpha_h) = g(\alpha_h) = 0$, hence $\sum_{h=0}^{\infty} \alpha_h^n \lambda_{h,j} = 0$. Besides, it is seen that $g_j(\alpha_j) = 1$, hence $\sum_{n=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1$. So we conclude that $MP = I_{\infty}$ and this finishing proving (4).

Now, we will check that $P(b) \in \mathcal{E}_1$ for all $b \in \mathcal{E}_0$. Let $b := (b_n) \in \mathcal{E}_0$, let $a := (a_n) = P(b)$ and let $f(t) = \sum_{n=0}^{\infty} a_n t^n$. For each $j \in \mathbb{N}$ we put $f_j(t) = \sum_{m=0}^{j} b_m g_m(t)$. Then $f_j$ belongs to $A(d(0, 1^-))$ for all $j \in \mathbb{N}$. Let $r \in ]0, 1[$. Like the family $|\lambda_{n,j}| r^n$, the family $|\lambda_{n,j} b_j| r^n$ tends to zero uniformly with respect to $n$ when $j$ tends to $+\infty$. That way, in $H(d(0, r))$ we have $\lim_{j \to \infty} \|f - f_j\|_{d(0, r)} = 0$ and therefore $f$ belongs to $H(d(0, r))$. This is true for all $r \in ]0, 1[$ and therefore $f$ belongs to $A(d(0, 1^-))$. Hence $P(b) \in \mathcal{E}_1$. This shows (5).

Let us show (6). Let $b := (b_0, \ldots, b_n, \ldots)$ be a bounded sequence. Let $a = Pb$, and let $a = (a_0, \ldots, a_n, \ldots)$. We will show

$$\limsup_{n \to \infty} |a_n|^{1/n} \leq 1.$$

Without loss of generality, we may assume $|b_j| \leq 1$, whenever $j \in \mathbb{N}$. Then we have $|a_n| \leq \sup_{j \in \mathbb{N}} |\lambda_{n,j}| = s_n$, therefore $\limsup_{n \to \infty} |a_n|^{1/n} \leq \limsup_{n \to \infty} s_n^{1/n} \leq 1$. Now, by (12), it is seen that for all $j \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} a_n \alpha_j^n$ is convergent and therefore we may consider
Ma = M(Pb). By definition, for each \( i \in \mathbb{N} \), we have \( a_i = \sum_{j=0}^{\infty} \lambda_{i,j} b_j \). Let \( M(a) = (x_h)_{h \in \mathbb{N}} \).

For each \( h \in \mathbb{N} \) we have \( x_h = \sum_{m=0}^{\infty} \alpha_m^h a_m = \sum_{m=0}^{\infty} \alpha_m^h (\sum_{j=0}^{\infty} \lambda_{m,j} b_j) \). Let \( r = |\alpha_h| \). As we saw, the family \( |\lambda_{m,j} b_j| r^m \) tends to 0 when \( m \) tends to \( +\infty \), uniformly with respect to \( j \). Hence by the Limits Inversion Theorem, we have

\[
\sum_{m=0}^{\infty} \alpha_m^h (\sum_{j=0}^{\infty} \lambda_{m,j} b_j) = \sum_{j=0}^{\infty} b_j (\sum_{m=0}^{\infty} \lambda_{m,j} \alpha_m^h).
\]

Hence by (3), we see that \( x_j = b_j \) and this finishes proving (6). Then by (6) \( \psi_P \) is clearly injective.

Finally we will prove the last statement of the theorem. Let \( \phi(x) = \sum_{n=0}^{\infty} \nu_n x^n \). The function \( \phi \) belongs to \( A(d(0,1^-)) \) and is invertible in \( A(d(0,1^-)) \) thanks to the inequality \( |\nu_0| > |\nu_n| \) whenever \( n > 0 \). Hence the function \( G(x) = g(x) \phi(x) \) is easily seen to satisfy i), ii), iii), iv) like \( g \). Then \( G \) factorizes in \( A(d(0,1^-)) \) and can be written as \( \phi_j(x)(1 - x/\alpha_j) \) with \( \phi_j(x) = \psi_j(x) \phi(x) \). Hence we put \( G_j(x) = \frac{\phi_j(x)}{\phi_j(\alpha_j)} = \frac{g_j(x) \phi(x)}{\phi(\alpha_j)} \). Now it is clearly seen that the power series of \( G_j \) is \( \sum_{n=0}^{\infty} \mu_{n,j} x^n \). By definition, the matrix \( Q \) satisfies the same properties as \( P \). But when \( \phi \) is not a constant function, for each fixed \( j \in \mathbb{N} \), we do not have \( \mu_{n,j} = \lambda_{n,j} \) for all \( n \in \mathbb{N} \). Hence \( Q \) is different from \( P \). As a consequence we see that \( \psi_M \) is not surjective, it would be an automorphism of \( \mathcal{E}_0 \) and therefore \( \psi_P \) would also be an automorphism of \( \mathcal{E}_0 \) and it would be unique. This ends the proof of Theorem 3.

4. PROVING THEOREMS 4 AND 5

Notation. For each integer \( q \in \mathbb{N}^* \), we will denote by \( G(q) \) the group of the \( q \)-roots of 1.

Lemma 7: Let \( (a_n) \) be a sequence in \( d(0,1^-) \) such that \( \lim_{n \to \infty} |a_n| = 1 \). For each \( s \in \mathbb{N} \), there exists a prime integer \( q > p \) and \( \zeta \in G(q) \) such that \( |\zeta^h a_s - a_j| = \max(|a_s|, |a_j|) \) for every \( j \in \mathbb{N} \), for every \( h = 1, \ldots, q - 1 \).

Proof. Let \( r = |a_s| \). Since \( \lim_{n \to \infty} |a_n| = 1 \), the circle \( C(0,r) \) contains finitely many terms of the sequence \( (a_n) \). Without loss of generality we may assume \( |a_n| < r \) whenever \( n < s \), \( |a_n| > r \) whenever \( n > t \) and \( |a_n| = r \), whenever \( n = s, \ldots, t \) (with obviously \( l \leq s \leq t \)). Whatever \( q \in \mathbb{N} \), \( \zeta \in G(q) \) are, it is seen that we have \( |\zeta^h a_s - a_j| = |a_s| \) for all \( j < l \) and \( |\zeta^h a_s - a_j| = |a_j| \) for all \( j > t \). In the residue class field \( k \) of \( K \), for every \( j = l, \ldots, t \), let \( \gamma_j \) be the class of \( a_j/a_s \). There does exist a prime integer \( q > p \) such that the polynomial \( p(x) = x^q - 1 \) admits none of the \( \gamma_j \) \( (l \leq j \leq t) \) as a zero. Hence, for
every $q$-root $\nu$ of 1 in $k$, we have $\nu^h \neq \gamma_j$ whenever $j = l, \ldots, t$, whenever $h = 1, \ldots, q - 1$. Now let $\zeta$ be a $q$-th root of 1 in $K$. Then by classical properties of the polynomials, we have $|\frac{\zeta^h - a_j}{a_s} - \alpha_j| = 1$, hence $|\zeta^h a_s - a_j| = |a_s| = r$ whenever $h = 1, \ldots, q - 1$, whenever $j = l, \ldots, t$. This completes the proof of Lemma 7.

Lemma 8 : Let $(a_n)$ be a regular sequence and let $\rho = \inf_{n \neq m} |a_n - a_m|$. There exists a sequence $(b_n)$ in $d(0,1^-)$ satisfying:

1. $\lim_{n \to \infty} |b_n| = 1$.
2. $|b_n - b_m| \geq \rho$ whenever $n \neq m$.
3. $(a_n)$ is a subsequence of $(b_n)$,
4. There exists a sequence $(q_n)$ of prime integers different from $p$ satisfying $\lim_{n \to \infty} q_n = +\infty$, such that for every $m \in \mathbb{N}$, $\zeta \in G(q_n)$, $\zeta b_n$ is another term of the sequence $(b_n)$,
5. There exists $f \in A(d(0,1^-))$ admitting each $b_n$ as a simple zero and having no other zero in $d(0,1^-)$, satisfying

$$\lim_{|x| \to 1^-} |f(x)| = +\infty.$$ 

Proof. First we will construct a sequence $(b'_n)$ satisfying (1), (2), (3), (4). Let $(q_j)$ be a strictly increasing sequence of prime integers strictly bigger than $p$ and, for each $j \in \mathbb{N}$, let $S_j = \{=o \ q_t, \}$ let $(j \in \mathbb{N}$ and let $= \zeta^h a_j (o  h  qJ -1)$. We will show that a good choice of the sequence $(q_j)$ enables us to obtain

$$|b'_n - b'_m| = \max(|b'_n|, |b'_m|)$$

for every couple $(n, m)$ satisfying $n \neq m$ and $(n, m) \neq (s_t, s_j)$ whenever $(i, j) \in \mathbb{N} \times \mathbb{N}$. In other words $|b'_n - b'_m| = \max(|b'_n|, |b'_m|)$ must be true all time except when $n = m$ and when $(b'_n, b'_m)$ is equal to some couple $(a_{s_t}, a_{s_j})$. For each $t \in \mathbb{N}$, let $F_t = \{s_0, s_1, \ldots, s_t\}$ and let $E_t$ be $\{0,1, \ldots, s_t-1\} \setminus F_t$. Assume that $q_0, q_1, \ldots, q_{r-1}$ have been chosen to satisfy the following properties $(\alpha_r)$ and $(\beta_r)$

$$(\alpha_r) \quad |b'_n - a_{s_j}| = \max(|b'_n|, |a_{s_j}|) \text{ for all } j \in \mathbb{N}, \text{ for all } n \in E_t.$$ 

$$(\beta_r) \quad |b'_n - b'_m| = \max(|b'_n|, |b'_m|) \text{ for all } (n, m) \in E_t \times E_t \text{ such that } n \neq m.$$ 

We will choose $q_t$ such that both $(\alpha_{r+1})$, $(\beta_{r+1})$ are satisfied. Indeed, by Lemma 7 we can take a prime integer $u$ such that, given $\zeta \in G(u)$, we have $|\zeta^h a_t - a_j| = \max(|a_t|, |a_j|)$ for all $j \in \mathbb{N}$, for all $h = 1, \ldots, u - 1$, $|\zeta^h a_t - b'_n| = \max(|a_t|, |b'_n|)$ for all $n < s_t$, for all $h = 1, \ldots, u - 1$. Thus we can take $q_t = u$ and we see that both $(\alpha_{r+1})$, $(\beta_{r+1})$ are satisfied. Hence we can construct the sequence $(q_t)$ by induction and, therefore, the sequence $(b'_n)$ satisfying (6) is now constructed. Then it is easily checked that the sequence $(b'_n)$ so obtained satisfies (1), (2), (3), (4).
Now let \( \{r_0, \ldots, r_n, \ldots\} = \{a_j : j \in \mathbb{N}\} \) and let \( D = \Omega(b_n) \). The infinite product \( g(x) = \prod_{j=0}^{\infty} \left( 1 - \frac{x}{a_j} \right)^{q_j} \) converges in \( A(d(0,1^-)) \) and has no zero in \( d(0,r) \cap D \) because, by construction of the sequence \( (b_n') \), each zero of \( g \) is one of the points \( b_m' \) for some \( m \in \mathbb{N} \).

Hence it is seen that we have \( |g(x)| \geq 1 \) for every \( x \in d(0,1^-) \setminus \left( \bigcup_{n=0}^{\infty} C(0,r_n) \right) \). For each \( n \in \mathbb{N} \), let \( \Sigma_n = D \cap C(0,r_n) \), let \( \tau_n = \inf_{x \in \Sigma_n} |g(x)| \), let \( \sigma_n \in (r_n, r_{n+1}) \cap |K| \), let \( c_n \in C(0, \sigma_n) \), and let \( u_n > \min(p, \sigma_n) \) be a prime integer such that \( \tau_n(c_n+1)^{u_n} > n + 1 \). Since \( \lim_{n \to \infty} u_n = +\infty \), it is seen that the infinite product \( h(x) = \prod_{n=0}^{\infty} \left( 1 - \frac{x}{c_n} \right)^{u_n} \) converges in \( A(d(0,1^-)) \). Let \( D' = \Omega((c_n), \rho) \) and let \( D'' = D' \cap D \). Let \( h(x) = \sum_{n=0}^{\infty} \lambda_n x^n \) and, for each \( r \in (0,1) \), let \( M(r) = \sup_{n \in \mathbb{N}} |\lambda_n| r^n \). Each pole of \( h \) is simple and is of the form \( \zeta c_n \) with \( \zeta \in G(u_n) \). Hence it is seen that \( h \) satisfies \( |h(x)| \geq M |x| / \rho \) for all \( x \in D' \). Hence if \( x \in D'' \setminus \left( \bigcup_{n=0}^{\infty} \Sigma_n \right) \), then we have

\[
|g(x)h(x)| = M(r_n)\tau_n \geq \frac{r_n}{\tau_{n-1}} \tau_n > n \quad \text{and finally we have}
\]

\[
(7) \quad \lim_{|x| \to 1, x \in D''} |g(x)h(x)| = +\infty.
\]

Now let \( (b''_n) \) be the sequence of the zeros of \( g \). Clearly \( (b''_n) \) satisfies (1) and (4) and also satisfies \( |b''_n - b'_m| = \max(|b''_n|, |b'_m|) \) whenever \( n, m \in \mathbb{N} \) and \( |b''_n - b''_m| = \max(|b''_n|, |b'_m|) \) whenever \( n \neq m \). Now we put \( b_{2n} = b'_n \) and \( b_{2n+1} = b''_n \). The sequence \( (b_n) \) clearly satisfies (1), (2), (3), (4) and also satisfies (5) because the zeros of \( h \) are the \( b''_n \) while those of \( g \) are the \( b'_n \). Thus the zeros of \( f \) are just the \( b_n \), and then, by (7), we have \( \lim_{|x| \to 1, x \in \Omega(b_n)} |f(x)| = +\infty. \)

This ends the proof of Lemma 8.

**Proof of Theorem 4.** Without loss of generality we may obviously assume \( |\alpha_n| \leq |\alpha_{n+1}| \) whenever \( n \in \mathbb{N} \). Let \( \rho = |\alpha_0| \). Hence by hypothesis each disk \( d(\alpha_q, \rho^-) \) contains no point \( \alpha_n \) for each \( n \neq q \). Let \( D = \Omega((\alpha_n), \rho^-) \).

For each \( n \in \mathbb{N} \), let \( T_n \) be the hole \( d(\alpha_n, \rho^-) \) of \( D \). Since \( |\alpha_n| = 0 \), it is shortly checked that the sequence \( (T_n, 1) \) is a \( T \)-sequence of \( D \) ([8]). Then, since \( K \) is spherically complete, by [4], Theorem 4, there exists \( g \in A(d(0,1^-)) \) admitting each \( \alpha_n \) as a simple zero and having no zero else in \( d(0,1^-) \). Therefore, as \( \prod_{n=0}^{\infty} |\alpha_n| = 0 \), is is seen that \( g \)
satisfies \( \lim_{|x| \to 1^-} |g(x)| = +\infty \). Now we can apply Theorem 3, which shows that the matrix \( M = M(a_n) \) admits inverses \( P \). Then the sequence \( (a_n) \) satisfies \( \sum_{n=0}^{\infty} a_n \alpha_j^n = b_j \) for every \( j \in \mathbb{N} \) and this clearly ends the proof of Theorem 4.

**Proof of Theorem 5.** By Lemma 8, there exists a regular sequence \( (\gamma_n) \) of \( d(0,1^-) \) such that \( (a_n) \) is a subsequence of \( (\gamma_n) \) together with an analytic function \( g \in A(d(0,1^-)) \) admitting each \( \gamma_m \) as a simple zero and having no other zero in \( d(0,1^-) \), satisfying \( \lim_{|x| \to 1^-} |g(x)| = +\infty \) with \( \rho = \inf_{n \neq m} |\gamma_n - \gamma_m| \). Then, by Theorem 3, the matrix \( M = M(\gamma_n) \) admits line-vanishing inverses \( M' \) satisfying \( M(M'(b)) = b \) for all bounded sequence \( b = (b_n) \). Let \( a := (a_n) = M'(b) \). Thus we have \( M(a) = b \) and therefore \( \sum_{n=0}^{\infty} a_n \gamma_j^n = b_j \) whenever \( j \in \mathbb{N} \). This ends the proof of Theorem 5.

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