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ORTHONORMAL BASES FOR P-ADIC CONTINUOUS AND
CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper we adapt the well-known Mahler and van der Put base of the Banach space of continuous functions to the case of the n-times continuously differentiable functions in one and several variables.

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1. Introduction

Let $K$ be an algebraic extension of $\mathbb{Q}_p$, the field of p-adic numbers. As usual, we write $\mathbb{Z}_p$ for the ring of p-adic integers and $C(\mathbb{Z}_p \rightarrow K)$ for the Banach space of continuous functions from $\mathbb{Z}_p$ to $K$. We have the following well-known bases for $C(\mathbb{Z}_p \rightarrow K)$: on one hand, we have the Mahler base $\left(\frac{x}{n}\right)$ ($n \in \mathbb{N}$), consisting of polynomials of degree $n$ and on the other hand we have the van der Put base $\{e_n \mid n \in \mathbb{N}\}$ consisting of locally constant functions $e_n$ defined as follows: $e_0(x) = 1$ and for $n > 0$, $e_n$ is the characteristic function of the ball $\{\alpha \in \mathbb{Z}_p \mid |\alpha - n| < 1/n\}$. For every $f \in C(\mathbb{Z}_p \rightarrow K)$ we have the following uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{n}\right)$$

where $a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j)$

$$f(x) = \sum_{n=0}^{\infty} b_n e_n(x)$$

where $b_0 = f(0)$ and $b_n = f(n) - f(n-)$. Here $n_-$ is defined as follows. For every $n \in \mathbb{N}_0$, we have a Hensel expansion $n = n_0 + n_1p + \ldots + n_sp^s$ with $n_s \neq 0$. Then $n_- = n_0 + n_1p + \ldots + n_{s-1}p^{s-1}$. We further put $\gamma_0 = 1$, $\gamma_n = n - n_- = n_sp^s$, $\delta_0 = 1$, $\delta_n = p^s$ and $n_- = n - \delta_n$. Remark that $|\delta_n| = |\gamma_n|$. 


In the sequel, we will also use the following notation, for \( m, x \in \mathbb{Q}_p \), \( x = \sum_{j=-\infty}^{\infty} a_j p^j : m \triangleleft x \)
if \( m = \sum_{j=-\infty}^{i} a_j p^j \) for some \( i \in \mathbb{Z} \). We sometimes refer to the relation \( \triangleleft \) between \( m \) and \( x \) as "\( m \) is an initial part of \( x \)" or "\( x \) starts with \( m \)".

Let \( f : \mathbb{Z}_p \to K \). The (first) difference quotient \( \phi_1 f : \nabla^2\mathbb{Z}_p \to K \) is defined by \( \phi_1 f(x,y) = \frac{f(y) - f(x)}{y - x} \), where \( \nabla^2\mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(x,x) \mid x \in \mathbb{Z}_p \} \). \( f \) is called continuously differentiable (or strictly differentiable, or uniformly differentiable) at \( a \in \mathbb{Z}_p \) if \( \lim_{(x,y) \to (a,a)} \phi_1 f(x,y) \) exists. We will also say that \( f \) is \( C^1 \) at \( a \). In a similar way, we may define \( C^n \)-functions as follows: for \( n \in \mathbb{N} \), we define \( \nabla^{n+1}\mathbb{Z}_p = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}_p^{n+1} \mid x_i \neq x_j \text{ if } i \neq j \} \) and the \( n \)-th difference quotient \( \phi_n f : \nabla^{n+1}\mathbb{Z}_p \to K \) by \( \phi_0 f = f \) and
\[
\phi_n f(x_1, x_2, \ldots, x_{n+1}) = \frac{\phi_{n-1} f(x_2, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_1, x_3, \ldots, x_{n+1})}{x_2 - x_1}.
\]

A function \( f \) is called a \( C^n \)-function if \( \phi_n f \) can be extended to a continuous function \( \phi_n f \) on \( \mathbb{Z}_p^{n+1} \). Recall from [4],[5] that \( \phi_n f(x, x, \ldots, x) = \frac{f^{(n)}}{n!} \), for all \( x \in \mathbb{Z}_p \). The set of all \( C^n \)-functions from \( \mathbb{Z}_p \) to \( K \) will be denoted by \( C^n(\mathbb{Z}_p \to K) \). For any \( C^n \)-function \( f \), we define \( \|f\|_n = \max \{ \|\phi_j f\|_s \mid 0 \leq j \leq n \} \) where \( \| \cdot \|_s \) is the sup norm. (For \( f : X \to K, \|f\|_s = \max_{x \in X} |f(x)| \) \( \| \cdot \|_n \) is a norm on \( C^n \), making \( C^n \) into a Banach space.

2. Generalization of the Mahler base for \( C(\mathbb{Z}_p \to \mathbb{Q}_p) \)

One can construct other orthonormal bases of \( C(\mathbb{Z}_p \to K) \) by generalizing the procedure used to define the Mahler base as did Y. Amice. In general, we have the following characterization of the polynomial sequences \( e_n \in K[x], n \geq 0 \) such that \( \deg(e_n) = n \) and which are orthonormal bases of the space \( C(B \to K) \), where \( B = \{ x \in K \mid |x| \leq 1 \} \).

**Theorem:** Let \( (e_n)_{n \geq 0} \) be a sequence of polynomials in \( K[x] \) of degree \( n \). They form an orthonormal base of \( C(B \to K) \) if and only if \( \|e_n\|_s = 1 \) and \( ||e_n||_G = |\text{coeff } x^n| = |\pi^{-(n-s(n))}/(q-1)| \) where \( \pi \) is a uniformizing parameter of \( K \), \( q \) the cardinality of the residue class field of \( K \) and \( s(n) \) the sum of the digits of \( n \) in base \( q \). By the way, for a polynomial
\[
f(x) = \sum_{i=0}^{n} a_i x^i, \|f\|_G = \max_{i \leq n} |a_i|.
\]

Given an orthonormal base, we can construct other orthonormal bases by taking a certain linear combination of the given base as will be stated in the following theorem.

**Theorem:** Let \( e_n(n \in \mathbb{N}) \) be an orthonormal base of \( C(\mathbb{Z}_p \to K) \) and put \( p_n = \sum_{j=0}^{n} a_{n,j} e_j \) where \( a_{n,j} \in K \) and \( a_{n,n} \neq 0 \). The \( p_n(n \in \mathbb{N}) \) form an orthonormal base for
$C(\mathbb{Z}_p \to K)$ if and only if $|a_{n,j}| \leq 1$ for all $j \leq n$ and $|a_{n,n}| = 1$.

We can generalize the Mahler base also by changing the degree of the polynomials as follows.

**Theorem:** The polynomials $q_n(x) = \left(\frac{px}{pn}\right) (n \in \mathbb{N})$ form an orthonormal base for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ and every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} a_{pn}\left(\frac{px}{pn}\right)$

with $a_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \left(\frac{pn}{pk}\right) \alpha_{n-k}(p)f(k)$

and $\alpha_0^{(p)} = 1, \alpha_m^{(p)} = \sum_{1 \leq i_1 \leq \ldots \leq i_r \leq m, 0 \leq i_1 \leq \ldots \leq i_r \leq m} (-1)^{r+m} \left(\frac{pm}{p_1 \ldots p_r}\right)$

If we mix the Mahler and van der Put base together, we obtain a new orthonormal base.

**Theorem:** The sequence $g_n(x) = \left(\frac{x}{n}\right) \varepsilon_n(x) (n \in \mathbb{N})$ forms an orthonormal base for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Moreover, every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum_{i=0}^{\infty} a_i \left(\frac{x}{i}\right) \varepsilon_i(x)$

with $a_i = \sum_{j \varepsilon_i} \alpha_i f(j)$

and $\alpha_i, i = 1, \alpha_i,j = \sum_{j=k_0 \leq k_1 \leq \ldots \leq k_n=i} (-1)^n \left(\begin{array}{c} i \\ k_{n-1} \end{array}\right) \left(\begin{array}{c} k_{n-1} \\ k_{n-2} \end{array}\right) \ldots \left(\begin{array}{c} k_1 \end{array}\right)$

3. Differentiable functions

For $C^n$-functions the polynomials $\left(\frac{x}{i}\right) (i \in \mathbb{N})$ still remain a base, we only have to add the factor $\gamma_i^{-1} \gamma_{[i/2]} \ldots \gamma_{[i/n]}$ where $\gamma_i = i - i_-$ and $[\alpha]$ denotes the integer part of $\alpha$, to obtain the orthonormal base $\gamma_i^{-1} \gamma_{[i/2]} \ldots \gamma_{[i/n]} \left(\frac{x}{i}\right)$. The proof is based on the following lemma in case $n=2$.

**Lemma** Let $f$ be a continuous function with interpolation coefficients $a_n$. Then $f$ is a $C^2$-function if and only if $\left|\frac{a_{i+j+k+2}}{(i+k+2)}\right| \to 0$ as $i + j + k$ approach infinity.

**Corollary** If $f$ is a $C^2$-function, then $||\phi_2 f||_2 = \sup_n \left|\frac{a_n}{\gamma_{n+2}}\right|$.

A similar property does not hold for the van der Put base.

In case $n=1$, we know that $\{ \gamma_i \varepsilon_i(x) | i \in \mathbb{N} \} \cup \{(x-i) \varepsilon_i(x) | i \in \mathbb{N} \}$ is an orthonormal base for $C^1(\mathbb{Z}_p \to K)$. Therefore every continuously differentiable function $f$ can be written
under the form $f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x)$ where $a_0 = f(0)$, $a_n = f(n) - f(n_0) - (n - n_0) f'(n_0)$, $b_0 = f'(0)$ and $b_n = f'(n) - f'(n_0)$. For details we refer to [6]. The case $n = 2$, can be treated as follows.

**Theorem:** Let $f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) \in C^1(Z_p \to K)$.

$f \in C^2(Z_p \to K)$ if and only if $\lim \frac{a_n}{\gamma_n^2}$ and $\lim \frac{b_n}{\gamma_n}$ exist for all $a \in Z_p$, and $\lim \frac{b_n}{\gamma_n} = 2 \lim \frac{a_n}{\gamma_n^2}$.

**Theorem:** $\{ \gamma_n^2 e_n(x), \gamma_n(x-n)e_n(x), (x-n)^2 e_n(x) \mid n \in N \}$ is an orthonormal base for $C^2(Z_p \to K)$ and for every $f \in C^2(Z_p \to K)$ we have

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) + \sum_{n=0}^{\infty} c_n \frac{(x-n)^2}{2} e_n(x)$$

with

- $a_0 = f(0)$
- $a_n = f(n) - f(n_0) - (n - n_0) f'(n_0)$ for $n \neq 0$
- $b_0 = f'(0)$
- $b_n = f'(n) - f'(n_0) - (n - n_0) f''(n_0)$ for $n \neq 0$
- $c_0 = f''(0)$
- $c_n = f''(n) - f''(n_0)$ for $n \neq 0$

The construction of this orthonormal base, which is very technical, is based on the use of an antiderivation map $P_n : C^{n-1}(Z_p \to K) \to C^n(Z_p \to K)$ defined by $P_n f(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f(j)(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}$ with $x_m = \sum_{j=-\infty}^{+\infty} a_j p_j$ and on the two following lemmas.

**Lemma:** For $(t_1, \ldots, t_k) \in \nabla^k X = \{(x_1, x_2, \ldots, x_k) \mid x_i \neq x_j \text{ if } i \neq j \}$ with $t_1 = x, t_i = y$ and $t_k = z$, we have

$$\phi_2 f(x, y, z) = \sum_{j=2}^{k-1} \mu_j \phi_2 f(t_{j-1}t_j, t_{j+1})$$

with $\mu_j = \begin{cases} \frac{(t_{j+1} - t_{j-1})(t_1 - t_k)}{(z-x)(y-z)} & \text{for } j \geq i \\
\frac{(t_i - t_{j-1})(t_{j+1} - t_k)}{(z-x)(y-z)} & \text{for } j \leq i \end{cases}$

Moreover, $\sum_{j=2}^{k-1} \mu_j = 1$

**Lemma:** Let $S$ be a ball in $K$ and $f \in C(Z_p \to K)$.

Suppose that $\phi_2 f(n, n - \delta_n, n + p^k \delta_n) \in S$ for all $n \in N_0, k \in N$, then $\phi_2 f(x, y, z) \in S$ for all $x, y, z \in Z_p, x \neq y, x \neq z, y \neq z$.

4. **Several variables**

We can also construct the Mahler and van der Put base for functions of several variables. This brings us to the following results.
Theorem: The family \( \max\{\gamma_n, \gamma_m\}. (x_n, y_m) (n, m \in \mathbb{N}) \) forms an orthonormal base for \( C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \). The proof is based on

Theorem: \( f(x, y) = \sum_{n,m} a_{n,m} \binom{x}{n} \binom{y}{m} \) is a \( C^1 \)-function if and only if \( \left| \frac{a_{i+j+1,k}}{j+1} \right| \to 0 \)

and \( \left| \frac{a_{i,j+k+1}}{k+1} \right| \to 0 \) as \( i+j+k \) approach infinity or equivalently \( \left| \frac{a_{n,m}}{\gamma_n} \right| \to 0 \) and \( \left| \frac{a_{n,m}}{\gamma_m} \right| \to 0 \) as \( n+m \) approach infinity.

Starting with the van der Put base \( e_n(n \in \mathbb{N}) \) of \( C(\mathbb{Z}_p \to K) \), we get

Theorem: The family \( e_n(x)e_m(y), (x-n)e_n(x)e_m(y), (y-m)e_n(x)e_m(y) \)

\( (n, m \in \mathbb{N}) \) forms an orthogonal base for \( C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \) and every \( C^1 \)-function \( f \) can be written as

\[
f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}e_i(x)e_j(y) + b_{i,j}(x-i)e_i(x)e_j(y) + c_{i,j}(y-j)e_i(x)e_j(y)
\]

with

\[
a_{0,0} = f(0,0)
\]

\[
a_{n,0} = f(n,0) - f(n_-,0) - \gamma_n \frac{\partial f}{\partial x}(n_-,0) \quad \text{for} \ n \neq 0
\]

\[
a_{0,m} = f(0,m) - f(0,m_-) - \gamma_m \frac{\partial f}{\partial y}(0,m_-) \quad \text{for} \ m \neq 0
\]

\[
a_{n,m} = f(n,m) - f(n_-,m) - f(n,m_-) + f(n_-,m_-) - \gamma_n \left( \frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n_-,m_-) \right) - \gamma_m \left( \frac{\partial f}{\partial y}(n,m_-) - \frac{\partial f}{\partial y}(n_-,m_-) \right) \quad \text{for} \ n \neq 0 \ \text{and} \ m \neq 0
\]

\[
b_{0,0} = \frac{\partial f}{\partial x}(0,0)
\]

\[
b_{n,0} = \frac{\partial f}{\partial x}(n,0) - \frac{\partial f}{\partial x}(n_-,0) \quad \text{for} \ n \neq 0
\]

\[
b_{0,m} = \frac{\partial f}{\partial x}(0,m) - \frac{\partial f}{\partial x}(0,m_-) \quad \text{for} \ m \neq 0
\]

\[
b_{n,m} = \frac{\partial f}{\partial x}(n,m) - \frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n,m_-) + \frac{\partial f}{\partial x}(n_-,m_-) \quad \text{for} \ n \neq 0 \ \text{and} \ m \neq 0
\]

\[
c_{0,0} = \frac{\partial f}{\partial y}(0,0)
\]

\[
c_{n,0} = \frac{\partial f}{\partial y}(n,0) - \frac{\partial f}{\partial y}(n_-,0) \quad \text{for} \ n \neq 0
\]

\[
c_{0,m} = \frac{\partial f}{\partial y}(0,m) - \frac{\partial f}{\partial y}(0,m_-) \quad \text{for} \ m \neq 0
\]

\[
c_{n,m} = \frac{\partial f}{\partial y}(n,m) - \frac{\partial f}{\partial y}(n_-,m) - \frac{\partial f}{\partial y}(n,m_-) + \frac{\partial f}{\partial y}(n_-,m_-) \quad \text{for} \ n \neq 0 \ \text{and} \ m \neq 0
\]

Remark: To obtain an orthonormal base, the \( e_i(x)e_j(y) \) should be multiplied by
max\{\gamma_i, \gamma_j\}; the \((x - i)e_i(x)e_j(y)\) by \(\text{max}\left\{\frac{1}{p^\gamma_i}, 1, \frac{\gamma_j}{p^\gamma_i}\right\}\) in case \(i \neq 0\) and by \(\gamma_j\) in case \(i = 0\) and analogous for \((y - j)e_i(x)e_j(y)\).

**Generalization:** The sequence \((x - i)^{k}(y - j)^{l}e_i(x)e_j(y)\) with \(0 \leq k + l \leq n, i \in \mathbb{N}\) and \(j \in \mathbb{N}\) forms an orthogonal base for \(C^n(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow K)\) whereby every \(C^n\)-function \(f\) can be written as \(f(x, y) = \sum_{i,j=0}^{\infty} \sum_{k+l=0}^{n} \frac{a_{i,j}^{k,l}(x-i)^{k}y^{-l}}{k!}e_i(x)e_j(y)\) with

\[
a_{i,j}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k \partial y^l}(i,j) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha}f}{\partial x^k \partial y^l+\alpha}(i-,j)\frac{\gamma_i^\alpha}{\alpha!} - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta}f}{\partial x^k \partial y^l+\beta}(i,j-,\beta)\frac{\gamma_j^\beta}{\beta!} +
\]

\[
\sum_{\alpha+\beta=0}^{n-k-l} \frac{\partial^{k+l+\alpha+\beta}f}{\partial x^k \partial y^l+\alpha+\beta}(i-,j-,\alpha+\beta)\frac{\gamma_i^\alpha \gamma_j^\beta}{\alpha!\beta!}
\]

for \(i \neq 0\) and \(j \neq 0\)

\[
a_{i,0}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k \partial y^l}(i,0) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha}f}{\partial x^k \partial y^l+\alpha}(i-,0)\frac{\gamma_i^\alpha}{\alpha!}
\]

for \(i \neq 0\)

\[
a_{0,j}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k \partial y^l}(0,j) - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta}f}{\partial x^k \partial y^l+\beta}(0,j-,\beta)\frac{\gamma_j^\beta}{\beta!}
\]

for \(j \neq 0\)

and \(a_{0,0}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k \partial y^l}(0,0)\)

The previous theorems show that \(C^n(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow K)\) is not the complete tensor product of \(C^n(\mathbb{Z}_p \rightarrow K)\) with \(C^n(\mathbb{Z}_p \rightarrow K)\) as one may expect, considering the case \(C(\mathbb{Z}_p \rightarrow \mathbb{K})\).

Therefore we define a finer structure for functions of two variables.

**Definition:**

\(\phi_{0,0}f(x_0, y_0) = f(x_0, y_0)\)

\(\phi_{1,0}f(x_0, x_1, y_0) = \frac{f(x_0, y_0) - f(x_1, y_0)}{x_0 - x_1}\) for \(x_0 \neq x_1\)

\(\phi_{0,1}f(x_0, y_0, y_1) = \frac{f(x_0, y_0) - f(x_0, y_1)}{y_0 - y_1}\) for \(y_0 \neq y_1\)

\(\vdots\)

\(\phi_{i,j}f(x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_j)\)

\(= \phi_{i-1,j}f(x_0, \ldots, x_{i-2}, x_{i-1}, y_0, \ldots, y_j) - \phi_{i-1,j}f(x_0, \ldots, x_{i-2}, x_{i-1}, y_0, \ldots, y_j)\)

\(= \phi_{i,j-1}f(x_0, \ldots, x_i, y_0, \ldots, y_{j-2}, y_{j-1}) - \phi_{i,j-1}f(x_0, \ldots, x_i, y_0, \ldots, y_{j-2}, y_{j-1})\)

for \((x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_j) \in \mathbb{V}^{i+1}\mathbb{Z}_p \times \mathbb{V}^{j+1}\mathbb{Z}_p\) is the difference quotient of order \(i\) in the first variable and order \(j\) in the second variable of the function \(f\) from \(\mathbb{Z}_p \times \mathbb{Z}_p\) to \(K\).

**Definition:** \(f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow K\) is \(m\) times strictly differentiable in his first variable and \(n\) times strictly differentiable in his second variable (for short: a \(C^{m,n}\)-function) if and
only if \( \phi_{m,n} \) can be extended to a continuous function \( \overline{\phi_{m,n} f} \) on \( \mathbb{Z}_p^{m+n+2} \). The set of all \( C^{m,n} \)-functions \( f : \mathbb{Z}_p \times \mathbb{Z}_p \to K \) is denoted \( C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \). For \( f : \mathbb{Z}_p \times \mathbb{Z}_p \to K \), set \( \|f\|_{m,n} = \max_{0 \leq i,j \leq m} \|\phi_{i,j} f\|_s \).

For these functions, we get the following equivalent of the Mahler base.

**Theorem:** The family \( \gamma_i \gamma_{[i/2]} \cdots \gamma_{[i/m]} \gamma_j \gamma_{[j/2]} \cdots \gamma_{[j/n]} \left( \begin{array}{c} x \\ i \\ \end{array} \right) \left( \begin{array}{c} y \\ j \\ \end{array} \right) \) \((i,j \in \mathbb{N})\) forms an orthonormal base for \( C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \).

Since it can be easily seen that there is an isometry between the complete tensor product \( C^m(\mathbb{Z}_p \to K) \otimes C^n(\mathbb{Z}_p \to K) \) and \( C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \), the van der Put base for \( C^{m,n} \)-functions is given as follows.

**Theorem:** The family \( \gamma_i^{m-k} (x-i)^k \gamma_j^{n-l} (y-j)^l e_i(x)e_j(y) \) with \( 0 \leq k \leq m, 0 \leq l \leq n, \) \( i \in \mathbb{N} \) and \( j \in \mathbb{N} \) forms an orthonormal base for \( C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \) whereby every \( C^{m,n} \)-function \( f \) can be written as \( f(x,y) = \sum_{i,j=0}^m \sum_{k=0}^n \sum_{l=0}^n a_{i,j}^{k,l} (x-i)^k (y-j)^l e_i(x)e_j(y) \) with

\[
\begin{align*}
a_{i,j}^{k,l} &= \frac{\partial^{k+l} f}{\partial x^k \partial y^l} (i,j) - \sum_{\alpha=0}^{m-k} \frac{\partial^{k+l+\alpha} f}{\partial x^k+\alpha \partial y^l} (i-\alpha,j) \frac{\gamma_{i-\alpha}^\alpha}{\alpha!} - \sum_{\beta=0}^{n-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^l+\beta} (i,j-\beta) \frac{\gamma_j^\beta}{\beta!} \\
&\quad + \sum_{\alpha=0}^{m-k} \sum_{\beta=0}^{n-l} \frac{\partial^{k+l+\alpha+\beta} f}{\partial x^k+\alpha \partial y^l+\beta} (i-\alpha,j-\beta) \frac{\gamma_i^\alpha \gamma_j^\beta}{\alpha! \beta!} \\
&\quad \text{for } i \neq 0 \text{ and } j \neq 0.
\end{align*}
\]

\[
\begin{align*}
a_{i,0}^{k,l} &= \frac{\partial^{k+l} f}{\partial x^k \partial y^l} (i,0) - \sum_{\alpha=0}^{m-k} \frac{\partial^{k+l+\alpha} f}{\partial x^k+\alpha \partial y^l} (i-\alpha,0) \frac{\gamma_{i-\alpha}^\alpha}{\alpha!} \quad \text{for } i \neq 0 \\
\end{align*}
\]

\[
\begin{align*}
a_{0,j}^{k,l} &= \frac{\partial^{k+l} f}{\partial x^k \partial y^l} (0,j) - \sum_{\beta=0}^{n-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^l+\beta} (0,j-\beta) \frac{\gamma_j^\beta}{\beta!} \quad \text{for } j \neq 0 \\
\end{align*}
\]

\[
\begin{align*}
a_{0,0}^{k,l} &= \frac{\partial^{k+l} f}{\partial x^k \partial y^l} (0,0)
\end{align*}
\]

**REFERENCES**


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