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## *p*-adic almost periodicity and representations

# P-ADIC ALMOST PERIODICITY

# AND REPRESENTATIONS

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**Abstract**– In the first international conference on $p$-adic functional analysis, the question whether it is possible to get the structure of the Banach Algebra $A_c(G)$ of p-adic valued continuous almost periodic functions on a totally disconnected topological IB-group $G$ through the structure of its non-archimedean Bohr compactification $\hat{G}$ was raised. We affirmatively answer this question here. This structure of $A_c(G)$ helps one to study the p-adic regular representation of $G$ using the known theory of representations for compact groups.

# 1 Introduction

Let $G$ be a group and $K$ a complete ultra-metric valued field. When $G$ carries a topology under which $G$ is a topological group, we have studied in earlier papers Rangan [5], [6], [7] and [8] continuous almost periodic functions on $G$ with values in $K$. In Rangan [8] we conjectured that a structure theory for the Banach algebra $A = A_c(G)$ of continuous almost periodic functions on $G$ can be obtained using the known structure theory of the group algebra of a compact group by going to the Bohr compactification $\hat{G}$ of $G$. In this paper we give an affirmative answer to the conjecture. The observation that $G$ is an IB-group if and only if the Bohr compactification $\hat{G}$ is an IB-group or equivalently a $p$-free group, where $p$ is the characteristic of the residue class field of $K$, which is implicitly contained in the results proved in Rangan [7], helps us to establish the conjecture.

When $G$ is an arbitrary group and $K$ is a locally compact field we consider the subgroup topology on $G$ defined by the normal subgroups of finite index in $G$ under which

$G$ becomes a $O$-dimensional group. The space of continuous almost periodic functions on $G$ described above coincides with the space of almost periodic functions $AP(G \to K)$ defined by Schikof [10] using compactoid. This enables us to prove that there exists an invariant mean on $AP(G \to K)$ or equivalently the pair $(G, K)$ is a.p.i.m. in the sense of Diarra [2](p.23, N.B.(i)) if and only if $G$ is a IB-group or equivalently a $p$-free group (see Rangan [7]). Thus in the case when the base field is locally-compact, the problem of characterising $(G, K)$ pairs which are a.p.i.m posed by Diarra is solved. The problem still remains open for non-locally compact fields. This also gives rise to the structure theory for $AP(G \to K)$ which is got by going to its Bohr compactification.

The structure theory so arrived at for the algebra of almost periodic functions gives rise to a study of representations of $G$ taking the base space for representation to be the space of almost periodic functions on $G$. This may give rise to an alternative approach to representation theory developed by Diarra [1] using Hopf algebras. We intend discussing the details in another paper. Using the structure theory of $AP(G \to K)$, we prove that the regular representation decomposes as a direct sum of finite-dimensional representations.

# 2   Notations and Definitions

$G$ is a group and $K$ is a complete ultra metric rank one valued field, $p$ denotes the characteristic of the residue class field. For $f : G \to K$, $x, s \in G$ we put $f_s(x) := f(s^{-1}x)$, $f^s(x) := f(xs), f^\vee(x) := f(x^{-1}), f_G = \{f_s : s \in G\}$ and $f^G := \{f^s : s \in G\}$. A function $f$ defined on $G$ is called almost periodic if $f_G$ is precompact or equivalently if for every $\epsilon > 0$ there exists a covering of $G$ by a finite collection of subsets $A_1, A_2, \cdots, A_n$ such that for $x, y \in A_i$ for $i = 1, 2, \cdots, n$ $|f(cxd) - f(cyd)| < \epsilon$ for all $c, d \in G$ (See Maak [4]). Interestingly it turns out that for a given $\epsilon > 0$ and an almost periodic function $f$ on $G$, the covering consisting of minimum number of subsets $A_1, A_2 \cdots, A_n$ such that for $x, y \in A_i, |f(cxd) - f(cyd)| < \epsilon$ for $i = 1, 2, \cdots, n$ is the covering by cosets of a suitable normal subgroup $H(f, \epsilon)$ called the $\epsilon$-kernel of finite index $n$ in $G$. If $f$ is a continuous almost periodic function on a topological group $G, H(f, \epsilon)$ is also an open and closed subgroup of finite index in $G$. A (topological) group is called an IB-group (Index Bounded group) if $\inf |n| > 0$, as $n$ varies over all the indices of (closed) subgroups of finite index of $G$. We take $c = \inf |n|$. $G$ is $p$-free if only if $c = 1$ or equivalently $|n| = 1$ for each index $n$. There exists a Mean $M$ with $\| M \| = 1$ (sup norm) on $A_c(G)$ if and only if $G$ is a $p$-free group.

Schikhof [10] calls a function $f : G \to K$ almost periodic if $f_G$ is a compactoid in $B(G, K)$, the space of bounded functions on $G$ with the supremum norm. The set of all almost periodic functions from $G$ to $K$ is denoted by $AP(G \to K)$. The almost periodic functions which are analogous of the classical case discussed earlier are called strictly almost periodic and the space of such functions is denoted by $SAP(G \to K)$. When $G$ is a topological group the space of continuous strictly almost periodic functions is the space $A_c(G)$ of

the earlier papers of the author. In general $SAP(G \to K) \subset AP(G \to K)$ ; however when the base field is locally compact $SAP(G \to K) = AP(G \to K)$. Diarra [1] has shown that $\chi_N$ the characteristic function of a normal subgroup $N$ belongs to $AP(G \to K)$ if and only if $N$ is of finite index in $G$.

# 3   Existence of Mean

**Theorem 3.1** *If $G$ is a topological $O$-dimensional group then $G$ is an IB-group if and only if its Bohr compactification $\hat{G}$ is an IB-group or equivalently a p-free group.*

**Proof:** Let $G$ be an IB-group. Then Theorem 3.3. [5] implies that there exists a Mean $M$ on $A_c(G)$. Again by Theorem 3.8. [7] $M$ defines an invariant integral for continuous functions on $\hat{G}$ and so $\hat{G}$ is a $p$-free group or equivalently an IB-group.

   Conversely if $\hat{G}$ is an IB-group or equivalently a $p$-free group, the integral on $\hat{G}$ induces an invariant mean on $A_c(G)$. and so $G$ is a $p$-free group or an IB-group with $c = 1$.   ∎

**Remark 1:** When $G$ is compact the collection of open and closed subgroups coincides with the collection of closed subgroups of finite index in $G$ and so the $p$-free condition in the usual sense coincides with the IB-condition on $G$.

**Remark 2:** When the base field $K$ is locally-compact Diarra has given (corollary 2, p.13, [1]) several equivalent criteria for the existence of mean on $AP(G \to K)$ in terms of almost periodic representations, existence of Haar measure on the Bohr compactification etc. The above theorem which gives a criterion for the existence of mean in $AP(G \to K)$ enables one to conclude that Diarra's equivalent formulations holds when and only when the group is $p$-free.

   If $G$ is an arbitrary group. Let $\tau_B$ be the subgroup topology on $G$ for which the collection of all normal subgroups of finite index is a fundamental system of neighbourhoods at the identity of $G$. With this topology, $G$ is a topological group.

**Proposition 3.2** *When $K$ is locally compact and $G$ is an arbitrary group, $AP(G \to K) = SAP(G \to K) = A_c(G)$, where $A_c(G)$ is the space of all continuous (in the subgroup topology defined above) of almost periodic functions in the sense of Maak.*

**Proof:** When $K$ is locally compact every closed bounded subset of $K$ is compact and so $SAP(G \to K) = AP(G \to K)$ (See Schikhof [10], p.3); clearly $A_c(G) \subset AP(G \to K)$. If $f \in AP(G \to K), f \in SAP(G \to K)$. Hence for $\epsilon > 0$, there exists a normal subgroup of finite index $H = H(f, \epsilon)$ such that

(i)      $G = \cup_{i=1}^{n} H x_i, x_i \in G$

(ii)     for  $x, y \in H x_i, \ i = 1, 2, \cdots, n$

         $|f(cxd) - f(cyd)| < \epsilon$ for all $c, d \in G$.

In particular for $x, y \in H, |f(x) - f(y)| < \epsilon$, i.e. $f$ is uniformly continuous with respect to the subgroup topology $\tau_B$ on $G$ and so $f \in A_c(G)$. This proves the proposition.    ∎

The next theorem gives a necessary and sufficient condition for the existence of Mean on $AP(G \to K)$ in tune with the earlier conditions for the existence of Haar measure etc. (see van Rooij [8]) where $G$ is an arbitrary group which solves the problem posed by Schikhof [10] in the case of the locally compact base field $K$. See also Diarra [1] theorem 4 and Schikhof [10], Theorem 8.2.

**Theorem 3.3** *Let $K$ be a locally compact field. An invariant Mean $M$ on $AP(G \to K)$ exists if and only if $G$ is p-free.*

**Proof:** We consider the subgroup topology $\tau_B$ on $G$ given by the normal subgroups of finite index as a neighbourhood base at the identity. By the earlier proposition 3.2, $AP(G \to K) = SAP(G \to K) = A_c(G)$. Now the Theorem follows from Theorem 3.3 of Rangan [5].    ∎

**Example:** Let $G$ be any free-group. Then for every $x \in G, x$ different from the identity of $G$, there exists a normal subgroup of finite index $N, x \notin N$. (See Hewitt and Ross [3]). Hence the subgroup topology on $G$ given by the family of normal subgroups of finite index as a neighbourhood base is a Hausdorff topology on $G$. Hence $AP(G \to K) = SAP(G \to K) = A_c(G)$. $G$ is a maximally almost periodic group. An invariant Mean exists on $AP(G \to K)$ if and only if $G$ is p-free.

**Remark:** When $K$ is locally compact for the study of continuous almost periodic functions on a totally disconnected topological group, only the topology $\tau_B$ on $G$ matters. For if $(G, \tau)$ be a totally-disconnected topological group. $G$ is a totally disconnected topological group also with respect to the topology $\tau_B$ defined by closed (in $\tau$) normal subgroups of finite index in $(G, \tau)$. The topology $\tau_B$ is weaker than $\tau$. By Theorem 4.1 Rangan [6], and proposition 3.2 above it follows that $A_c(G, \tau) = A_c(G, \tau_B) = AP(G \to K)$.

# 4   Structure of $A = A_c(G)$

Throughout this section we assume that $K$ is locally compact and $G$ is either a totally disconnected topological group or an arbitrary group $G$ considered as a topological

group with respect to the subgroup topology $\tau_B$ defined by the normal subgroups of finite index in $G$. So $A_c(G, \tau) = A_c(G, \tau_B) = AP(G \to K)$. We assume $G$ to be a $p$-free group.

**Theorem 4.1** *The algebra* $A = A_c(G)$ *is isometrically isomorphic to the group algebra* $L(\hat{G})$ *of the Bohr compactification* $\hat{G}$ *of* $G$.

**Proof:** The map $\theta : A \to L(\hat{G})$ given by $f \to \hat{f}$ where $\hat{f}$ is the associated continuous function on the compact group $\hat{G}$ to $f$ (see Rangan [6], Theorem 4.4). If $\rho$ is the homomorphism which imbeds $G$ in $\hat{G}$, for $x \in G, f(x) = \hat{f}(\rho(x))$. $\theta$ is one-to-one: For $\theta(f) = \theta(g) \Rightarrow \hat{f} = \hat{g} \Rightarrow f(x) = g(x)$ for all $x \in G \Rightarrow f = g$. $\theta$ is onto: if $h \in L(\hat{G}), h$ is a continuous function on $\hat{G}$. Define $f(x) = h(\rho(x))$ for $x \in G$ then $\hat{f} = h$. $\theta$ is an algebra homomorphism: For

$$
\begin{aligned}
f * g(x) &= M_y(f(y)g(y^{-1}x)) \\
&= \int_G f(y)g(y^{-1}x)dy = \hat{f} * \hat{g}(x)
\end{aligned}
$$

where the integral is the Haar integral and it exists since $\hat{G}$ is $p$-free, $G$ being so.

$\theta$ is an isometry: When $G$ is $p$-free $|n| = 1$ for every normal subgroup of finite index and so $c = 1$. Hence for $f \in A$,

$$
\| f \| = \sup_{x \in G} |f(x)| = \sup_{x \in G} |\hat{f}(\rho(x))| = \sup_{x \in \hat{G}} |f(t)|
$$

since $\rho(G)$ is dense in $\hat{G}$.     ∎

**Proposition 4.2** *A is the closure of the* $K$-*linear span of the idempotents of* $A$.

**Proof:** Since $A = A_c(G) = A_c(G, \tau_B) = AP(G \to K) = SAP(G \to K)$ the proposition follows from Lemma 4.4, Schikhof [10], which is now easily seen to be a restatement of the approximation Theorem 7.4 of Rangan [5].     ∎

**Theorem 4.3** *For a* $p$-*free group* $G, A = \oplus A_e$ *where* $A_e = e * A$ *is a finite-dimensional two sided ideal of* $A$ *and for every* $f \in A$,

$$
f = \sum_{e \in E} e * f \text{ and } \| f \| = \sup_{e \in E} \| e * f \|
$$

*and every non-zero minimal two sided ideal in* $A$ *is an* $A_e$ *for a suitable* $e \in E$. *If* $I$ *is a closed two sided ideal in* $A$ *then*

$$
I = cl \sum_{e \in I} A_e
$$

*where* $E$ *is the set of all minimal non-zero central idempotents of* $A$.

**Proof:** Follows from 8.14 Theorem van Rooij [9] since by the earlier theorem $A$ and $L(\hat{G})$ are isometrically isomorphic. ∎

It is not difficult to prove, using the existence of the approximate identity $(U_H)$, $(H$ varying over the collection $\Gamma'_G$ of normal subgroups of finite index in $G$) that the closed ideals in $A$ are same as closed invariant subspaces. For $f \in A$, defining $(L_a f)(x) = f(a^{-1}x)$ for $x \in G$, we get the (left) regular representation $a \to L_a$ on $G$. $A_e$ being invariant subspaces in view of Theorem 4.3, $L_a$ decomposes as a direct sum of finite-dimensional representations. Thus we get the following result.

**Theorem 4.4** *The regular Representation decomposes as a direct sum of finite-dimensional representations.*

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