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WEIGHTED MEANS IN NON-ARCHIMEDEAN FIELDS

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§1. INTRODUCTION.

In developing summability methods in non-archimedean fields, Srinivasan [6] defined the analogue of the classical weighted means $(\overline{N}, p_n)$ under the assumption that the sequence $\{p_n\}$ of weights satisfies the conditions :

$$|p_0| < |p_1| < |p_2| < \ldots < |p_n| < \ldots ; \tag{1}$$

and

$$\lim_{n\to\infty} |p_n| = \infty . \tag{2}$$

However, it turned out that these weighted means were equivalent to convergence. In the present paper, an attempt is made to remedy the situation by assuming that the sequence $\{p_n\}$ of weights satisfies the conditions :

$$p_n \neq 0, \quad n = 0,1,2,\ldots; \tag{3}$$

and

$$|p_i| \leq |P_j|, \; i = 0,1,2,\ldots,j, \quad j = 0,1,2,\ldots, \tag{4}$$

where $P_j = \sum_{k=0}^{j} p_k$, $j = 0,1,2,\ldots$. Note that (3) and (4) imply $P_n \neq 0$, $n = 0,1,2,\ldots$.

(4) is equivalent to

$$\max_{0\leq i\leq j} |p_i| \leq |P_j|, \; j = 0,1,2,\ldots .$$

Since the valuation is non-archimedean,

$$|P_j| \leq \max_{0\leq i\leq j} |p_j|$$

so that (4) is equivalent to

$$|P_j| = \max_{0\leq i\leq j} |p_j| = |p_j|. \tag{4'}$$

The assumptions (3) and (4) make the method of summability arising out of the weighted means non-trivial in certain cases (Remark 4) and further make it possible to compare two regular weighted means (Theorem 3) or compare a regular weighted mean with a regular matrix method (Theorem 4 and Theorem 5). This helps us to obtain (§4) a strictly increasing scale of regular summability methods in $\mathbf{Q}_p$, the p-adic field for a prime p; analogous to the scale of Cesàro means in $\mathbb{R}$ (the field of real numbers). These arise out of taking the weights

$$\begin{aligned} p_n &= p^{nk}, \quad \text{if } n \text{ is odd;} \\ &= \frac{1}{p^{nk}}, \quad \text{if } n \text{ is even ,} \end{aligned}$$

$n = 0, 1, 2, \ldots, \; k = 0, 1, 2, \ldots$.

For a knowledge of $(\overline{N}, p_n)$ methods in the classical case, the reader may refer [2],[5] and for analysis in non-archimedean fields [1].

§2. PRELIMINARIES .

Throughout this paper, K denotes a complete, non-trivially valued, non-archimedean field and infinite matrices and sequences have their entries in K. Given an infinite matrix $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ and a sequence $\{x_k\}$, $k = 0, 1, 2, \ldots$, by the A-transform of $\{x_k\}$, we mean the sequence $\{(Ax)_n\}$ where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \; n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = s$, we say that $\{x_k\}$ is *A-summable* (or summable by the infinite matrix method A) to s. If $\lim_{n\to\infty} (Ax)_n = s$ whenever $\lim_{k\to\infty} x_k = s$, the matrix method A is said to be *regular*. It is well-known (see [3], [4]) that A is regular if and only if

$$\left.\begin{aligned} &\text{(a)} \quad \sup_{n,k} |a_{nk}| < \infty \quad ; \\ &\text{(b)} \quad \lim_{n\to\infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots \quad ; \\ \text{and} \quad &\text{(c)} \quad \lim_{n\to\infty} \Big(\sum_{k=0}^{\infty} a_{nk} \Big) = 1 \quad . \end{aligned}\right\} \qquad (5)$$

(cf. For criterion for the regularity of a matrix method in the classical case see [2], p.43, Theorem 2). If a regular matrix A is such that $\lim_{n\to\infty} (Ax)_n = s$ implies $\lim_{k\to\infty} x_k = s$, the matrix method A is said to be *trivial.* Given two infinite matrix methods A, B, we say that *A is included in B*, written as $A \subset B$, if any sequence $\{x_k\}$ that is A-summable to s is also B-summable to s. An infinite matrix $A = (a_{nk})$ is said to be *triangular* (or, more precisely, lower triangular) if $a_{nk} = 0, \quad k > n, \quad n = 0, 1, 2, \ldots$.

Definition 1. The $(\overline{N}, p_n)$ method is defined by the infinite matrix (a_{nk}) where

$$\left.\begin{aligned} a_{nk} &= \frac{p_k}{P_n}, \ k \leq n \ ; \\ &= 0, \ k > n \ . \end{aligned}\right\} \tag{6}$$

Remark 1. If $\left|\frac{P_{n+1}}{P_n}\right| > 1$, $n = 0, 1, 2 \ldots$ and $\lim_{n\to\infty} |P_n| = \infty$ i.e. $|P_n|$ strictly increases to infinity, then the method $(\overline{N}, p_n)$ is trivial. For $|p_n| = |P_n - P_{n-1}| = |P_n|$, since $|P_n| > |P_{n-1}|$. So (1) is satisfied. Since $\lim_{n\to\infty} |P_n| = \infty$, $\lim_{n\to\infty} |p_n| = \infty$ so that (2) is satisfied too. Hence $(\overline{N}, p_n)$ is trivial because of Theorem 4.2 of [6].

In the sequel we shall suppose that the sequence $\{p_n\}$ of weights satisfies conditions (3) and (4).

An example of such an $(\overline{N}, p_n)$ method corresponds to $\{p_n\}$ defined by

$$\begin{aligned} p_n &= p^n, \quad \text{if} \quad n \quad \text{is odd;} \\ &= \frac{1}{p^n}, \quad \text{if} \quad n \quad \text{is even ,} \end{aligned}$$

where $K = \mathbf{Q}_p$.

Remark 2. We note that (4) is equivalent to

$$|P_{n+1}| \geq |P_n|, \quad n = 0, 1, 2, , \ldots . \tag{7}$$

Proof. Let (4) hold. Now

$$\begin{aligned} |P_{n+1}| &= \max_{0\leq i\leq n+1} |p_i| \\ &= \max \left[\max_{0\leq i\leq n} |p_i|, \quad |p_{n+1}| \right] \\ &= \max \left[\, |P_n|, \quad |p_{n+1}| \, \right] \\ &\geq |P_n|, \quad n = 0, 1, 2, \ldots \quad . \end{aligned}$$

Conversely, let (7) hold. For a fixed integer $j \geq o$ let $0 \leq i \leq j$. Then

$$\begin{aligned} |p_i| &= |P_i - P_{i-1}| \\ &\leq \max \left[\, |P_i|, \, |P_{i-1}| \, \right] \\ &\leq |P_i| \\ &\leq |P_j| \quad , \end{aligned}$$

by (7).

§3. Main results.

Theorem 1. $(\overline{N}, p_n)$ is regular if an only if

$$\lim_{n\to\infty} |P_n| = \infty \tag{8}$$

Proof. Let the $(\overline{N}, p_n)$ method be regular Using (6) and (5)(b), we note that (8) holds. Conservely, let (8) hold. In view of (6) and (8) it follows that $\lim_{n\to\infty} a_{nk} = 0,\ k = 0, 1, 2, \ldots$.
Now, $|a_{nk}| = 0,\ k > n$. If $k \leq n$, $|a_{nk}| = \dfrac{|p_k|}{|P_n|} \leq 1$, in view of (4).
Also $\sum_{k=0}^{\infty} a_{nk} = 1,\ n = 0, 1, 2, \ldots$ so that $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty} a_{nk}\right) = 1$. Thus, by (5) the method $(\overline{N}, p_n)$ is regular.

Remark 3. If $(\overline{N}, p_n)$ is non-trivial, then (1) cannot be satisfied. Suppose (1) holds, then $|p_n| = |P_n|$ so that (2) also holds. Thus $(\overline{N}, p_n)$ is trivial by Theorem 4.2 of [6], a contradiction. This establishes the claim.

Remark 4. There are non-trivial $(\overline{N}, p_n)$ methods. Let $\alpha \in K$ such that $0 < c = |\alpha| < 1$, this being possible since K is non-trivially valued. Let

$$\{p_n\} = \left\{\alpha, \frac{1}{\alpha^2}, \alpha^3, \frac{1}{\alpha^4}, \ldots\right\}$$

and

$$\{s_n\} = \left\{\frac{1}{\alpha}, \alpha^2, \frac{1}{\alpha^3}, \alpha^4, \ldots\right\}$$

It is clear that $\{s_n\}$ does not converge. If $\{t_n\}$ is the $(\overline{N}, p_n)$ transform of $\{s_k\}$,

$$\begin{aligned}
|t_{2k}| &= \left|\frac{2k}{\alpha + \frac{1}{\alpha^2} + \alpha^3 + \ldots + \frac{1}{\alpha^{2k}}}\right| \\
&= \frac{|2k|}{\left(\frac{1}{c^{2k}}\right)} \\
&\leq c^{2k} \\
|t_{2k+1}| &= \left|\frac{2k+1}{\alpha + \dfrac{1}{\alpha^2} + \alpha^3 + \ldots + \dfrac{1}{\alpha^{2k}} + \alpha^{2k+1}}\right| \\
&= \frac{|2k+1|}{\left(\frac{1}{c^{2k}}\right)} \\
&\leq c^{2k}
\end{aligned}$$

so that $\lim_{n\to\infty} t_n = 0$. Thus $\{s_n\}$, though non convergent, is summable $(\overline{N}, p_n)$ (in fact, to 0). This establishes our claim.

Theorem 2. (Limitation theorem) If $\{s_n\}$ is summable $(\overline{N}, p_n)$ to s, then

$$|s_n - s| = o\Big(\Big|\frac{P_n}{p_n}\Big| \Big),\ n \to \infty.$$

Proof. If $\{t_n\}$ is the $(\overline{N}, p_n)$ transform of $\{s_k\}$, then

$$\begin{aligned}
\Big|\frac{p_n(s_n - s)}{P_n}\Big| &= \Big|\frac{p_n s_n - p_n s)}{P_n}\Big| \\
&= \Big|\frac{P_n t_n - P_{n-1}t_{n-1} - s(P_n - P_{n-1}}{P_n}\Big| \\
&= \Big|\frac{P_n(t_n - s) - P_{n-1}(t_{n-1} - s)}{P_n}\Big| \\
&\leq \max\Big[\,|t_n - s|,\ \Big|\frac{P_{n-1}}{P_n}\Big|\ \ |t_{n-1} - s|\,\Big] \\
&\leq \max\,[\,|t_n - s|,\ |t_{n-1} - s|\,]
\end{aligned}$$

since $\Big|\frac{P_{n-1}}{P_n}\Big| \leq 1$, by (7) . Since $\lim_{n\to\infty} t_n = s$, it follows that $\lim_{n\to\infty}\Big|\frac{p_n(s_n - s)}{P_n}\Big| = 0$. Thus

$$|s_n - s| = o\Big(\Big|\frac{P_n}{p_n}\Big| \Big), n \to \infty.$$

Theorem 3. (Comparison theorem for two regular weighted means). If $(\overline{N}, p_n)$, $(\overline{N}, q_n)$ are two regular methods and if

$$\Big|\frac{P_n}{p_n}\Big| \leq H\Big|\frac{Q_n}{q_n}\Big|, \quad n = 0,1,2,\dots\ , \tag{9}$$

where $H > 0$ is a constant and $Q_n = \sum_{k=0}^{\infty} q_k$, then $(\overline{N}, p_n) \subset (\overline{N}, q_n)$.

Proof. Let, for a given sequence $\{s_n\}$,

$$\begin{aligned}
t_n &= \frac{p_0 s_0 + p_1 s_1 + \dots + p_n s_n}{P_n}\ , \\
u_n &= \frac{q_0 s_0 + q_1 s_1 + \dots + q_n s_n}{Q_n}\ ,\ n = 0,1,2,\dots\ .
\end{aligned}$$

Then $p_0s_0 = P_0t_0,\ p_ns_n = P_nt_n - P_{n-1}t_{n-1},\ n = 1,2,\dots$. Now,

$$\begin{aligned} u_n &= \frac{1}{Q_n}\Big[\frac{q_0}{p_0}P_0t_0 + \frac{q_1}{p_1}(P_1t_1 - P_0t_0) + \dots + \frac{q_n}{p_n}(P_nt_n - P_{n-}t_{n-1})\Big] \\ &= \sum_{k=0}^{\infty} c_{nk}t_k\ , \end{aligned}$$

where
$$\begin{aligned} c_{nk} &= \Big(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\Big)\frac{P_k}{Q_n},\ k < n; \\ &= \frac{q_k}{p_k}\frac{P_k}{Q_k},\ k = n; \\ &= 0,\ k > n. \end{aligned}$$

Since $\lim_{n\to\infty}|Q_n| = \infty,\ \lim_{n\to\infty} c_{nk} = 0,\ k = 0,1,2,\dots$. If $s_n = 1,\ n = 0,1,2,\dots,$

$t_n = u_n\ =\ 1,\ n = 0,1,2,\dots$ so that $\sum_{k=0}^{\infty} c_{nk} = 1,\ n = 0,1,2,\dots$ and so $\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty} c_{nk}\Big) = 1.$

Let $k < n$.

$$\begin{aligned} |c_{nk}| &= \Big|\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\Big|\ \Big|\frac{P_k}{Q_n}\Big| \\ &\le \max\Big[\ \Big|\frac{q_k}{p_k}\Big|\ \Big|\frac{P_k}{Q_n}\Big|,\ \Big|\frac{q_{k+1}}{p_{k+1}}\Big|\ \Big|\frac{P_k}{Q_n}\Big|\ \Big] \\ &\le \max\Big[\ \Big|\frac{q_k}{p_k}\Big|\ \Big|\frac{P_k}{Q_k}\Big|,\ \Big|\frac{q_{k+1}}{p_{k+1}}\Big|\ \Big|\frac{P_{k+1}}{Q_{k+1}}\Big|\ \Big] \\ &\le H\ , \end{aligned}$$

by (9) , since $k < n$ implies $|Q_k|, |Q_{k+1}| \le |Q_n|$ and so $\frac{1}{Q_n} \le \frac{1}{Q_k}, \frac{1}{Q_{k+1}}$ and $|P_k| \le |P_{k+1}|$.

If $k = n,\ |c_{nn}| = \Big|\frac{q_n}{p_n}\frac{P_n}{Q_n}\Big| \le H$ and $|c_{nk}| = 0 \le H,\ k > n$. Consequently $\sup_{n,k}|a_{nk}| \le H$..
The method (c_{nk}) is thus regular, using (5) and so $(\overline{N}, p_n) \subset (\overline{N}, q_n)$. The proof of the theorem is now complete.

Remark 5. Note that the classical counterpart of Theorem 3 (see [2], p.58, Theorem 14) has an additional hypothesis.

Theorem 4. (Comparison theorem for a regular $(\overline{N}, p_n)$ method and a regular matrix). Let $(\overline{N}, p_n)$ be a regular method and A be a regular matrix . If

$$\lim_{k\to\infty}\frac{a_{nk}P_k}{p_k} = 0, \quad n = 0,1,2,\dots\ ; \tag{10}$$

and

$$\sup_{n,k}\Big|\ \Big(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}}\Big)P_k\Big| < \infty\ , \tag{11}$$

then $(\overline{N}, p_n) \subset A$.

Proof. Let $\{s_n\}$ be any sequence , $\{t_n\}$, $\{\tau_n\}$ be its $(\overline{N}, p_n)$, A transforms respectively so that

$$t_n = \frac{p_0 s_0 + p_1 s_1 + \ldots + p_n s_n}{P_n},$$

$$\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k, \; n = 0, 1, 2, \ldots .$$

Now,

$$s_n = \frac{P_n t_n - P_{n-1} s_1 t_{n-1}}{p_n}, P_{-1} = 0$$

Let $\lim_{n\to\infty} t_n = s$. $\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k$ exists , $n = 0, 1, 2 \ldots$ and in fact

$$\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k = \sum_{k=0}^{\infty} a_{nk} \left\{ \frac{P_k t_k - P_{k-1} t_{k-1}}{p_k} \right\}$$
$$= \sum_{k=0}^{\infty} \left(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) P_k t_k ,$$

since $\lim_{k\to\infty} \frac{a_{n,k+1}}{p_{k+1}} P_k t_k = 0$ by (10) and using the fact that $\{t_k\}$ is convergent and so bounded and $\left| \frac{P_k}{P_{k+1}} \right| \leq 1$. We can now write

$$\tau_n = \sum_{k=0}^{\infty} b_{nk} t_k ,$$

where

$$b_{nk} = \left(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) P_k .$$

By (11) , $\sup_{n,k} |b_{nk}| < \infty$. Since A is regular , $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \ldots$ so that $\lim_{n\to\infty} b_{nk} = 0$, $k = 0, 1, 2, \ldots$. Let $s_n = 1$, $n = 0, 1, 2, \ldots$. Then $t_n = 1$, $n = 0, 1, 2, \ldots$. It now follows that $\sum_{k=0}^{\infty} b_{nk} = \sum_{k=0}^{\infty} a_{nk}$, $n = 0, 1, 2, \ldots$. Consequently $\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} b_{nk} \right) = \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} a_{nk} \right) = 1$. The method (b_{nk}) is thus regular and so $\lim_{n\to\infty} t_n = s$ implies $\lim_{n\to\infty} \tau_n = s$. i.e. $(\overline{N}, p_n) \subset A$.

Theorem 5. $(\overline{N}, p_n)$ is a regular method and $A = (a_{nk})$ is a regular triangular matrix. Then $(\overline{N}, p_n) \subset A$ if and only if (11) holds.

Proof . Let (11) hold. Since A is a triangular matrix, (10) clearly holds. In view of Theorem 4, we have $(\overline{N}, p_n) \subset A$. Conversely, let $(\overline{N}, p_n) \subset A$. Following the notation of Theorem 4, let $\lim_{n\to\infty} t_n = s$. As in the proof of Theorem 4,

$$\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k = \sum_{k=0}^{\infty} b_{nk} t_k,$$

where

$$b_{nk} = \left(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}}\right) P_k \, .$$

Since $(\overline{N}, p_n) \subset A$, for every sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = s$, $\lim_{n\to\infty} \tau_n = s$. This means that (b_{nk}) is a regular matrix and so (11) holds. This complices the proof.

§4. A SCALE OF STRICTLY INCREASING WEIGHTED MEANS.

We conclude the present paper by obtaining a strictly increasing scale of regular summability methods in $\mathbf{Q}_p$. We define, for $k = 0, 1, 2, \ldots$, the method $(\overline{N}, p_n^{(k)})$ by

$$\begin{aligned} p_n^{(k)} &= p^{nk}, \quad \text{if } n \text{ is odd ;} \\ &= \frac{1}{p^{nk}}, \quad \text{if } n \text{ is even ;} \end{aligned}$$

We now establish that

$$(\overline{N}, p_n^{(k)}) \underset{\neq}{\subset} (\overline{N}, p_n^{(k+1)}). \tag{12}$$

We apply Theorem 3 to prove this assertion. For convenience, let $p_n = p_n^{(k)}$ and $q_n = p_n^{(k+1)}$, $n = 0, 1, 2, \ldots$. If n is odd,

$$\begin{aligned} \left|\frac{P_n}{p_n}\right| &= \frac{1}{c^{(n-1)k}} \cdot \frac{1}{c^{nk}} &&= \frac{1}{c^{(2n-1)k}} \\ \left|\frac{Q_n}{q_n}\right| &= \frac{1}{c^{(n-1)(k+1)}} \cdot \frac{1}{c^{n(k+1)}} &&= \frac{1}{c^{(2n-1)(k+1)}}, \quad c = |p| < 1, \end{aligned}$$

so that

$$\left|\frac{P_n}{p_n}\right| \leq \left|\frac{Q_n}{q_n}\right|$$

If n is even,

$$\begin{aligned} \left|\frac{P_n}{p_n}\right| &= \frac{1}{c^{nk}} \cdot c^{nk} = 1 \\ \left|\frac{Q_n}{q_n}\right| &= \frac{1}{c^{n(k+1)}} \cdot c^{n(k+1)} = 1. \end{aligned}$$

Thus

$$\left|\frac{P_n}{p_n}\right| \leq \left|\frac{Q_n}{q_n}\right|$$

in this case too. Consequently, by Theorem 3, $(\overline{N}, p_n^{(k)}) \subset (\overline{N}, p_n^{(k+1)})$. Let now

$$\begin{aligned} s_n &= 0, && \text{if } n \text{ is even ;} \\ &= \frac{1}{p^{n(k+1)+k(n-1)}}, && \text{if } n \text{ is odd .} \end{aligned}$$

Let $\{\tau_n\}$ be the $(\overline{N}, q_n)$ transform of $\{s_n\}$.
If n is odd,

$$\begin{aligned} |\tau_n| &= \left| \frac{0 + p^{k+1} \cdot \frac{1}{p^{k+1}} + 0 + p^{3(k+1)} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{n(k+1)} \cdot \frac{1}{p^{n(k+1)+k(n-1)}}}{1 + p^{k+1} + \frac{1}{p^{2(k+1)}} + \ldots + \frac{1}{p^{(n-1)(k+1)}} + p^{n(k+1)}} \right| \\ &= \frac{\frac{1}{c^{k(n-1)}}}{\frac{1}{c^{(k+1)(n-1)}}} \\ &= c^{n-1} \end{aligned}$$

If n is even,

$$\begin{aligned} |\tau_n| &= \left| \frac{0 + p^{k+1} \cdot \frac{1}{p^{k+1}} + 0 + p^{3(k+1)} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{(n-1)(k+1)} \cdot \frac{1}{p^{(n-1)(k+1)+k(n-2)}} + 0}{1 + p^{k+1} + \frac{1}{p^{2(k+1)}} + \ldots + p^{(n-1)-(k+1)} + \frac{1}{p^{n(k+1)}}} \right| \\ &= \frac{\frac{1}{c^{k(n-2)}}}{\frac{1}{c^{n(k+1)}}} \\ &= c^{n+2k} \end{aligned}$$

In both the cases , $\lim_{n\to\infty} \tau_n = 0$. Thus $\{s_n\}$ is summable $(\overline{N}, q_n)$ to 0. Let , now , $\{t_n\}$ be the $(\overline{N}, p_n)$ transform of $\{s_n\}$.

If n is odd

$$
\begin{aligned}
|\tau_n| &= \left| \frac{0 + p^k \cdot \frac{1}{p^{k+1}} + 0 + p^{3k} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{nk} \cdot \frac{1}{p^{n(k+1)+k(n-1)}}}{1 + p^k + \frac{1}{p^{2k}} + \ldots + \frac{1}{p^{(n-1)k}} + p^{nk}} \right| \\
&= \frac{\frac{1}{c^{n+k(n-1)}}}{\frac{1}{c^{(n-1)k}}} \\
&= \frac{1}{c^n}
\end{aligned}
$$

Since $\frac{1}{c} > 1$, $\lim_{n \to \infty} |t_n| = \infty$ that $\{t_n\}$ cannot converge. Thus $\{s_n\}$ is not $(\overline{N}, p_n)$ summable and consequently (12) holds.

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