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N. DE GRANDE-DE KIMPE
C. PEREZ-GARCIA

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LIMITED SPACES

N. De Grande-De Kimpe and C. Perez-Garcia *

Abstract. We introduce and study two new classes of non-archimedean locally convex Hausdorff spaces: the limited spaces and the BL-spaces. In particular we have:

$$E \textit{ is nuclear} \Rightarrow E \textit{ is limited} \Rightarrow E \textit{ is } BL.$$

We also characterize the nuclear spaces among the limited spaces and the limited spaces among the BL-spaces. Finally we compare the non-archimedean results with the classical (= real or complex) ones.

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1. PRELIMINARIES

Throughout this paper K is a non-archimedean valued field that is complete for the metric induced by the non-trivial valuation $| \, . \, |$. Also, E, F are Hausdorff locally convex spaces over K.

A subset A of E is called *compactoid* if for every zero-neighbourhood U in E there exists a finite set $S \subset E$ such that $A \subset coS + U$, where coS is the absolutely convex hull of S. Every compactoid set is bounded.

An other interesting subclass of the bounded subsets of E consists of the limited sets (Definition 2.1). It turns out that every compactoid subset is limited (2.2.ii)) and spaces in which all the limited subsets are compactoid are called GP-spaces (they have been studied in [9]).

By $L(E, F)$ we will denote the vector space of all continuous linear maps (or operators) from E into F. $T \in L(E, F)$ is called *compact* if there exists a zero-neighbourhood U in E for which $T(U)$ is a compactoid subset of F. Also, T is called *compactifying* if for every bounded subset A of E, $T(A)$ is a compactoid subset of F. By $C(E, F)$ (resp. $CF(E, F)$)

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we will denote the vector space of all compact (resp. compactifying) operators from E into F (These spaces of operators and the relation between them were studied in [4]).

The compact (compactifying) operators between normed spaces form an operator ideal (the concept of operator ideal is the same as in the classical theory, see e.g. [22]). Also, the limited operators (Definition 2.1) constitute another interesting example of an operator ideal, which has been studied in [10].

We finish this section with some facts concerning the normed spaces associated with a locally convex space E.

i) For a continuous seminorm p on E we put $E_p = E/Kerp$ and we denote by π_p the canonical surjection $\pi_p : E \longrightarrow E_p$. Then E_p is a normed space for the norm $\| \, . \, \|_p$ defined by $\| \pi_p(x) \|_p = p(x)$, $x \in E$.

ii) If p, q are continuous seminorms on E with $q \geq p$ then there exists a unique continuous linear map $\varphi_{pq} : E_q \longrightarrow E_p$ with $\varphi_{pq} \circ \pi_q = \pi_p$.

iii) If U is a zero-neighbourhood in E and U^o is the polar of U in E' we put $E'_{U^o} = \bigcup_{\lambda \in K} \lambda U^o$ and for $f \in E'_{U^o}$ we define $\| f \|_{U^o} = inf\{| \alpha | : f \in \alpha U^o\}$. Then $E'_{U^o}, \| \, . \, \|_{U^o}$ is a Banach space.

iv) If U, V are zero-neighbourhoods in E with $V \subset U$ we denote by φ_{UV} the continuous canonical injection $\varphi_{UV} : E'_{U^o}, \| \, . \, \|_{U^o} \longrightarrow E'_{V^o}, \| \, . \, \|_{V^o}$.

v) Let p be a continuous seminorm on E and put $U = \{x \in E : p(x) \leq 1\}$. Let $I_p : E'_{U^o} \longrightarrow (E_p)'$ be defined by

$$I_p(f)(\pi_p(x)) = f(x) \qquad f \in E'_{U^o}.$$

Then I_p is a linear homeomorphism from E'_{U^o} onto $(E_p)'$. So, E'_{U^o} and $(E_p)'$ can be identified as locally convex spaces.

vi) Let p, q be as in ii), U as in v) and $V = \{x \in E : q(x) \leq 1\}$. Then, making the identifications as in v), we have that $\varphi_{UV} = (\varphi_{pq})^*$ (the transposed of φ_{pq}).

vii) A sequence $(f_n)_n$ in E' is said to be *locally convergent* to zero if there exists a zero-neighbourhood U in E such that $(f_n) \subset E'_{U^o}$ and $lim_n \| f_n \|_{U^o} = 0$.

For unexplained terms and background we refer to [25] (normed spaces) and [23] (locally convex spaces).

2. LIMITED SPACES

Definition 2.1:

i) (compare [18]) A bounded subset A of E is called *limited* in E if every equicontinuous $\sigma(E', E)$-null sequence in E' converges to zero uniformly on A.

ii) (compare [3]) An operator $T \in L(E, F)$ is called *limited* if there exists a zero-neighbourhood U in E such that $T(U)$ is limited in F. We denote by $Lim(E, F)$ the vector space of all limited operators from E to F.

The following properties of limited sets and operators will be needed in the sequel.

Proposition 2.2:

i) A bounded subset $A \subset E$ is limited in E iff for every $T \in L(E, c_0)$ $T(A)$ is compactoid in c_0. In particular, if E is a normed space then the closed unit ball of E, B_E, is limited iff $L(E, c_0) = C(E, c_0)$.

ii) Every compactoid set is limited and hence $C(E,F) \subset Lim(E,F)$ for all locally convex spaces E,F.

iii) If A is limited in E and $B \subset A$ then B is limited in E.

iv) If $A \subset E$, then A is limited in E iff A is limited on $\hat{E}$ (where $\hat{E}$ denotes the completion of E).

v) If $A \subset E$ is limited in E and $T \in L(E,F)$, then $T(A)$ is limited in F.

vi) If A is limited in E then its closure, $\overline{A}$, is limited in E.

vii) If $A, B \subset E$ are limited in E then $A+B$ is limited in E.

viii) If E,F are normed spaces and $T \in L(E,F)$ then, T is limited iff T^* (the transposed of T) transforms equicontinuous weak-convergent sequences in F' into norm-convergent sequences in E'.*

ix) The limited operators between normed spaces form an operator ideal.

Proof: Properties i), ..., v) are proved in [9], 2.2, 2.3. Also, vi) and vii) follow directly from i) and the properties of compactoid sets (see e.g. [6], 1.2). Finally, viii) and ix) are proved in [10], 4.10 and 4.12 respectively.

Definition 2.3: (compare [17]) We say that E is a *limited space* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $q \geq p$ such that the canonical operator $\varphi_{pq} : E_q \longrightarrow E_p$ is limited.

Applying 2.2 and similarly to Proposition 2.5 of [7], we can give the following characterization of limited spaces in terms of operators.

Theorem 2.4: *The following are equivalent.*

i) E is limited.

ii) $L(E,F) = Lim(E,F)$ for all Banach spaces F.

iii) $L(E,F) = Lim(E,F)$ for all normed spaces F.

iv) For every continuous seminorm p on E the canonical operator $\pi_p : E \longrightarrow E_p$ is limited.

Corollary 2.5: *If E is limited then*
every bounded subset of E is limited. (*)
(spaces E with property (*) will be called *BL-spaces*. They are studied in section 3).

Proof: We have $L(E, c_0) = Lim(E, c_0)$ and hence $L(E, c_0) = C(E, c_0)$ ([9], 2.8.i)). Then, apply 2.2.i).

Examples 2.6:

i) Recall that E is *nuclear* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $q \geq p$ such that the corresponding map φ_{pq} is compact. Then, it follows directly from 2.2.ii) that every nuclear space is limited.

ii) If the valuation on K is discrete, then every locally convex space over K is GP ([9], 2.8.iii)) and so E is limited iff E is nuclear.

iii) If E is a normed space then E is limited iff $L(E, c_0) = C(E, c_0)$ (apply 2.2, 2.4 and 2.5).

iv) If the valuation on K is dense then l^∞ is a limited (non-nuclear) space. Indeed, by [25], 5.19 $L(l^\infty, c_0) = C(l^\infty, c_0)$. Now apply iii).

Analogously, if K is small then $L(l^\infty(I), c_0) = C(l^\infty(I), c_0)$ for all sets I ([21], 6.4) and so $l^\infty(I)$ is limited.

v) Let X be a zero-dimensional Hausdorff space and consider the following spaces of continuous functions :

$PC(X) = \{f : X \longrightarrow K : f$ is continuous and $f(X)$ is a precompact subset of $K\}$, endowed with the supremum norm.

$C(X) = \{f : X \longrightarrow K : f$ is continuous$\}$, endowed with the compact-open topology.

$BC(X) = \{f : X \longrightarrow K : f$ is bounded and continuous $\}$, endowed with the strict topology. This is the topology generated by the seminorms $p_\phi(f) = sup_{x\in X} \mid \phi(x).f(x) \mid$, where $\phi : X \longrightarrow K$ is a bounded function vanishing at infinity.

Then,

a) $PC(X)$ is limited (resp. a BL-space) iff $PC(X)$ is finite-dimensional (or equivalentely, X is finite). Indeed, observe that $PC(X)$ is a GP-space ([9], 3.1).

b) $C(X)$ (resp. $BC(X)$) is limited iff $C(X)$ (resp. $BC(X)$) is nuclear. Even more, $C(X)$ (resp. $BC(X)$) is a BL-space iff it is nuclear (see [9], 3.7). For several characterizations of the nuclearity of $C(X)$ (resp. $BC(X)$) see [7], 3.3 (resp. [8], 3.4).

Proposition 2.7: *(Permanence properties)*

i) If the valuation on K is discrete and E is a limited space over K, then every subspace of E is a limited space.

ii) A subspace of a limited space is not in general a limited space.

iii) If $\{E_i : i \in I\}$ is a family of limited spaces, then the product $E = \prod_{i\in I} E_i$ is a limited space.

iv) If $\{E_n : n \in N\}$ is a sequence of limited spaces, then the locally convex direct sum $E = \bigoplus_{n\in N} E_n$ is a limited space. However, the locally convex direct sum of an uncountable family of limited spaces does not need to be limited (e.g., take the locally convex direct sum of an uncountable family of copies of K when K is discretely valued and apply 2.6.ii)).

v) If E is a limited space and M is a closed subspace of E, then the quotient E/M is a limited space.

Proof: i) For discretely valued fields limited spaces coincide with nuclear spaces (2.6.ii)). Now apply [6], 5.7.ii).

ii) Suppose that the valuation on K is dense. Then, l^∞ is limited (2.6.iv)), and since $L(c_0, c_0) \neq C(c_0, c_0)$ we have that $c_0 \subset l^\infty$ is not limited (see 2.6.iii)).

iii) Let F be a normed space and let $T \in L(E, F)$. We shall prove that $T \in Lim(E, F)$ (see 2.4). Since T is bounded on some zero-neighbourhood of E we can assume that I is finite. Now, applying 2.2.vii) and the fact that every E_i is limited we can easily see that $T \in Lim(E, F)$.

iv) Let E_n, E be as in iv) and $u_n : E_n \longrightarrow E$, $n = 1, 2, \ldots$ the canonical maps from E_n into E. Let F be a normed space and take $T \in L(E, F)$. It suffices to prove that T is limited (see 2.4). Again by 2.4 all the maps $T \circ u_n$, $n = 1, 2, \ldots$ are limited. Hence, for each n there exists a zero-neighbourhood U_n in E_n such that $(T \circ u_n)(U_n)$ is a limited subset of F contained in B_F. Now choose $(\delta_n)_n \in c_0$ and define $U = co(\bigcup_n u_n(\delta_n U_n))$. Then, U is a zero-neighbourhood in E. It is now left to prove that $T(U)$ is limited in F, or equivalently, that $S(T(U))$ is compactoid in c_0 for every $S \in L(F, c_0)$ with $\| S \| \leq 1$ (2.2.i)). For such an S we have $S(T(U)) \subset \sum_n \delta_n(S((T \circ u_n)(U_n)))$. Take $\epsilon > 0$. Then there exists $n_0 \in N$ such that $\delta_n(S((T \circ u_n)(U_n))) \subset \{x \in c_0 : \| x \| \leq \epsilon\} = B_{c_0}(0, \epsilon)$ for all $n \geq n_0$. Hence $S(T(U)) \subset \sum_{n=1}^{n=n_0} \delta_n(S((T \circ u_n)(U_n))) + B_{c_0}(0, \epsilon)$. Since the first term of this sum is compactoid we are done.

v) Let F be a normed space and let $T \in L(E/M, F)$. Since E is limited, there exists a zero-neighbourhood U in E such that $(T \circ Q)(U)$ is limited in F (where $Q : E \longrightarrow E/M$ is the canonical surjection). Then, $T \in Lim(E/M, F)$. By 2.4, E/M is limited.

We now characterize the nuclear spaces among the limited spaces.

Theorem 2.8: (compare 2.6.ii)) *Suppose that the valuation on K is dense. Then, the following are equivalent.*

i) E is nuclear.

ii) E is limited and of countable type (Recall that E is *of countable type* if for every continuous seminorm p on E the normed space E_p is of countable type).

iii) E is limited and for every continuous seminorm p on E, E_p is a GP-space.

Proof: *i)* $\Rightarrow$ *ii)* It follows from 2.6.i) and [23], 1.3.

ii) $\Rightarrow$ *iii)* If E is of countable type, then for every continuous seminorm p on E, E_p is a GP-space ([9], 2.8.i)).

iii) $\Rightarrow$ *i)* Direct consequence of the definition of nuclear space.

Remark 2.9:

i) In [11] we constructed an example of a non-nuclear Fréchet space E of countable type in which every bounded subset is compactoid (hence limited). By 2.8 this space is not limited. Hence the converse of 2.5 does not hold in general.

ii) There is in general no relation between "E is limited" and "E is of countable type".

Indeed, l^∞ is limited when the valuation on K is dense (2.6.iv)). On the other hand, c_0 is of countable type but not limited (2.8).

iii) The space E is called *quasinormable* if for every zero-neighbourhood U in E there exists a zero-neighbourhood V in E with $V \subset U$ such that on U^o the strong topology $\beta(E', E)$ on E' coincides with the topology induced by the norm $\| \, . \, \|_{V^o}$ on E'_{V^o} (where $\beta(E', E)$ is the topology of uniform convergence on the bounded sets of E). One can easily see that if E is polar (i.e., its topology is defined by a family Π of seminorms such that $p = sup\{| f |: f \in E', | f | \leq p\}$ for all $p \in \Pi$) then, E is quasinormable iff for every zero-neighbourhood U in E there exists a zero-neighbourhood V in E with $V \subset U$ such that for non-zero μ in K there exists a bounded set A in E with $V \subset \overline{A + \mu U}^{\sigma(E,E')}$. Taking into account this fact in conjunction with [23], 5.8 we can prove, like in [14], 5.3 that

"If E is polar, then E is nuclear iff E is quasinormable and every bounded subset of E is compactoid".

However, there is in general no relation between "E is limited" and "E is quasinormable" (compare with 3.11).

Indeed, c_0 is quasinormable (every normed space is quasinormable) but not limited. On the other hand, suppose that the valuation on K is dense and consider the space $E = l^\infty$ equipped with the topology defined by all the $\sigma(E, E^*)$-contiuous seminorms together with the usual norm $\| \, . \, \|_\infty$ (where E^* denotes the algebraic dual of E). By 2.6.iv) this space is limited. Also, E is a polar space for which every bounded set is compactoid ([23], Remark following 10.11). But E is not nuclear, and hence E is not quasinormable.

Observe that this example also shows that the condition "E is of countable type" in 2.8.ii) cannot be substituted in general by "E is GP".

3. BL-SPACES

Recall (2.5) that E is called a *BL-space* if every bounded subset of E is limited in E.

Before characterizing BL-spaces we give the following Lemma wich will be very useful in the sequel.

Lemma 3.1: *If T is a linear map from E into c_0, then*

i) T is continuous iff T can be written as

$$T(x) = (f_n(x))_n \qquad (x \in E)$$

where $(f_n)_n$ is an equicontinuous $\sigma(E', E)$-null sequence in E'.

ii) T is compact iff T can be written as in i) with $(f_n)_n$ converging locally to zero.

iii) T is compactifying iff T can be written as in i) with $(f_n)_n$ converging to zero in $\beta(E', E)$.

Proof: Properties i) and ii) are proved in [5], Lemma 2.

iii) Let T be written as in i). It follows from [20], 2.1 that T is compactifying iff for every bounded subset A of E, $(sup_{x \in A} \mid f_n(x) \mid)_n$ is majorized by an element of c_0. But this means that $lim_n f_n = 0$ in $\beta(E', E)$.

Theorem 3.2: *The following are equivalent.*

i) E is a BL-space.

ii) Every equicontinuous $\sigma(E',E)$-null sequence in E' converges to zero in $\beta(E',E)$.

iii) $L(E,c_0) = CF(E,c_0)$.

iv) If $(T_n)_n$ is an equicontinuous sequence in L(E,F), F any Banach space, such that $T_n(E)$ is of countable type for all n and $T_n \longrightarrow T$ pointwise, then $T \in CF(E,F)$.

If the valuation on K is dense, then properties i),...,iv) are equivalent to

v) $CF(E,c_0)$ is complemented in $L_\beta(E,c_0)$ (where $L_\beta(E,c_0)$ denotes the vector space $L(E,c_0)$ endowed with the topology of uniform convergence on the bounded sets of E).

Note that if the valuation on K is discrete then $v) \Rightarrow iii)$ does not hold. Indeed, $CF(c_0,c_0) = C(c_0,c_0) \neq L(c_0,c_0)$. However ([25], 4.14) $C(c_0,c_0)$ is complemented in $L(c_0,c_0)$.

Proof: The equivalence $i) \Leftrightarrow ii)$ follows directly from the definition of a limited set. Also, $ii) \Leftrightarrow iii)$ is a direct consequence of 3.1.

$iii) \Rightarrow iv)$ Obviously T is continuous. Now, put $Z = [\bigcup_n T_n(E)] \subset F$. Then Z is of countable type, hence linearly homeomorphic to c_0 and by iii) $T : E \longrightarrow Z$ is compactifying. Since $T(E) \subset Z$, we conclude that $T : E \longrightarrow F$ is compactifying.

Obviously $iv) \Rightarrow iii)$ and $iii) \Rightarrow v)$.

$v) \Rightarrow ii)$ If ii) does not hold, there exists an equicontinuous sequence $(f_n)_n$ in E' with $lim_n f_n = 0$ in $\sigma(E',E)$, a bounded subset A of E, $\lambda, \mu \in K - \{0\}$, and for all n an $x_n \in A$ such that $|\lambda| < |f_n(x_n)| < |\mu|$ for all n.

Assume that there exists a continuous linear projection $Q : L_\beta(E,c_0) \longrightarrow L_\beta(E,c_0)$ whose range is $CF(E,c_0)$.

We now construct a continuous linear surjection $P : l^\infty \longrightarrow c_0$. This will give us a contradiction by [25], 5.19. The map P will have the form $P = L \circ H \circ Q \circ J$, where $J : l^\infty \longrightarrow L_\beta(E,c_0)$, $H : CF_\beta(E,c_0) \longrightarrow J(c_0)$ and $L : J(c_0) \longrightarrow c_0$, are defined as follows.

Definition of J: For $\alpha = (\alpha_n)_n \in l^\infty$ we put $J(\alpha) = T_\alpha$ where $T_\alpha(x) = (\alpha_n f_n(x))_n$ $(x \in E)$. Then J is linear and by 3.1.i) it is well defined. Also, for every bounded subset D of E and every $\alpha \in l^\infty$ we have that

$$sup_{x\in D} sup_n |\alpha_n| . |f_n(x)| \leq M. \|\alpha\|_\infty,$$

where $M = sup_{x\in D} sup_n |f_n(x)| < \infty$. Hence J is continuous.

For latter use we also prove that $J : l^\infty \longrightarrow J(l^\infty)$ is an homeomorphism. To do that observe that $B = \{x_1, x_2, \ldots\}$ is a bounded subset of E such that $|\lambda| . \|\alpha\|_\infty \leq p_B(J(\alpha))$ for all $\alpha \in l^\infty$ (where p_B is the seminorm on $L(E,c_0)$ associated with B).

Also, note that for $\alpha \in c_0$ we have that $(\alpha_n f_n)_n$ converges locally to zero and hence (3.1.ii)) $T_\alpha \in C(E,c_0) \subset CF(E,c_0)$.

Definition of H: Let $T \in CF(E,c_0)$. Then, there exists an equicontinuous sequence $(g_n)_n$ in E' with $lim_n g_n = 0$ in $\beta(E',E)$ such that $T(x) = (g_n(x))_n$ for all $x \in E$ (see 3.1.iii)). Then $(g_n(x_n))_n \in c_0$ and we define $H(T) = J((g_n(x_n))_n)$. Clearly H is well defined and linear. Also, for every bounded subset D of E and every $T \in CF(E, c_0)$ we have that $p_D(H(T)) \leq s.p_B(T)$ with $B = \{x_1, x_2, \ldots\}$ and $s = sup_{x \in D} sup_n \mid f_n(x) \mid < \infty$, which implies that H is continuous.

Definition of L: For L we take the inverse of the linear homeomorphism $J \mid c_0 : c_0 \longrightarrow J(c_0)$.

It is now left to prove that the map $P = L \circ H \circ Q \circ J$ is surjective. Take $\alpha = (\alpha_n)_n \in c_0$ and put $\beta = (\alpha_n / f_n(x_n))_n$. Then $\beta \in c_0$ and from the above it follows that $P(\beta) = \alpha$.

Remark 3.3: The equivalence *iii)* $\Leftrightarrow$ *v)* of 3.2 constitutes an extension to locally convex spaces of the result previously proved by T. Kiyosawa in [15], Theorem 14.

Examples 3.4:

i) Every limited space is a BL-space (2.5). But there are BL-spaces that are not limited (see 2.9.i)).

ii) If E is a semi-Montel space (i.e., every bounded subset of E is compactoid), then E is a BL-space.

iii) If E is a GP-space (e.g. when the valuation on K is discrete or when E is of countable type, see [9], 2.8), then E is a BL-space iff E is a semi-Montel space.

iv) Suppose that the valuation on K is dense. Then. l^∞ is a (non semi-Montel) BL space (see 2.6.iv)).

v) Finally recall that in 2.6.v) we studied when certain spaces of continuous functions are BL-spaces.

Proposition 3.5: *(Permanence properties).*

i) If the valuation on K is discrete and E is a BL-space over K, then every subspace of E is a BL-space.

ii) A subspace of a BL-space is not in general a BL-space.

iii) If $\{E_i : i \in I\}$ is a family of BL-spaces, then the product $\prod_{i \in I} E_i$ and the locally convex direct sum $\bigoplus_{i \in I} E_i$ are BL-spaces.

iv) A quotient of a BL-space by a closed subspace is not in general a BL- space.

Proof: The proof for subspaces and products follows, by 2.2 and 3.4, like in 2.7.

Let $\{E_i : i \in I\}$ be as in iii) and let A be a bounded subset in $\bigoplus_{i \in I} E_i$. Then, A is contained in $\bigoplus_{i \in J} E_i$ for some finite set $J \subset I$ ([16], 18.5.4) and also bounded in that space. Since $\bigoplus_{i \in J} E_i$ is linearly homeomorphic to $\prod_{i \in J} E_i$, we conclude that A is limited in $\bigoplus_{i \in J} E_i$ and hence in $\bigoplus_{i \in I} E_i$.

For iv), consider the example 4.1 in [11]. The space constructed there is a Fréchet BL-space of countable type. However, it has a quotient which is linearly homeomorphic to c_0 (which is not BL, 2.7.ii)). The proof is esentially the same as the classical theory (see e.g. [16], 31.5) taking into account that the topology of a space of countable type is the topology of uniform convergence on the subsets of E' that are complete metrizable edged and compactoid with respect to the topology $\sigma(E', E)$ ([24], 2.1).

We now want to characterize the limited spaces among the BL-spaces. We therefore need.

Definition 3.6: (compare 2.9.iii)) E is called *sequentially-quasinormable* (s.q. normable in short) if for every zero-neighbourhood U in E there exists a zero-neighbourhood V in E with $V \subset U$ such that on U^o the topology $\beta(E', E)$ and the topology induced by the norm on E'_{V^o} have the same convergent sequences.

Examples 3.7:

i) Every quasinormable space is s.q.normable. In particular, every normed space is s.q.normable.

ii) If $E', \beta(E', E)$ is metrizable then E is s.q.normable iff E is quasinormable. Note that $E', \beta(E', E)$ is metrizable if there is in E a fundamental sequence of bounded sets (e.g. when E is the strong dual of a Fréchet space).

iii) In general "s.q.normable" does not imply "quasinormable". Indeed, take $E = l^\infty$ endowed with the topology considered in 2.9.iii). Then E is limited (and hence sq-normable, see 3.11), but not quasinormable.

However, we have.

Theorem 3.8: *Let E be a Fréchet space. Suppose E is strongly polar (i.e., for every continuous seminorm p on E, $p = sup\{\mid f \mid : f \in E', \mid f \mid \leq p\}$; e.g. if E is of countable type or K is spherically complete, see [23]). Then, the following are equivalent.*

i) E is s.q.normable.

ii) E is quasinormable.

iii) $C(E, c_0) = CF(E, c_0)$.

Proof: Clearly $ii) \Rightarrow i)$ (see 3.7.i)).

The proof of $ii) \Leftrightarrow iii)$ is long and laborious but it is, mutatis mutandis, the same as in the complex case (see [1]).

$i) \Rightarrow iii)$ Assume E is s.q.normable. Let $(f_n)_n$ be a sequence in E' with $lim_n f_n = 0$ in $\beta(E', E)$. Then, there exists a zero-neighbourhood V in E such that $(f_n)_n$ converges to zero in $E'_{V^o}, \| \cdot \|_{V^o}$. It follows from 3.1 that $CF(E, c_0) = C(E, c_0)$.

In 2.9.i) we refer to an example of a non-nuclear semi-Montel Fréchet space. Now, as a consequence of 3.8 we can give the following description of the nuclearity of a Fréchet semi-Montel space.

Corollary 3.9: *For a Fréchet space E the following are equivalent.*

i) E is nuclear.

ii) E is semi-Montel and $C(E, c_0) = CF(E, c_0)$.

iii) E is semi-Montel and every $\sigma(E', E)$-null sequence in E' is locally convergent to zero.

Proof: *i)* $\Leftrightarrow$ *ii)* Every Fréchet semi-Montel space is of countable type ([11], 3.1). Then the conclusion follows from 3.8 and the characterization of nuclear spaces given in 2.9.iii) (Recall that every space of countable type is polar [23], 4.4).

The equivalence *ii)* $\Leftrightarrow$ *iii)* is a direct consequence of 3.1.

Remark 3.10: Applying 3.8 we deduce that the space E considered in 2.9.i) is not s.q.normable.

Theorem 3.11: *The following are equivalent.*

i) E is limited.

ii) E is s.q.normable and BL.

Proof: *i)* $\Rightarrow$ *ii)* We only have to prove that E is s.q.normable (see 2.5).

Let U be a zero-neighbourhood in E. We can assume that $U = \{x \in E : p(x) \leq 1\}$ for some continuous seminorm p on E. Take a continuous seminorm q on E with $q \geq p$ and such that $\varphi_{pq} : E_q \longrightarrow E_p$ is limited. Clearly $V = \{x \in E : q(x) \leq 1\}$ is a zero-neighbourhood in E with $V \subset U$.

Now, let $(f_n)_n$ be a sequence in U^0 converging to zero in $\beta(E', E)$ (and hence, $lim_n f_n(x) = 0$ for all $x \in E$). Then, (see Section 1.v)) $lim_n f_n = 0$ in $\sigma(E'_{U^o}, E_p)$. Applying 2.2.viii) and Section 1.vi) we deduce that $(f_n)_n$ converges to zero in E'_{V^o}. Therefore, E is s.q.normable.

ii) $\Rightarrow$ *i)* Let p be a continuous seminorm on E and put $U = \{x \in E : p(x) \leq 1\}$. Then take a zero-neighbourhood V in E with $V \subset U$ as in the definition of s.q.normable space. Then , there exists a continuous seminorm q on E with $q \geq p$ such that $W = \{x \in E : q(x) \leq 1\} \subset V$ and we shall prove that $\varphi_{pq} : E_q \longrightarrow E_p$ is limited, or that (see Section 1.vi) and 2.2.viii)) the canonical injection $\varphi_{UW} : E'_{U^o} \longrightarrow E'_{W^o}$ transforms equicontinuous $\sigma(E'_{U^o}, E_p)$-null sequences in E'_{U^o} into null-sequences in $E'_{W^o}, \| \cdot \|_{W^o}$.

Let $(f_n)_n$ be an equicontinuous (and hence norm-bounded) sequence in E'_{U^o} with $lim_n f_n = 0$ in $\sigma(E'_{U^o}, E_p)$ (and hence $lim_n f_n = 0$ in $\sigma(E', E)$) and choose $\alpha \in K - \{0\}$ such that $f_n \in \alpha U^o$ for all n. Since E is a BL-space we then have (see 3.2.*i*) $\Leftrightarrow$ *ii*)) that $lim_n f_n = 0$ in $\beta(E', E)$. By the choice of V we obtain that $lim_n f_n = 0$ in E'_{V^o}. But $V^o \subset W^o$ and so the canonical injection $E'_{V^o} \longrightarrow E'_{W^o}$ is continuous. Thus, $lim_n f_n = 0$ in E'_{W^o} and we are done.

As a direct consequence of 3.8 and 3.11, we derive.

Corollary 3.12: (compare 2.9.iii)) *If E is a strongly polar Fréchet space then: E is limited* $\Rightarrow$ *E is quasinormable.*

4. COMPARISON WITH THE COMPLEX CASE

B. Josefson [13] and N. Nissenzweig [19] proved that if E is a Banach space over the real or complex field such that every $\sigma(E', E)$-null sequence in E' converges to zero in $\beta(E', E)$

(or equivalently, every bounded subset of E is limited), then E is finite-dimensional. H. Jarchow ([12], p. 247) asked if this result can be extended to Fréchet spaces. This question has been solved in a positive way. Some of the more important generalizations of the Josefson-Nissenzweig theorem appearing in the archimedean literature are collected in the following result.

Theorem 4.1: *Let E be a locally convex space over the real or complex field.*

a) (see [17]) *The following are equivalent.*

i) E is Schwartz.

ii) E is limited.

iii) E is quasinormable and BL.

b) (see [2]) *Suppose that E is a Fréchet space. Then,*

i) E is semi-Montel iff E is BL.

ii) E is Schwartz iff every $\sigma(E', E)$-null sequence in E' is locally convergent to zero.

The non-archimedean counterpart of this theorem is not true in general when the valuation on K is dense. Indeed, l^∞ is a quasinormable and limited space (2.6.iv)) that is not nuclear. Also, the space E considered in 3.7.iii) provides an example of a non-quasinormable limited space.

However, when the valuation on K is discrete the situation is completely different. The non-archimedean version of 4.1 holds in this case, and we have.

Theorem 4.2: *Let E be a locally convex space over a discretely valued field K.*

a) The following are equivalent.

i) E is nuclear.

ii) E is limited.

iii) E is quasinormable and BL.

b) Suppose that E is a Fréchet space. Then,

i) E is semi-Montel iff E is BL (i.e., every $\sigma(E', E)$-null sequence in E' converges to zero in $\beta(E', E)$).

ii) E is nuclear iff every $\sigma(E', E)$-null sequence in E' is locally convergent to zero.

Proof: a) The equivalence $i) \Leftrightarrow ii)$ was proved in 2.6.ii). For the proof of $i) \Leftrightarrow iii)$ apply the characterization of nuclearity given in 2.9.iii) and the fact that every locally convex space over K is GP ([9], 2.8.iii)) and polar ([23], Remark preceding 4.1).

b) Property i) follows directly from 3.4.iii).

To prove ii), suppose that every $\sigma(E', E)$-null sequence in E' is locally convergent to zero. Then, $L(E, c_0) = C(E, c_0)$ (see 3.1) and by 3.2, E is a BL-space. Since E is GP we derive that it is a semi-Montel space and so E is of countable type ([11], 3.1). Now the conclusion follows from [23], 1.3.

Remark 4.3:

i) The crucial fact to prove 4.1.a) is that

"The product of three limited operators between Banach spaces over the real or complex field, is a compact operator" ([17], Theorem 1).

In the non-archimedean case this result remains true when the valuation on K is discrete (In this case $Lim(E,F) = C(E,F)$ for all Banach spaces E, F over K [9], 2.8.iii)). However, for densely valued fields the result is false in general (e.g., by 2.6.iv) the identity map Id on l^∞ is a limited operator such that Id^3 is clearly a non compact operator).

ii) Also, the crucial fact to prove 4.1.b) is that

"Every Fréchet BL-space over the real or complex field is reflexive"

We have again that this result remains true in the non-archimedean case when the ground field K is discretely valued (Every Fréchet BL-space over K is semi-Montel and hence reflexive [23], 10.3). However, for densely valued fields the result is false in general (e.g., l^∞ is a BL-space which is not reflexive if K is spherically complete [25], 4.16).

iii) It follows from 4.2.b).i) that the condition "E is of countable type" in 3.3.iv) of [11] can be eliminated when the valuation on K is discrete.

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N. De Grande-De Kimpe
Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2 (10F7)
B1050 BRUSSEL
BELGIUM

C. Perez-Garcia
Departamento de Matematicas
Facultad de Ciencias
Universidad de Cantabria
Avda. de los Castros s/n
39071 SANTANDER
SPAIN