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Exponential-Bessel partial differential equation and Fox’s $H$-function


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EXponential-Bessel Partial Differential Equation and Fox's H-Function

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ABSTRACT: In this paper, we present and solve a two dimensional Exponential-Bessel partial differential equation, and obtain a particular solution of it involving Fox's \( H \)-function.

1. INTRODUCTION. The object of this paper is to formulate a two dimensional Exponential-Bessel partial differential equation and obtain its double series solution. We further present a particular solution of our Exponential-Bessel equation involving Fox's \( H \)-function. It is interesting to note that the particular solution also yields a new two dimensional series expansion for Fox's \( H \)-function involving exponential functions and Bessel functions.

The \( H \)-function introduced by Fox [5, p. 408], will be represented as follows:

\[
\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \equiv A, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv B.
\]

The following formulae are required in the proof:
The integral [2, p. 704, (2.2)] :

\[ (1.2) \quad \int_0^\Pi \cos 2ux (\sin \frac{x}{2})^{-2w_1} H_{p,q}^{m,n} \left[ z (\sin \frac{x}{2})^{-2h} \left( \frac{a_p e_p}{b_q f_q} \right) \right] dx \]

\[ = \sqrt{(\Pi)} H_{p+2,q+2}^{m+1,n+1} \left[ z \left( \frac{1-w_1-2u,h}{1/2-w_1,h} \right) \left( a_p e_p, (b_q f_q) \right) \right], \]

where \( h > 0, \sum_{j=1}^p e_j - \sum_{j=1}^q f_j \equiv A \leq 0, \sum_{j=n+1}^p e_j + \sum_{j=1}^m e_j - \sum_{j=m+1}^q f_j \equiv B > 0, \]

\[ |\arg z| < 1/2\Pi, Re(1 - 2w_1) - 2h \max_{1 \leq j \leq n} [Re(a_j - 1)/e_j] > 0. \]

The integral [7, p. 94, (2.2)] :

\[ (1.3) \quad \int_0^\infty y^{w_2-1} \sin y J_v(y) H_{p,q}^{m,n} \left[ z y^{2k} \left( \frac{a_p e_p}{b_q f_q} \right) \right] dy \]

\[ = 2^{w_2-1} \sqrt{(\Pi)} H_{p+4,q+1}^{m+1,n+1} \left[ z \left( 1 - \frac{1}{2} - w_2, 2k \right), \left( \frac{1}{2} - w_2, 2k \right), \left( 1 + v - w_2, k \right), \left( 1 + v - w_2, k \right), \left( 1 + v - w_2, k \right) \right] \]

where \( k > 0, A \leq 0, B > 0, |\arg z| < 1/2\Pi, Re(w_2 + v) + 2k \min_{1 \leq j \leq m} [Re(b_j)/f_j] > 0. \]

The orthogonality property of the Bessel functions [6, p. 291, (6)] :

\[ (1.4) \quad \int_0^\infty x^{-1} J_{a+2n+1}(x) J_{a+2m+1}(x) dx \]

\[ = \begin{cases} 
0, & \text{if } m \neq n; \\
(4n + 2a + 2)^{-1}, & \text{if } m = n, \ Re a + m + n > -1.
\end{cases} \]

The following orthogonality property :

\[ (1.5) \quad \int_0^{\Pi} e^{2imx} \cos 2nx dx = \begin{cases} 
0, & m \neq n; \\
\Pi/2, & m = n \neq 0; \\
\Pi, & m = n = 0.
\end{cases} \]
2. TWO DIMENSIONAL EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION

Let us consider

\[
\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + y^2u,
\]

where \( u \equiv u(x, y, t) \) and \( u(x, y, 0) = f(x, y) \).

To solve (2.1), we assume that (2.1) has a solution of the form:

\[
(2.2) \quad u(x, y, t) = e^{4ct + (v + 2s + 1)^2 t}X(ix)Y(y).
\]

The substitution of (2.2) into (2.1) yields:

\[
(2.3) \quad -c \left[ X'' + 4r^2 X \right] + X \left[ y^2 Y'' + y Y' + \left\{ y^2 - (v + 2s + 1)^2 \right\} Y \right] = 0.
\]

We see that \( X'' + 4r^2 X = 0 \) has a solution \( X = e^{2r^2 x} \) and \( y^2 Y'' + y Y' + \left\{ y^2 - (v + 2s + 1)^2 \right\} Y = 0 \) is Bessel equation \([1, p. 200, (6.25)]\), with solution \( Y = J_{v+2s+1}(y) \). Therefore the solution of (2.1) is of the form:

\[
(2.4) \quad u(x, y, t) = e^{4ct + (v + 2s + 1)^2 t} e^{2riz} J_{v+2s+1}(y).
\]

In view of the principle of superposition, the general solution of (2.1) is given by

\[
(2.5) \quad u(x, y, t) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{4ct + (v + 2s + 1)^2 t + 2riz} J_{v+2s+1}(y).
\]

In (2.5), putting \( t = 0 \), we get

\[
(2.6) \quad f(x, y) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2riz} J_{v+2s+1}(y).
\]

Multiplying both sides of (2.6) by \( y^{-1} \cos 2ux J_{v+2s+1}(y) \), integrating with respect to \( y \) from 0 to \( \infty \) and with respect to \( x \) from 0 to \( \Pi \), then using (1.4) and (1.5), the Fourier Exponential-Bessel coefficients are given by

\[
(2.7) \quad A_{r,s} = \frac{4}{\Pi} (v + 2s + 1) \times \int_{0}^{\Pi} \int_{0}^{\infty} f(x, y) y^{-1} \cos 2ux J_{v+2s+1}(y) dy dx.
\]
In view of the theory of double and multiple Fourier series given by Carslaw and Jaeger [3, pp. 180-183], and many other references, such as Erdélyi [4, pp. 64-65] etc..., the double series (2.6) is convergent, provided the function \( f(x, y) \) is defined in the region \( 0 < x < \pi, 0 < y < \infty \). In brief, the double series (2.6) converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take \( f(x, y) \) as Fox's \( H \)-function and present another method to obtain Fourier exponential-Bessel coefficients \( A_{r,s} \).

3. PARTICULAR SOLUTION INVOLVING FOX'S \( H \)-FUNCTION

The particular solution to be obtained is

\[
(3.1) \quad u(x, y, t) = 2^{W_2+1} \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} e^{4cr^2t+(v+2s+1)^2t+2r\text{i}x(v+2+s+1)}j_{v+2s+1}(y)
\]

\[
\times H_{p+6,q+3}^{m+2,n+2}
\]

\[
2^{2k}\begin{bmatrix}
(1 - w_1 - 2r, h), \left(\frac{-w_1+v+2s}{2}, k\right), (a_p, c_p), \\
(1 - w_1 + 2r, h), \left(1 + \frac{v+2s+1-w_1}{2}, k\right), \\
(1 + \frac{v+2s+1+w_2}{2}, k), \left(1 + \frac{v+2s-w_2}{2}, k\right), \\
(\frac{1}{2} - w, h), \left(\frac{1}{2} - w_2, k\right), (b_q, f_q), (1 - w_1, h)
\end{bmatrix}
\]

valid under the conditions of (1.2), (1.3) and (1;4).

**Proof.** Let

\[
(3.2) \quad f(x, y) = \left(\sin \frac{x}{2}\right)^{-2w_1} y^w \sin y H_{p,q}^{m,n} \left[ z \left(\sin \frac{z}{2}\right)^{-2h} y^{2k} \right]
\]

\[
= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2irx} j_{v+2s+1}(y).
\]

Equation (3.2) is valid, since \( f(x, y) \) is defined in the region \( 0 < x < \Pi, 0 < y < \infty \). Multiplying both sides of (3.2) by \( y^{-1} J_{v+2w+1}(y) \) and integrating with respect to \( y \) from 0 to \( \infty \), then using (1.3) and (1;4). Now multiplying both sides of the resulting
expression by \( \cos 2ux \) and integrating with respect to \( x \) from 0 to \( \pi \), then using (1.2) and (1.5), we obtain the value of \( A_{r,s} \). Substituting this value of \( A_{r,s} \) in (2.5), the expansion (3.1) is obtained.

**NOTE 1**: The value of \( A_{0,s} \) is one-half the value of \( A_{r,s} \).

**NOTE 2**: If we put \( t = 0 \) in (3.1), it reduces to a new two dimensional series expansion for Fox's \( H \)-function involving exponential functions and Bessel functions.

Since on specializing the parameters Fox's \( H \)-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result (3.1) presented in this paper is of a general character and hence may encompass several cases of interest.
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