ANNALES MATHÉMATIQUES

BLAISE PASCAL

Gaël Meigniez

Matching Cells

Volume 30, nº 1 (2023), p. 97-106.

https://doi.org/10.5802/ambp.417

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

ANNALES MATHÉMATIQUES BLAISE PASCAL **30**, 97-106 (2023)

Matching Cells

GAËL MEIGNIEZ

Abstract

A (total) matching of the cells of a triangulated manifold can be thought as a combinatorial or discrete version of a nonsingular vector field. This note gives several methods for constructing such matchings.

Cellules couplées

Résumé

Un couplage (total) entre les cellules d'une variété triangulée peut être pensé comme une version combinatoire, discrète, d'un champ de vecteurs non singulier. Cette note décrit plusieurs méthodes pour construire de tels couplages.

1. Introduction

On a polyhedral complex, a "partial matching" is a family of disjoint pairs of cells such that in each pair, one of the two cells is a hyperface (a face of codimension 1) of the other. Following Forman [5, 6, 7, 8], such objects are regarded as a combinatorial equivalent to vector fields. In the literature, most attention has been given to "discrete Morse theory", which concerns partial matchings whith strongly constrained dynamics, and their relations to the homology of the ambiant complex. The present note is about *total* matchings involving all the cells, or whose unmatched cells constitute a prescribed subcomplex; regardless of their dynamics; we are interested on the *existence* of such objects. We provide some construction methods, mainly on triangulated manifolds, either allowing oneself to subdivide the triangulation, or not. The author feels that the methods are more important than the existence results themselves. A first approach is combinatorial and linear-algebraic, making Hall's "marriage theorem" play with cellular homology; two other ones are geometric: a matching is deduced from an ambient nonsingular vector field transverse to the cells, or from a round handle decomposition of the manifold.

Here are two results. All manifolds and triangulations are understood smooth (C^∞).

Theorem 1.1 (Rational homology sphere)**.** *Let M be a rational homology sphere of odd dimension.*

Keywords: Matching, Triangulation, Discrete vector field.
2020 *Mathematics Subject Classification*: 05C70, 05E45, 37C10, 37F20, 57Q15.

Then, every triangulation, and more generally every polyhedral cellulation of M is (totally) matchable.

Theorem 1.2 (Matchable subdivision)**.** *Let M be a compact connected manifold with smooth boundary, and let $\partial_0 M$ be a union of connected components of ∂M such that $\chi(M, \partial_0 M) = 0$.*

Then, every triangulation of M admits a subdivision matchable rel. $\partial_0 M$.

See below for the meaning of relative matchings. Either of the two sets $\partial_0 M$ and $\partial M \setminus \partial_0 M$ may be empty, or both. In particular:

Corollary 1.3. *Every closed connected manifold whose Euler characteristic vanishes admits a (totally) matchable triangulation.*

For manifolds of dimension 3, this is due to E. Gallais [9].

Question 1.4. *Let n be an odd integer. Is every triangulation of every closed n-manifold (totally) matchable?*

The question is open even for $n = 3$.

2. Matching cells, and obstructions to do so

The rest of this note progressively investigates some methods to construct matchings, and also some obstructions; starting with immediate, elementary remarks; and finishing with the proofs of Theorems 1 and 2.

Notation 2.1. One denotes by $|A|$ the cardinality of the set A, by $\mathbf{D}^n \subset \mathbf{R}^n$ the compact unit disk, and by $\mathbf{S}^{n-1} := \partial \mathbf{D}^n$ the $(n-1)$-sphere.

In a first phase, manifolds are not mandatory, nor simplices. Consider generally a polyhedral cellular complex X (the cells are convex polyhedra, finiteness is understood everywhere) and a subcomplex $Y \subset X$. Call two cells *incident* to each other if one is a hyperface of the other. Write $\Sigma(X,Y)$ (resp. $\Sigma^n(X,Y)$) (resp. $\Sigma_0(X,Y)$) (resp. $\Sigma_1(X,Y)$) for the set of the cells (resp. the n-dimensional cells) (resp. the even-dimensional cells) (resp. the odd-dimensional cells) of X not lying in Y.

Definition 2.2. A *matching* on X *relative to Y,* or a matching *on the pair* (X,Y)*,* is a partition of $\Sigma(X,Y)$ into incident pairs.

For $Y = \emptyset$, we write $\Sigma(X)$ (resp. $\Sigma^n(X)$) (resp. $\Sigma_0(X)$) (resp. $\Sigma_1(X)$) instead of $\Sigma(X,\emptyset)$ (resp. $\Sigma^n(X,\emptyset)$) (resp. $\Sigma_0(X,\emptyset)$) (resp. $\Sigma_1(X,\emptyset)$); and we speak of "a matching on X".

The cases of the complexes of dimension 1 and of the triangulations of surfaces will easily follow from a few general remarks.

Remark 2.3 (Euler characteristic). If (X, Y) is matchable, then the relative Euler characteristic $\chi(X, Y) = \chi(X) - \chi(Y)$ vanishes.

Indeed, $\chi(X, Y) = |\Sigma_0(X, Y)| - |\Sigma_1(X, Y)|$.

Remark 2.4 (Collapse). Every collapse of a polyhedral complex X onto a subcomplex Y gives a matching on X rel. Y.

Indeed, a collapse is nothing but a filtration of X by subcomplexes (X_n), where $0 \leq n \leq N$, such that $X_0 = Y$ and $X_N = X$; and that $\Sigma(X_n, X_{n-1})$ consists, for each $1 \leq n \leq N$, of exactly two incident cells.

(More precisely, an *orbit* in a matching is defined as a finite sequence

$$\sigma_0, \sigma_1, \cdots \in \Sigma(X, Y)$$

such that for every odd k, the cells σ_{k-1} and σ_{k+1} are two distinct hyperfaces of σ_k, and σ_{k-1} is the mate of σ_k. A collapse of X onto Y amounts to a matching of X rel. Y without *cyclic* orbit.)

Remark 2.5 (Top-dimensional cycle). Every cellulation of the circle admits exactly two matchings.

More generally, let X be a polyhedral cellulation of a manifold; let ℓ be a simple loop in the 1-skeleton of the dual cellulation; let $Y \subset X$ be the union of the cells of X disjoint from ℓ. Then, X admits exactly two matchings rel. Y.

This is obvious.

Example 2.6 (Graphs). *Every connected graph whose Euler characteristic vanishes is matchable.*

Indeed, such a graph collapses onto a circle.

Example 2.7 (Surfaces). *Let M be a compact, connected* 2*-manifold such that $\chi(M) = 0$. Then, every polyhedral cellulation X of M is matchable absolutely, and relatively to ∂M.*

Proof. First case: M is the annulus or the Möbius strip. Then, the 1-skeleton of the cellulation dual to X contains an essential simple loop ℓ such that M cut along ℓ is an annulus or two annuli. So, the union $Y \subset X$ of the cells of X disjoint from ℓ collapses onto ∂M. The pair (X, Y) is matchable (Remark 2.5), the pair $(Y, \partial M)$ is matchable (Remark 2.4), and $X|\partial M$ is matchable (Remark 2.5).

Second case: M is the 2-torus or the Klein bottle. Then, the 1-skeleton of the cellulation dual to X contains an essential simple loop ℓ such that M cut along ℓ is an annulus.

Consider the union $Y \subset X$ of the cells of X disjoint from ℓ. The pair (X, Y) is matchable (Remark 2.5) and the annulus Y is matchable (first case). $\square$

Next, recall Hall's so-called "marriage theorem". Let $\Sigma := \Sigma_0 \sqcup \Sigma_1$ be a finite, $\mathbf{Z}/2\mathbf{Z}$-graded set and let $I \subset \Sigma \times \Sigma$ be a symmetric relation in Σ, of degree 1. For every subset $A \subset \Sigma$, consider the subset $I(A) \subset \Sigma$ of the elements I-related to at least one element of A. A *matching* on Σ with respect to I is a partition of Σ into I-related pairs.

Theorem 2.8 (Hall [10])**.** *The following properties are equivalent:*

(1) *The relation I is matchable;*

(2) *One has $|A| \le |I(A)|$ for every $A \subset \Sigma$;*

(3) $|\Sigma_1| = |\Sigma_0|$ *and one has $|A| \le |I(A)|$ for every $A \subset \Sigma_0$.*

Also recall that the Ford–Fulkerson algorithm [4, 3] computes a matching, if any, in time $O(|\Sigma|^2|I|)$, thus giving some (moderate) effectiveness to our existence results.

Coming back to polyhedral complexes, some examples of unmatchable complexes will follow from the *trivial* sense of Hall's criterion.

Example 2.9. A connected simplicial 2-complex whose Euler characteristic vanishes, unmatchable as well as every subdivision.

Let
$$X := S^2 \vee S^1 \vee S^1$$
be the bouquet, at some common vertex v, of a triangulated 2-sphere S^2 with two triangulated circles. Then, $\chi(X) = 0$, but X does not admit any matching. Indeed, for $A := \Sigma_0(S^2, v)$, one has $I(A) = \Sigma_1(S^2)$, hence $|I(A)| = |A| - 1$. The same holds for any subdivision of X.

Example 2.10. Some unmatchable triangulated closed connected orientable manifolds, whose Euler characteristic vanishes.

Let $n = 2k$ be even and at least 4. Start with a closed orientable n-manifold M whose Euler characteristic $\chi(M)$ is even, divided into two parts M_1, M_2 by a smooth hypersurface M_0. Fix a triangulation X_0 of M_0.

Recall that by Poincaré duality, the Euler characteristic of every closed odd-dimensional manifold vanishes. Moreover,
$$\chi(M) + \chi(M_0) = \chi(M_1) + \chi(M_2)$$
So, $\chi(M_1)$ and $\chi(M_2)$ share the same parity. Also recall that, n being even, the Euler characteristic of a connected sum of two n-manifolds V, V' is
$$\chi(V \sharp V') = \chi(V) + \chi(V') - 2$$

Since $\chi((\mathbf{S}^1)^n) = 0$ and $\chi((\mathbf{S}^2)^k) = 2^k$, after modifying M by some appropriate number of connected sums with $(\mathbf{S}^1)^n$ and/or with $(\mathbf{S}^2)^k$ performed on both sides of M_0, one can give arbitrary values to $\chi(M_1)$ and $\chi(M_2)$ in the same parity class as before, without changing M_0. In particular, one can arrange that

$$\chi(M_2) = -\chi(M_1) > |\Sigma_0(X_0)| \tag{2.1}$$

Finally, following Armstrong [1], extend X_0 to some triangulation X_1 of M_1 and to some triangulation X_2 of M_2, thus obtaining a global triangulation X of M.

Clearly, $\chi(M) = 0$. We claim that X does not admit any matching.

Indeed, for $A := \Sigma_0(X_2, X_0)$ one has obviously

$$|A| = |\Sigma_0(X_2)| - |\Sigma_0(X_0)|$$

$$I(A) = \Sigma_1(X_2)$$

Now, recall that

$$\chi(M_2) = |\Sigma_0(X_2)| - |\Sigma_1(X_2)|$$

Together with the above inequation (2.1), it follows that $|I(A)| < |A|$: the triangulation X is unmatchable.

Note that, after Theorem 1.2, X admits a matchable subdivision.

Lemma 2.11 (Acyclic pair)**.** *If $H_*(X,Y) = 0$, then the pair (X,Y) is matchable.*

Rational coefficients are understood everywhere; one could as well use $\mathbf{Z}/2\mathbf{Z}$, or any field.

Proof. This is an application of Hall's criterion in the realm of elementary algebraic topology. For $n \geq 0$, consider as usual the chain vector space $C_n(X,Y)$ of basis $\Sigma^n(X,Y)$; the differential

$$\partial_n : C_n(X,Y) \to C_{n-1}(X,Y)$$

and its kernel $Z_n(X,Y)$.

The following filtration of the pair (X,Y) by subcomplexes $X_n \subset X$ is classical. Put $X_0 := Y$. For $n \geq 1$, let X_n be the union of Y with the $(n-1)$-skeleton of X and with some n-cells which span a linear subspace complementary to $Z_n(X,Y)$ in $C_n(X,Y)$. Since $H_*(X,Y) = 0$, it is straightforwardly verified that $H_*(X_n,Y) = 0$ for every $n \geq 0$. Then, the long exact sequence for the relative homologies of the triad (X_n, X_{n-1}, Y) yields $H_*(X_n, X_{n-1}) = 0$ for every $n \geq 1$.

One is thus reduced to prove Lemma 2.11 in the case where moreover, the cells lying in X but not in Y are of only two dimensions:

$$\Sigma(X,Y) = \Sigma^n(X,Y) \sqcup \Sigma^{n-1}(X,Y)$$

for some $n \geq 1$. The pair being acyclic, necessarily

$$|\Sigma^n(X,Y)| = |\Sigma^{n-1}(X,Y)|$$

For every $A \subset \Sigma(X,Y)$, let

$$\langle A \rangle \subset C_*(X,Y)$$

denote the spanned linear subspace. If moreover $A \subset \Sigma^n(X,Y)$, recall the set

$$I(A) \subset \Sigma^{n-1}(X,Y)$$

of the cells incident to at least one cell belonging to A; hence

$$\partial_n \langle A \rangle \subset \langle I(A) \rangle$$

Since ∂_n is linear and one-to-one:

$$|A| = \dim(\langle A \rangle) = \dim(\partial_n \langle A \rangle) \leq |I(A)|$$

After the equivalence of (1) with (3) in the marriage theorem, the pair (X_n, X_{n-1}) is matchable. □

Corollary 2.12 (Subdivision)**.** *Let (X,Y) be a pair of polyhedral complexes. Assume that (X,Y) is matchable.*

Then, every polyhedral subdivision (X',Y') of (X,Y) is also matchable.

Proof. Consider a matching on (X,Y). For each matched pair $\sigma, \tau \in \Sigma(X,Y)$ with $\tau \subset \sigma$, consider the union

$$\widehat{\partial}\sigma := \partial\sigma \setminus \mathrm{Int}(\tau)$$

of the other hyperfaces of σ. The restriction

$$(X'|\sigma, X'|\widehat{\partial}\sigma)$$

is a pair of polyhedral complexes which does of course not always collapse, but which always admits a matching, by Lemma 2.11. Clearly, the collection of all these partial matchings constitutes a global matching for the pair of complexes (X',Y'). □

Proof of Theorem 1.1. Let X be a polyhedral cellulation of a rational homology sphere M of odd dimension n. One can assume that $n \geq 3$. Fix a $(n-1)$-cell σ of X and a hyperface $\tau \subset \sigma$. Consider in X the union Y of τ with the cells of X not containing τ. First, the pair (X,Y) is matchable (Remark 2.5). Second, $H_*(Y, \partial\sigma) = 0$, hence the pair $(Y, \partial\sigma)$ is matchable (Lemma 2.11). Third, the polyhedral complex $\partial\sigma$, being homeomorphic to the $(n-2)$-sphere, is matchable by induction on n. □

Corollary 2.13 (Betti number 1)**.** *Let M be a closed connected 3-manifold whose first Betti number is 1.*

Then, every polyhedral cellulation X of M is matchable.

Proof. The 1-skeleton of X (resp. of the dual cellulation) contains a simple loop ℓ (resp. ℓ^*) generating $H_1(M)$. Consider the union $Y \subset X$ of the cells of X disjoint from ℓ^*. The pair (X, Y) and the circle ℓ are both matchable (Remark 2.5). Also, $H_*(Y, \ell) = 0$, hence the pair (Y, ℓ) is matchable (Lemma 2.11). □

Now, consider a triangulation X of a compact manifold M of dimension $n \geq 1$ with smooth boundary ∂M (maybe empty). If a nonsingular vector field ∇ on M is transverse to every $(n-1)$-simplex of X, we say for short that ∇ is *transverse to* X. Note that in particular, ∇ is then transverse to ∂M; thus, ∂M splits as the disjoint union of $\partial_s(M, \nabla)$, where ∇ enters M, with $\partial_u(M, \nabla)$, where ∇ exits M.

Theorem 2.14 (Transverse nonsingular vector field)**.** *If the nonsingular vector field* ∇ *is transverse to the triangulation* X*, then* X *is matchable rel.* $\partial_u(M, \nabla)$.

Proof. Because of the transversality, for every simplex $\sigma \in \Sigma(X)$ of dimension less than n and not contained in $\partial_u(M, \nabla)$ (resp. $\partial_s(M, \nabla)$), there is a unique *downstream* (resp. *upstream*) n-simplex $d(\sigma)$ (resp. $u(\sigma)) \in \Sigma^n(X)$ containing σ and such that the vector field ∇ enters $d(\sigma)$ (resp. exits $u(\sigma)$) at every point of $Int(\sigma) := \sigma \setminus \partial\sigma$. For $\sigma \in \Sigma^n(X)$, we agree that $d(\sigma) := \sigma$ and $u(\sigma) := \sigma$.

Consider any n-simplex $\delta \in \Sigma^n(X)$ and any face $\sigma \subset \delta$ (the case $\sigma = \delta$ is included.) We call σ *stable* (resp. *unstable*) with respect to δ if σ does not lie in $\partial_u(M, \nabla)$ (resp. $\partial_s(M, \nabla)$) and if $d(\sigma) = \delta$ (resp. $u(\sigma) = \delta$). Note that

- Every *hyperface* of δ is either stable or unstable;
- δ has at least one stable hyperface and at least one unstable hyperface (for degree reasons);
- σ is stable if and only if every hyperface of δ containing σ is stable.

Next, for each $\delta \in \Sigma^n(X)$, pick arbitrarily a base vertex $v(\delta)$ *in the intersection* $\partial_-\delta$ *of the unstable hyperfaces of* δ (here of course, it is mandatory that δ is a simplex rather than a general convex polytope.) To this choice, there corresponds canonically a matching, as follows. For every simplex $\sigma \in \Sigma(X, \partial_u(M, \nabla))$, define its mate $\overline{\sigma}$ by:

(1) If $v(d(\sigma)) \in \sigma$ then $\overline{\sigma}$ is the hyperface of σ opposed to $v(d(\sigma))$;

(2) If $v(d(\sigma)) \notin \sigma$ then $\overline{\sigma}$ is the join of σ with $v(d(\sigma))$.

These rules do define a matching on the pair $(X, \partial_u(M, \nabla))$: the point here is that $\overline{\sigma}$ is also a stable face of $d(\sigma)$. Indeed, if not, then $\overline{\sigma}$ would be contained in some unstable hyperface η of $d(\sigma)$; but in both cases (1) and (2) above, this would imply that σ itself would be contained in η, a contradiction. In other words, $d(\overline{\sigma}) = d(\sigma)$. It is now clear that the map $\sigma \mapsto \overline{\sigma}$ induces locally, for each n-simplex δ, an involution in the set of the stable faces of δ; and thus globally a matching on $\Sigma(X, \partial_u(M, \nabla))$. □

Remark 2.15. It can be suggestive, for $n = 2$ and $n = 3$ and for each $0 \le i \le n-1$, to figure out in $\mathbf{R}^n$, endowed with the parallel vector field $\nabla := -\partial/\partial x_n$, a linear n-simplex δ in general position with respect to ∇ and such that $\dim(\partial_-\delta) = i$; to list the stable faces and the unstable faces; to choose a base vertex $v \in \partial_-\delta$; and to compute the corresponding matching between the stable faces.

Remark 2.16. We feel that the preceding natural construction is of special interest with respect to Forman's general question "Which smooth vector fields can be triangulated?" ([7, §3]).

Remark 2.17. In particular, the Hall cardinality conditions also constitute some *combinatorial* necessary conditions for a triangulation to admit a transverse nonsingular vector field. For example, in Example 2.10, not only X does not admit any transverse nonsingular vector field (which is obvious since such a field would be transverse to M_0, in contradiction with $\chi(M_1, M_0) \neq 0$), but this holds also for every triangulation of M *combinatorially* isomorphic with X; and in particular, for every jiggling of X.

Proof of Theorem 1.2. Since $\chi(M, \partial_0 M) = 0$, there is on M a nonsingular vector field ∇ transverse to ∂M, which exits M through $\partial_0 M$, and which enters M through $\partial M \setminus \partial_0 M$. Let X be any triangulation of M. Then, by W. Thurston's famous Jiggling lemma [11], one has on M a triangulation X' which is combinatorially isomorphic to some (iterated crystalline) subdivision of X, and which is transverse to ∇. By Theorem 2.14, X' is matchable. □

Another proof of Theorem 1.2 in high dimensions. Finally, we give an alternative construction for Theorem 1.2; this construction works in every dimension, but 3. Note that, by Corollary 2.12 and the Hauptvermutung for smooth triangulations, it is enough to construct *one* triangulation of M matchable relatively to $\partial_0 M$.

After Asimov [2], since $n \ge 4$ and $\chi(M, \partial_0 M) = 0$, the pair $(M, \partial_0 M)$ admits a "round handle decomposition". For each $0 \le i \le n-1$, the *round handle* of dimension n and index i is defined as

$$H^n_i := \mathbf{S}^1 \times \mathbf{D}^i \times \mathbf{D}^{n-i-1}$$

and one puts

$$\partial_0 H^n_i := \mathbf{S}^1 \times \mathbf{S}^{i-1} \times \mathbf{D}^{n-i-1}$$

(one agrees that $\mathbf{S}^{-1} = \emptyset$). By a *round handle decomposition* for M, one means a filtration of M by submanifolds dimension n, with boundaries and corners:

$$\partial_0 M \times [0,1] = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that, for each $1 \le k \le \ell$, one obtains M_k by attaching to M_{k-1} a round handle of dimension n and of some index $0 \le i \le n-1$; the attachment map is an embedding

$$\partial_0 H_i^n \hookrightarrow \partial M_{k-1} \setminus (\partial_0 M \times 0)$$

Fix such a decomposition. Then, choose a triangulation of M for which each handle is a subcomplex. One is thus reduced to the two cases

(1) $M = \partial_0 M \times [0,1]$; or

(2) $M = H_i^n$ and $\partial_0 M = \partial_0 H_i^n$, for some $0 \le i \le n-1$.

In case (1), any triangulation of M is matchable rel. $\partial_0 M$ (Lemma 2.11).

In case (2), one has a deformation retraction of $\mathbf{D}^i \times \mathbf{D}^{n-i-1}$ onto its subset

$$K(n,i) := (\mathbf{D}^i \times 0) \cup (\mathbf{S}^{i-1} \times \mathbf{D}^{n-i-1})$$

Hence, H_i^n retracts by deformation onto $\mathbf{S}^1 \times K(n,i)$. We choose a triangulation of M such that $\mathbf{S}^1 \times K(n,i)$ is a union of cells of the triangulation. Applying Lemma 2.11 to the pair $(M, \mathbf{S}^1 \times K(n,i))$, the proof is reduced to the case where $M = \mathbf{S}^1 \times \mathbf{D}^i$ and $\partial_0 M = \mathbf{S}^1 \times \mathbf{S}^{i-1}$. In that case, let X be any triangulation of M. The 1-skeleton of the dual subdivision contains a simple loop ℓ homologous to the core $\mathbf{S}^1 \times 0$. Consider the union $Y \subset X$ of the cells of X disjoint from ℓ. On the one hand, the pair (X, Y) is matchable (Remark 2.5). On the other hand, since $H_*(Y, \partial_0 M) = 0$, the pair $(Y, \partial_0 M)$ is matchable (Lemma 2.11). □

References

[1] Mark Armstrong. Extending Triangulations. *Proc. Am. Math. Soc.*, 18:701–704, 1967.

[2] Daniel Asimov. Round Handles and Non-Singular Morse-Smale Flows. *Ann. Math.*, 102(1):41–54, 1975.

[3] Jack Edmonds and Richard M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.*, 19(2):248–264, 1972.

[4] Lester R. Ford, Jr and Delbert R. Fulkerson. Maximal flow through a network. *Can. J. Math.*, 8:399–404, 1956.

[5] Robin Forman. Combinatorial vector fields and dynamical systems. *Math. Z.*, 228(4):228–629, 1998.

[6] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.

[7] Robin Forman. Some applications of combinatorial differential topology. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proceedings of Symposia in Pure Mathematics*, pages 281–313. American Mathematical Society, 2005.

[8] Robin Forman. Topics in combinatorial differential topology and geometry. In *Geometric combinatorics*, volume 13 of *IAS/Park City Mathematics Series*, pages 133–205. American Mathematical Society, 2007.

[9] Étienne Gallais. Combinatorial realization of the Thom-Smale complex via discrete Morse theory. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, 9(2):229–252, 2010.

[10] Philip Hall. On Representatives of Subsets. *J. Lond. Math. Soc.*, 10(1):26–30, 1935.

[11] William Thurston. The theory of foliations of codimension greater than one. *Comment. Math. Helv.*, 49:214–231, 1974.

Gaël Meigniez
Université d'Aix-Marseille
Centre de Mathématiques et d'Informatique
Technopôle de Château-Gombert
39, rue Frédéric Joliot-Curie
13453 MARSEILLE Cedex 13
FRANCE
Gael.Meigniez@univ-amu.fr